**23/2** (2025), 195–211
DOI: 10.5485/TMCS.2025.15646
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https://ojs.lib.unideb.hu/tmcs

Teaching
Mathematics and
Computer Science

# On the nine-point conic of hyperbolic triangles

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*Abstract.* In the Cayley–Klein model, we review some basic results concerning the geometry of hyperbolic triangles. We introduce a new definition of the circumcircle of a hyperbolic triangle, guaranteed to exist in every case, and describe its main properties. Our central theorem establishes, by means of purely elementary projective geometric arguments, that a hyperbolic triangle has a nine-point conic if and only if it is a right triangle.

*Key words and phrases:* Cayley–Klein plane, hyperbolic triangles, Feuerbach circle, eleven-point conic.

*MSC Subject Classification:* 51M09.

## Introduction

Hyperbolic geometry plays a crucial role in the education of future mathematics teachers. Besides its historical significance, it also helps understanding what it means for a statement to be independent of an axiom system. In the current teacher education curriculum, students usually encounter hyperbolic geometry after studying projective geometry, sometimes even within the same course. Therefore, among the classical models, discussing the Cayley–Klein model is the most practical choice: here the knowledge acquired in projective geometry finds direct application. Beyond this practical reason, there are additional advantages of choosing the Cayley–Klein model. It facilitates understanding by representing hyperbolic lines as straight line segments in the Euclidean sense. Moreover, it aligns well with modern worldviews: the "universe" is considered a bounded domain – the interior of a circle – whose geometry approaches Euclidean geometry as the circle's radius increases.

Thus, it is important to develop an exposition of the main theorems of hyperbolic geometry based on elementary projective geometry within this model. Since triangle geometry plays a central role in high school geometry education, it is reasonable to explain this topic in the hyperbolic plane. Then students encounter several similarities between Euclidean and hyperbolic geometry, such as the collinearity of the medians; but also fundamental differences arise: besides the well-known theorems about the angle sum of triangles, an important distinction is that there are no similar but non-congruent triangles in the hyperbolic plane. Another interesting fact is that a triangle does not necessarily have a circumcircle. Naturally, this prompts the investigation of whether hyperbolic triangles have analogues of the Euler line and the nine-point circle (Feuerbach circle), and it is important that these questions are addressed using elementary and projective geometric tools familiar to students.

In our previous work Szilasi (2012a), we investigated the geometry of hyperbolic triangles in this manner. We provided projective proofs for several well-known facts that are analogous to results in the Euclidean case; in particular, we have shown that the medians, the altitudes, and the perpendicular bisectors of the sides form pencils. We gave an elementary projective proof of a theorem of Baldus (1929) on the Euler line of hyperbolic triangles: a hyperbolic triangle has an Euler line if and only if it is isosceles. In this paper, we investigate the Feuerbach circle of hyperbolic triangles. In the Euclidean plane, the midpoints, the feet of the altitudes and the midpoints of the segments of the orthocenter and the vertices lie on a circle. Some hyperbolic generalizations of this theorem have already been known (see, e.g., Akopyan (2011), Vigara (2014)), but they use the so-called pseudo-midpoints instead of midpoints. We focus on the classical case. Since, generally, these nine points are not even on a conic in the hyperbolic plane, it is natural to ask: Under what conditions are they incident to a conic in the hyperbolic case? In this paper, we give an elementary projective proof of a recent result of Evers (2024) on the nine-point conics of hyperbolic quadrangles, that was proved by calculations, using coordinates. Having this result, we show that six of these points – the midpoints and the feet of altitudes – are always on a conic, but this conic contains a seventh point (any of the midpoints of the segments of the orthocenter and the vertices) only in the trivial case of right triangles.

## Preliminaries

For the preliminaries on projective geometry, we refer to Bamberg and Penttila (2023), Coxeter (1955) and O'Hara and Ward (1937), and on the Cayley–Klein plane, we refer to our paper Szilasi (2012a). First, we summarize some of the most important facts.

A nondegenerate conic in the real projective plane will be called the *absolute conic*. Its inner points are the *hyperbolic points*, and the subsets of the projective lines consisting of hyperbolic points are the *hyperbolic lines*. Then the projective line is called the projective line corresponding to the hyperbolic line. The *isometries* are defined as the restrictions of the automorphisms of the absolute conic to the set of hyperbolic points. Thus, we obtain an absolute plane, called the *Cayley–Klein plane*. It can be shown that every hyperbolic plane is isomorphic to the Cayley–Klein plane.

