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Teaching Mathematics and Computer Science

Apollonea.com project: integrating geometry and collaboration in education

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Abstract. This article presents the Apollonea.com project, which aims to make the solutions to Apollonius' problems accessible to students and teachers through modern technology. The web platform contains more than 150 interactive constructions created by students using GeoGebra, allowing for dynamic manipulation and visualization of solutions to various variants of Apollonius' problems. The project combines classical geometric problems with an interdisciplinary approach, teamwork, and the use of modern technology. The article describes the process of developing the Apollonea.com website, the use of GeoGebra in the project, the structure and functions of the website, and its educational benefits in enhancing students' geometric skills. The project demonstrates how traditional mathematics education can be connected with modern ICT tools.

Key words and phrases: Apollonius problems, dynamic geometry, GeoGebra, student project, interdisciplinary learning, geometric transformations.

MSC Subject Classification: 97U50, 97G40, 51M04, 68U05.

Introduction

Apollonius' problems provide a unique opportunity to develop a wide range of geometric skills and knowledge. They offer a range of difficulty levels, from simple to highly challenging problems, allowing for a gradual increase in complexity and the engaging of students of different ages and levels of expertise. Solving these problems supports not only logical thinking and spatial imagination but also introduces students to advanced geometric concepts such as the power of a point with respect to a circle, dilation, and circular inversion. These principles rarely appear in standard curricula, yet they find natural application in Apollonius' problems.

The project described in this article took place over three academic years (2022 to 2025) as part of an elective mathematics seminar in the final year of an eight-year grammar school in Prague. The goal of the project was to create a website at Apollonea.com featuring interactive constructions with solutions to various variants of Apollonius' problems. The constructions were created using GeoGebra, and the project website currently contains more than 150 of them. Each construction is accompanied by a written description in both Czech and English.

The project involved 29 students from three consecutive years of the seminar. The students created both the constructions solving the problems and the accompanying texts. The scope and complexity of the project required the establishment of an organizational structure and the distribution of tasks among the students within it. Two students programmed the website, while others collaborated on designing icons. Additionally, it was necessary to ensure the verification of completed constructions and the correction of identified errors. Another 25 students from a lower grade translated the Czech texts into English.

The result of this project is not only a publicly accessible educational resource but also an inspiring example of how modern technology, classical geometry, and teamwork can be integrated to enrich the educational process.

What are Apollonius' problems?

Apollonius' problems are named after Apollonius of Perga (262–190 BCE), one of the most significant mathematicians and geometers of antiquity, who was the first to study these problems (Court, 1961). The goal of each of Apollonius' problems is to find a circle or circles tangent to three given objects – points, lines, or circles (Budai, 2012).

There are ten basic types of Apollonius' problems, which differ based on the combination of given objects. Examples include finding circles tangent to three given circles, finding circles passing through given points and touching a given circle, and finding circles passing through a given point and touching a given circle and a given line. Each basic problem type has several additional subvariants, which vary depending on the relative positions of the given objects. Two given lines may be parallel or intersecting, a given point may be located inside, outside, or on a given circle, and so on. These variations increase the number of possible problems.

For example, in the problem where two points and one line are given, the Apollonea.com website presents four different subvariants depending on the relative positions of the given objects, while for the problem where one point and two circles are given, there are as many as twenty-one different problem instances. Variants of the same problem type may differ in the method used to solve them or in the number of possible solutions (Budai, 2012; Nocar & Dofková, 2020).

Exploring different types of Apollonius' problems and their subvariants encourages students to seek different approaches to solving similar problems, as the specific positioning of given objects often allows for alternate, and sometimes simpler, solutions.

The importance of Apollonius' problems in education

Apollonius' problems offer a unique opportunity to connect classical geometry with modern technology, thereby developing a wide range of student competencies. They are ideal for teaching geometry due to their varying levels of difficulty. Basic variants can be solved using elementary geometric skills, while more complex problems require advanced concepts such as the power of a point with respect to a circle or circular inversion (Budai, 2012; Cibien et al., 2023).

The use of tools such as GeoGebra not only simplifies constructions but also enables interactive exploration of geometric relationships. Students can experiment with different scenarios, which helps them discover new connections (Nocar & Dofková, 2020). This approach not only facilitates problem-solving but also contributes to the overall development of mathematical thinking. Dynamic geometry provides students with the ability to solve complex problems more quickly, accurately, and intuitively. Research shows that the use of tools such as GeoGebra not only improves students' success in mathematics but also contributes to a deeper understanding of concepts and a more lasting retention of knowledge (Alabdulaziz et al., 2021).

Due to their diversity, Apollonius' problems represent not only a mathematical challenge but also an opportunity to develop creativity and the ability to analyze complex problems. These problems become a valuable tool for teachers who aim to support not only students' geometric skills but also their broader ability to work with modern technologies (Nocar & Dofková, 2020).

The educational design of the Apollonea.com project is also consistent with established pedagogical frameworks. Its interdisciplinary, student-centered character reflects principles of constructivism, where learners actively build new knowledge through exploration and collaboration (Piaget, 1970; Vygotsky, 1978). Furthermore, the organization of the project corresponds to project-based learning (PBL), as students worked on authentic tasks, assumed diverse roles, and produced tangible outcomes in the form of interactive constructions and a functioning website. These features align with inquiry-based approaches in education, which emphasize problem-solving, experimentation, and scaffolded guidance as central tools for discovery (Hmelo-Silver et al., 2007). Positioning the project within these frameworks highlights its potential not only as a collection of successful classroom practices but also as an example of how classical geometry can be taught in accordance with contemporary theories of learning.

Methods for solving Apollonius' problems

As previously mentioned, Apollonius' problems can be solved using a wide range of methods, from basic geometric principles to advanced techniques such as similarity transformations, dilation, or the concept of the power of a point with respect to a circle. Simpler problem variants can be solved using elementary constructions, such as constructing the perpendicular bisectors of segments between two points or the angle bisectors. For example, the problem of finding a circle passing through three given points can be solved solely by constructing the perpendicular bisectors of the segments connecting the points. In fact, this problem is commonly addressed in schools as the construction of the circumference of a triangle.

On the other hand, more complex problems, such as constructing a circle tangent to three given circles, require students to have knowledge of more advanced geometric concepts, a deeper understanding of geometric properties, and strong visualization skills. The use of methods such as dilation or the power of a point with respect to a circle allows students to identify key relationships between geometric objects and analyze their relative positions. These methods lead to efficient construction procedures and enhance students' ability to apply more complex concepts to specific problems. Other geometric tools also play a role in solving Apollonius' problems, including translation, Thales' theorem, Euclid's theorems on segment lengths and the altitude in a right triangle, and others.

One of the key techniques used by students in the project was the previously mentioned circular inversion. This