We say that a set of hyperbolic lines forms a *pencil* if the corresponding projective lines are concurrent. If their common point is a hyperbolic point, then the hyperbolic lines are concurrent. Otherwise, if their common point is on the absolute conic, they are called *asymptotically parallel* lines, and, if their common point is an outer point of the absolute conic, we speak of *ultraparallel* lines.

- Two hyperbolic lines are perpendicular if and only if their corresponding projective lines are conjugate with respect to the absolute conic.
- A point $M$ is the midpoint of the segment $\overline{AB}$ if and only if there is a point $M_1$ on the projective line $\overleftrightarrow{AB}$ such that $M$ and $M_1$ are conjugate with respect to the absolute conic and the cross-ratio $(ABMM_1)$ equals $-1$. Then we say that $M_1$ is the *outer point corresponding to the midpoint.*

Let $ABC$ be a hyperbolic triangle. We will use the following notation (Figure 1):

- $A_1$, $B_1$, $C_1$ are the midpoints of the sides;
- $A_0$, $B_0$, $C_0$ are the outer points corresponding to the midpoints of the sides;
- $A'$, $B'$, $C'$ are the poles of the sides.

In this configuration the following statements hold:

(i) The altitudes $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ of the triangle form a pencil. (For a proof, see, for example, Szilasi (2012a).) Their intersection is the *orthocenter*, and we denote it by $H$.

(ii) The outer points $A_0$, $B_0$, $C_0$ are collinear.

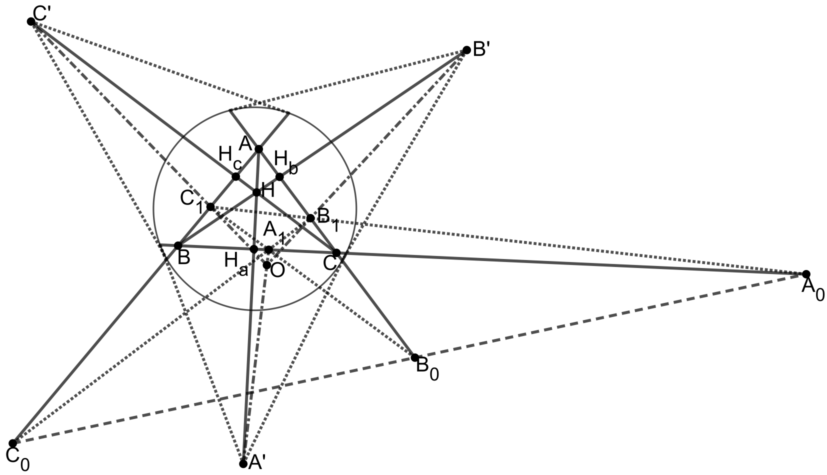


*Figure 1.* The configuration of a hyperbolic triangle

(iii) The points $A_0$, $B_1$, $C_1$ are collinear.

(iv) The perpendicular bisectors of the sides of the triangle are the polars of $A_0$, $B_0$ and $C_0$; therefore the perpendicular bisectors $\overleftrightarrow{A_1A'}$, $\overleftrightarrow{B_1B'}$, $\overleftrightarrow{C_1C'}$ form a pencil.

(v) The midlines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$ form a pencil.

From now on, considering a hyperbolic triangle, we will use these notations. Also, we will denote the feet of the altitudes by $H_a$, $H_b$, $H_c$, respectively.

An ordered quadruple $ABCD$ of points is called a *four-point* if no three of the points are collinear. A *complete quadrangle* is a four-point $ABCD$, together with the six lines, called *sides*, determined by pairs of the four points. The points

$$P := \overleftrightarrow{AB} \cap \overleftrightarrow{CD}, \qquad Q := \overleftrightarrow{AC} \cap \overleftrightarrow{BD}, \qquad R := \overleftrightarrow{AD} \cap \overleftrightarrow{BC}$$

are the *diagonal points*, the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{QR}$ are the *diagonals* of the quadrangle. The collinear points $X$, $Y$, $V$, $W$ form a *harmonic quadruple* if there exists a complete quadrangle whose vertices are $X$ and $Y$, $V$ is a diagonal point, and $W$ is incident to another diagonal of the quadrangle. It holds if and only if $(XYVW) = -1$, and in this case, we call $W$ the *harmonic conjugate* of $V$ with respect to $X$ and $Y$.

A bijective map from the set of points of a projective line to itself is called a *projectivity* of the line if it preserves cross-ratio. A projectivity of a line to itself is said to be an *involution* if its square is the identity. A point and its image are called *conjugate* points of the involution. From the definition, it is easy to see that two pairs of conjugate points uniquely determine an involution. It can be proved that an involution has either zero or two fixed points. It is called *elliptic* in the former case, and *hyperbolic* in the latter. If an involution is elliptic, the conjugate pairs of points separate each other; in the hyperbolic case they do not.

In the case of collineations, i.e., bijective line-preserving maps of the projective plane to itself, the role of involutions is played by harmonic homologies: a central collineation is called a *harmonic homology*, if every point and its image are harmonic conjugates with respect to the center and the intersection of their joining line with the axis. It is well-known (see, e.g., Coxeter (1955)) that the square of a collineation is the identity if and only if it is a harmonic homology.

We will use the classical theorem on the *eleven-point conic* (Figure 2), whose proof can be found, e.g., in Bamberg and Penttila (2023) or O'Hara and Ward (1937).

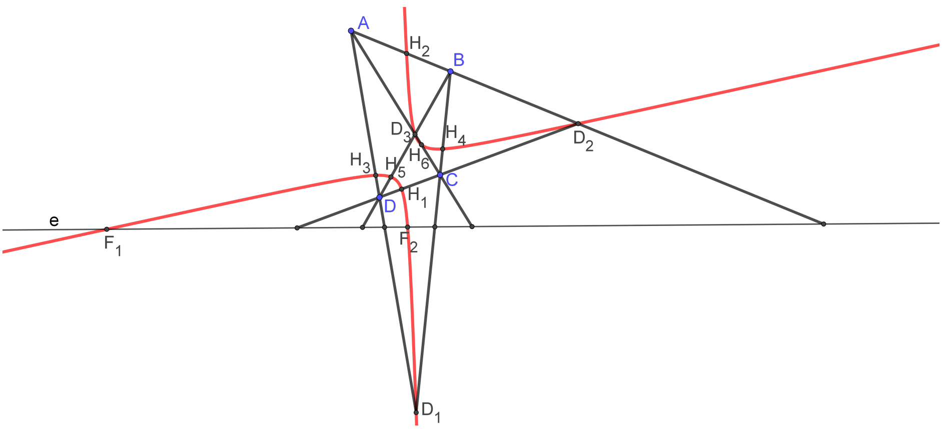


*Figure 2.* The eleven-point conic of $ABCD$ with respect to $e$: the diagonal points $D_1$, $D_2$, $D_3$, the harmonic conjugates $H_1$–$H_6$, and the fixed points $F_1$, $F_2$.

THEOREM 1. *Let $ABCD$ be a complete quadrangle, and suppose that $e$ is a line which is not incident to any of the vertices. Then the poles of the conics through the vertices of $ABCD$ with respect to $e$ are on a conic. The following points are also on this conic:*

- *the diagonal points $D_1$, $D_2$ and $D_3$ of the quadrangle;*
- *for any side of the quadrangle, the harmonic conjugate of its intersection with $e$ with respect to the vertices on the side;*
- *the fixed points of the involution determined by the intersections of $e$ and the conics that go through the vertices of the quadrangle.*

*This conic is called the* eleven-point conic *of $e$ with respect to $ABCD$.*

In the Euclidean plane, the eleven-point conic of the line at infinity with respect to $ABCH$, where $H$ is the orthocenter of the triangle $ABC$, is the Feuerbach circle. In this paper, we try to generalize this concept to the hyperbolic plane. Our question is if these nine points are on a conic in the hyperbolic case as well.

## Conics and circles in the Cayley–Klein plane

First, we have to clarify what a conic means in the hyperbolic plane. In the Cayley–Klein plane, we are going to use the most natural concept. Izmestiev in his study (2017) discussed that it is appropriate to play the role of the Euclidean conics in the hyperbolic case. We note that it is also possible to define conics of the hyperbolic plane using their metric properties, on this topic we refer to Molnár (1978) and Weiss (2016).

Definition 1. *In the Cayley–Klein plane a* hyperbolic conic *(or* conic*) is the set of hyperbolic points of a projective conic.*

The perpendicular bisectors of any hyperbolic triangle form a pencil, but their common point is not necessarily an inner point of the absolute conic. In any case, their intersection $O$ in the projective plane is called the *circumcenter* of the triangle. Since it is not always a hyperbolic point, not all hyperbolic triangles have a circumcircle in the classical sense. We generalize the concept of the circumcircle to hyperbolic triangles so that every hyperbolic triangle will have one.

Lemma 2. *Let $A$, $B$, $C$, $P$ be four points in the real projective plane in general position, and let $p$ be a line that is not incident to any of these points. There is a unique conic $c$ such that $A$, $B$, $C$ are on $c$, and the polar of $P$ with respect to $c$ is the line $p$.*

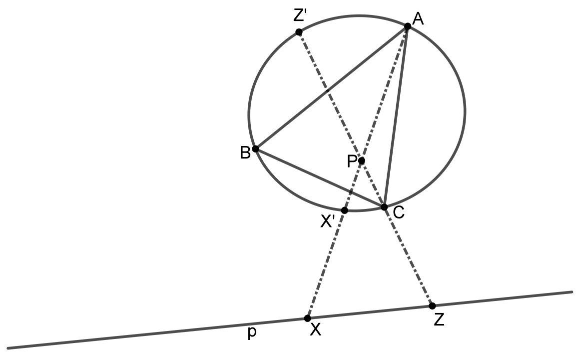


*Figure 3.* Conic through three points and with a given polar of a given point

Proof. As Figure 3 shows, let $\overleftrightarrow{AP} \cap p$ be $X$, and let $X'$ be the harmonic conjugate of $A$ with respect to $P$ and $X$. Similarly, let $\overleftrightarrow{CP} \cap p$ be $Z$, and let $Z'$ be the harmonic conjugate of $C$ with respect to $P$ and $Z$. Then the conic through $A$, $B$, $C$, $X'$ and $Z'$ is the desired conic, because, by our construction, the polar of $P$ is $\overleftrightarrow{XZ} = p$. It is uniquely determined by these five points. □

Definition 2. *Let $ABC$ be a hyperbolic triangle. The conic $c$ that contains $A$, $B$ and $C$ such that the polar of the circumcenter $O$ with respect to $c$ is the line $\overleftrightarrow{A_0B_0}$ of the outer midpoints is called the* circumcircle *of $ABC$.*

We mention some simple properties of the circumcircle $c$ (Figure 4, where $a$ denotes the absolute conic):

(1) The polar of any of the outer midpoints with respect to the circumcircle and to the absolute conic coincide.

Indeed, for example, the polar of the outer midpoint $A_0$ with respect to the absolute conic is $\overleftrightarrow{A_1O}$, where $A_1$ is the midpoint of $\overline{BC}$. Since $(A_0A_1BC)$ is harmonic, $A_0$ and $A_1$ are also conjugate with respect to $c$. By definition, $A_0$ and $O$ are conjugate with respect to $c$, whence our claim.

(2) On the line $\overleftrightarrow{A_0B_0}$ of the outer midpoints, the involutions of conjugate points with respect to $c$ and to the absolute conic coincide.

Indeed, the images of $A_0$ and $B_0$ with respect to the two involutions are the same points, as their polars with respect to the circumcircle and to

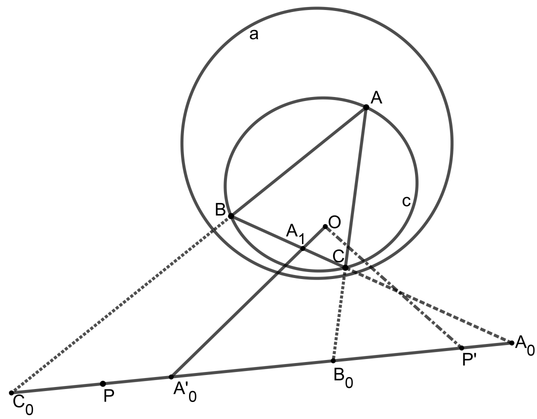


*Figure 4.* The circumcircle of a triangle

the absolute conic coincide. As an involution is uniquely determined by two conjugate point-pairs, our claim follows.

(3) Any point $P$ on the line $\overleftrightarrow{A_0B_0}$ of the outer midpoints has the same polar with respect to $c$ and the absolute conic.

This is clear, since if $P'$ is the image of $P$ by the involution of conjugate points, the polar of $P$ is $\overleftrightarrow{OP'}$.

(4) The hyperbolic reflection with center $P$ is the harmonic homology with axis $\overleftrightarrow{OP'}$. Therefore, the circumcircle is invariant by this reflection. (Evers (2024) uses this fact as a definition: all points of $\overleftrightarrow{A_0B_0}$ are *symmetry point*s of the circumcircle.)

(5) The circumcircle is uniquely determined by its center $O$ and one of its points $A$.

The circumcircle is a conic $c$ through $A$ such that the polar of $O$ with respect to $c$ is the polar $o$ of $O$ with respect to the absolute conic. Using the notation of Figure 5, let $l$ be an arbitrary line through $A$. We show that the other intersection $A'$ of $l$ with $c$ is uniquely determined. Let $P$ be the intersection of $l$ and $o$. Then $A'$ is the image of $A$ by the hyperbolic reflection with center $P$, i.e., the harmonic homology whose center is $P$ and axis is the polar of $P$ with respect to the absolute conic.

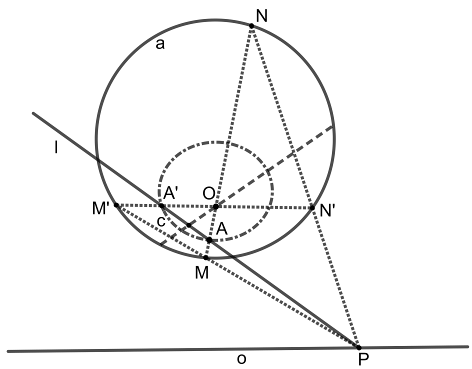


*Figure 5.* The circumcircle is uniquely determined by its center and one of its points.

(6) If the center of the circumcircle $O$ is a hyperbolic point, then for all $A$ points of the circumcircle, the hyperbolic distance $d(O, A)$ is constant. Therefore, if the circumcircle in the metrical sense exists, it is the same as the circumcircle we defined.

The hyperbolic distance $d(O, A)$ is defined as the logarithm of the cross-ratio $(OAMN)$, where $M$ and $N$ are the intersections of $\overleftrightarrow{OA}$ and the absolute conic. We can get an arbitrary point $A'$ of the circumcircle as in our previous proof, i.e., as its other intersection of an arbitrary line through $A$. We use the notation of Figure 5 again. As $\overleftrightarrow{OA'}$ is the hyperbolic reflection of $\overleftrightarrow{OA}$ from center $P$, and the absolute conic is invariant by this reflection, the images of $M$ and $N$ are the intersections $M'$ and $N'$ of $\overleftrightarrow{OA'}$ and the absolute conic. Therefore, $(OA'M'N') = (OAMN)$, which means that the hyperbolic distances $d(O, A')$ and $d(O, A)$ are equal.

(7) The circumcircle does not have points outside the absolute conic.

To see this, following the notation of Figure 6, suppose that $X$ is a point of the circumcircle, and let $X_1$ be the intersection of $\overleftrightarrow{AX}$ and $\overleftrightarrow{A_0B_0}$. Let the intersection of $\overleftrightarrow{AX}$ and the polar of $X_1$ be $X'$. Then $(AXX'X_1)$ is harmonic. Since $X'$ and $X_1$ are conjugate with respect to the absolute conic, $(PQX'X_1)$ is also harmonic, where $P$ and $Q$ are the intersections of their line and the absolute conic. Therefore, $(P, Q)$ and $(A, X)$ are conjugate pairs of the involution with fixed points $X'$, $X_1$. It is a hyperbolic involution, and $A$ is between the points $(P, Q)$ in the Euclidean sense. Therefore, $X$ also has to be between $P$ and $Q$, since the conjugate point-pairs of a hyperbolic involution do not separate each other.

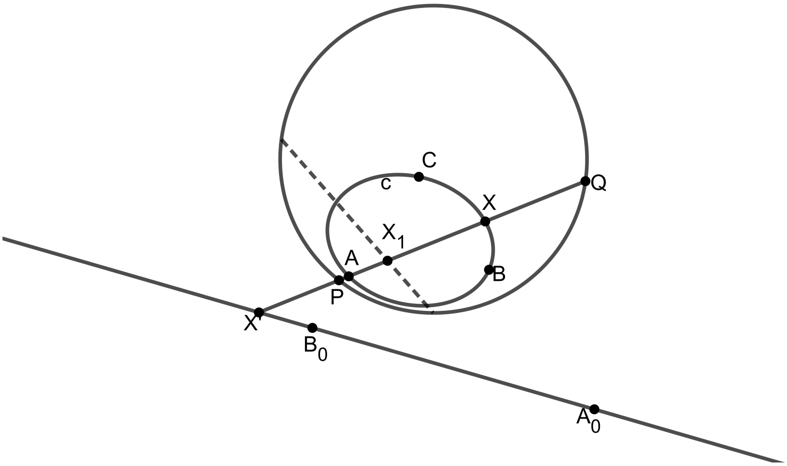


*Figure 6.* The circumcircle is an Euclidean ellipse.

(8) If the absolute conic is a circle, then the circumcircle is an ellipse in the Euclidean sense.

Indeed, it is a conic that has no points outside a circle.

## The six-point conic

First, we construct the eleven-point conic of the line $\overleftrightarrow{A_0B_0}$ of the outer points corresponding to the midpoints of the sides with respect to $ABCH$. The diagonals of the quadrangle are the feet of the altitudes. The harmonic conjugate of any outer point with respect to the vertices on the corresponding side is the midpoint of the side. Therefore, in this case, the eleven-point conic has these six points, and we obtain the following theorem.

Theorem 3. *The midpoints of the sides and the feet of the altitudes of a hyperbolic triangle lie on a conic.*

We call this conic the *six-point conic* of the triangle (Figure 7).

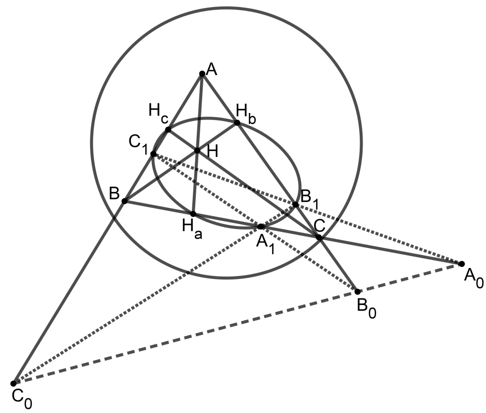


*Figure 7.* The six-point conic

We note that this theorem follows easily from the *Carnot theorem*. We recall this theorem here, and for an elementary proof, using the classical theorems of Ceva, Menelaos and Pascal, we refer to Szilasi (2012b):

> Let $ABC$ be an arbitrary triangle, and let $(A_1, A_2)$, $(B_1, B_2)$, $(C_1, C_2)$ be pairs points (different from the vertices) on the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$, respectively. Then the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ and $C_2$ are on a conic if and only if
>
> $$(ABC_1)(ABC_2)(BCA_1)(BCA_2)(CAB_1)(CAB_2) = 1.$$

Considering the configuration of the hyperbolic triangle, since $\overleftrightarrow{AA_1}$, $\overleftrightarrow{BB_1}$, $\overleftrightarrow{CC_1}$ are concurrent, by the Ceva theorem:

$$(ABC_1)(BCA_1)(CAB_1) = 1.$$

Similarly, since $\overleftrightarrow{AH_a}$, $\overleftrightarrow{BH_b}$, $\overleftrightarrow{CH_c}$ are concurrent, Ceva's theorem implies that

$$(ABH_c)(BCH_a)(CAH_b) = 1.$$

Therefore, we get

$$(ABH_1)(BCH_1)(CAH_1)(ABH_c)(BCH_a)(CAH_b) = 1,$$

which, by Carnot's theorem, is equivalent to the six points lying on a conic.

## Hyperbolic triangles with a nine-point conic

It is natural to ask, whether the six-point conic contains the midpoints of the segments formed by the vertices and the orthocenter just like in the Euclidean case. We are going to prove that *apart from the trivial case of right triangles,* in the hyperbolic case this is impossible, therefore *no hyperbolic triangles have a nine-point circle* (or even a seven-point conic).

We note that if $ABC$ is a right triangle where $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are perpendicular, then the "nine-point conic" has only 5 points (in this case, $A$ is the orthocenter, the feet of two altitudes coincide with $A$ and the midpoints of the segments connecting the orthocenter with the vertices do not exist or coincide with two midpoints), and 5 points of general position always lie on a conic.

Theorem 4. *Let $ABC$ be a hyperbolic triangle. Suppose that $ABC$ is not a right triangle, and let $H$ be its orthocenter. If one of the midpoints of the segments connecting $H$ with the vertices of $ABC$ are on the six-point conic of $ABC$, then all of them are, and the perpendicular bisectors of the sides and of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$ are concurrent.*

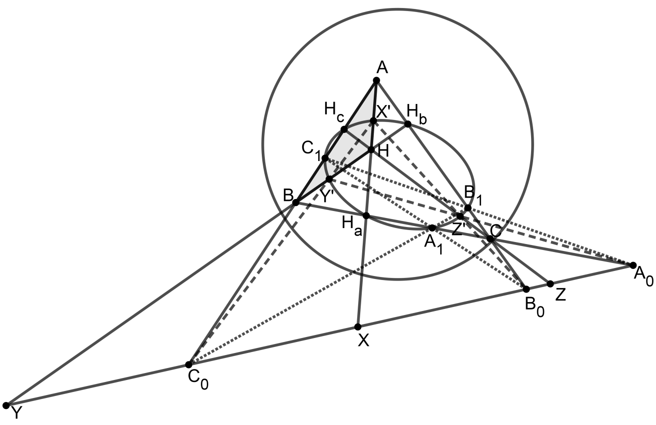


*Figure 8.* If the six-point conic contains one more midpoints, then it contains all three of them.

Proof. Following the notation of Figure 8, let $X$ be the intersection of $\overleftrightarrow{AH}$ and the line $\overleftrightarrow{A_0B_0}$ of the outer points corresponding to the midpoints of $ABC$.

If $X'$ is the harmonic conjugate of $X$ with respect to $A$ and $H$, then $X'$ is the intersection of the six-point conic and $\overleftrightarrow{AH}$ (which differs from $H_a$), because, by the definition of the six-point conic, it contains the harmonic conjugate of the intersection $X$ of $\overleftrightarrow{A_0B_0}$ and the side $\overleftrightarrow{AB}$ of the complete quadrangle $ABCH$ with respect to the vertices $A$ and $H$.

Suppose that $X'$ is the midpoint of $\overline{AH}$. In this case, $X$ is the outer point corresponding to $X'$, since $(AHXX') = -1$.

From this, it follows that $\overleftrightarrow{A_0B_0} = \overleftrightarrow{A_0X}$ is the line of the outer points corresponding to the midpoints of the triangles $ABC$ and $ABH$ as well, since it contains two of the outer points corresponding to the midpoints of $ABH$ as well.

The pole of the line of the outer points corresponding to the midpoints of a triangle is the intersection of the perpendicular bisectors of the triangle, so it follows that the intersection of the perpendicular bisectors of $ABC$ and $ABH$ coincide. This point is the intersection of the perpendicular bisectors of $BCH$ and $CAH$ as well.

This implies that the line $\overleftrightarrow{A_0B_0}$ is also the line of the outer points corresponding to the midpoints of the triangles $BCH$ and $CAH$, which concludes the proof. □

Our next theorem is formulated in a somewhat more general form by Evers (Theorem 4, 2024). Here, we present an elementary proof of that result.

We note that Evers defined the nine-point conic of a quadrangle: using our definition, it is the eleven-point conic of the line $\overleftrightarrow{A_0B_0}$ (or in the Euclidean case, the line at infinity) with respect to the quadrangle. His theorem is the following: *there exists a nine-point conic for the the quadrangle $ABCD$ if and only if the points $A$, $B$, $C$, $D$ are concyclic; in this case, the circle center is also on the nine-point conic.* In our case, the nine-point conic of the triangle $ABC$ is the nine-point conic of the quadrangle $ABCH$, where $H$ is the orthocenter of the triangle. Hence, the result of Evers takes the following form.

Theorem 5. *A hyperbolic triangle has a nine-point conic (i.e., a six-point conic that contains the midpoints of the segments connecting its orthocenter with the vertices) if and only if its orthocenter is on its circumcircle. In this case, the six-point conic also contains its circumcenter.*

Proof. Suppose that the triangle $ABC$ has a nine-point conic. Then all the midpoints of the segments connecting $H$ with the vertices are on the six-point conic. Therefore, by Theorem 4, the perpendicular bisectors of the sides and of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$ are concurrent. This point is the circumcenter of $ABC$, $ABH$, $BCH$ and $CAH$, so the circumcenters of these triangles coincide. Since we proved that the circumcircle is determined by the circumcenter and one of its points, these triangles have the same circumcircle. Therefore, $H$ is on the circumcircle of $ABC$.

The eleven-point conic of $\overleftrightarrow{A_0B_0}$ with respect to $ABCH$ contains, by definition, the poles of $\overleftrightarrow{A_0B_0}$ with respect to the conics through $A$, $B$, $C$ and $H$. Since, in this case, the circumcircle is one of these conics, the eleven-point conic contains the pole of $\overleftrightarrow{A_0B_0}$ with respect to the circumcircle, which is the circumcenter. □

Theorem 6. *If a hyperbolic triangle's orthocenter lies on its circumcircle, then it is a right triangle.*

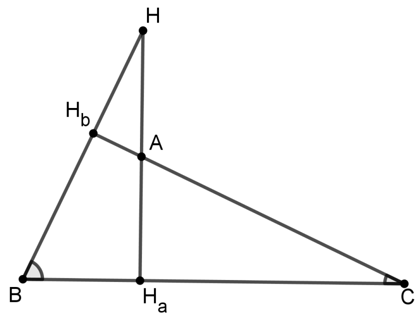


*Figure 9.* The third vertex and the orthocenter are in the same half-plane.

Proof. Let $ABC$ be a triangle in an absolute plane. Suppose that $\overline{BC}$ is the longest side of the triangle, then the angles at $B$ and $C$ are acute (Figure 9). We are going to show that the orthocenter $H$ is in the same half-plane determined by $\overleftrightarrow{BC}$ as $A$. Since $\overline{BC}$ is the longest side, by Theorem 6.4.3. of Millmann and Parker (1981), the foot of altitude $H_a$ from $A$ is between $B$ and $C$. The point $H$ is on the perpendicular to $\overleftrightarrow{BC}$ from $H_a$. Also, $H$ is on the perpendicular to $\overleftrightarrow{AC}$ from $B$. Let $H_b$ be the foot of this perpendicular. Since $BCA\angle$ is acute and $BCH_b$ is a right triangle, $CBH_B\angle$ is also acute. Therefore, the half-line $\overrightarrow{BH_B}$ is in the same half-plane as $A$, so its intersection with $\overleftrightarrow{AH_a}$, which is the orthocenter, is also in this half-plane.

Suppose now that $ABC$ is a hyperbolic triangle, and the absolute conic is a circle (Figure 10). Let $H_a$ be the foot of altitude from $A$ on $\overleftrightarrow{BC}$. Then the orthocenter is on $\overleftrightarrow{AH_a}$. So, if the orthocenter lies on the circumcircle, then it is one of the intersections of $\overleftrightarrow{AH_a}$ and the circumcircle. One of these intersections is $A$. If $A$ is the orthocenter, then $ABC$ is a right triangle.

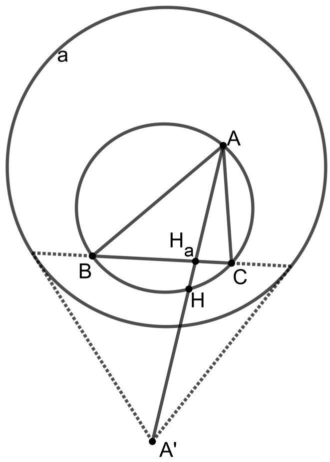


*Figure 10.* The intersection of an altitude and the circumcircle

Since the circumcircle is an ellipse in Euclidean sense, its other intersection with $\overleftrightarrow{AH_a}$ is not in the same half-plane determined by $\overleftrightarrow{BC}$ as $A$, therefore it cannot be the orthocenter. □

Thus, we have established that, in general, the Feuerbach circle of hyperbolic triangles does not exist in the classical sense. Nevertheless, it is of interest that, under a modified definition, one can construct certain "hyperbolic Feuerbach circles". We conclude the paper by briefly describing two such constructions.

Akopyan (2011) generalized the nice property of the Feuerbach circle in Euclidean geometry that it is tangent to the incircle and the excircles of the triangle. He defined the *pseudomedian*s of the hyperbolic triangle as the area-bisecting lines through the vertices, and denoted the feet of these lines on the corresponding sides of the triangle by $M_a$, $M_b$, $M_c$. He proved that the circle through these three points is tangent to the incircle and the excircles of the triangle, and therefore it is a possible hyperbolic generalization of the Feuerbach circle.

Vigara (2014) defined the nine-point conic of hyperbolic triangles, similarly to our approach, using the Cayley–Klein model and the theorem on the eleven-point conic. However, instead of using the line of the outer midpoints, as we did, he defines it as the eleven-point conic of the *line of the poles* $A_p$, $B_p$, $C_p$ *of the altitudes of triangle* with respect to $ABCH$. Using the definition of the eleven-point conic, the feet of altitudes of the triangle are incident to this conic (as they are the diagonals of $ABCH$), just like in the case of our six-point conic. This conic also contains the harmonic conjugates of $A_p$, $B_p$ and $C_p$ with respect to the vertices of the triangle, called the *pseudo-midpoints*. Three other points of the conic can be derived the following way: let $A_p$ and $B_p$ be two altitudes' poles, and intersect the lines $\overleftrightarrow{HA_p}$ and $\overleftrightarrow{HB_p}$ with the corresponding sides of the triangle; then the intersection of their line and the third altitude is also on the conic. The nine-point conic defined this way can also be a hyperbolic analogue of the Feuerbach circle. Using this approach, considering these pseudomidpoints instead of the midpoints of the sides, one can get a nice hyperbolic generalization of the Euler-line as well, which also does not exist for hyperbolic triangles in the classical sense.

## Acknowledgements

I am grateful to the anonymous reviewers for their valuable comments and suggestions.

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*(Received April, 2025)*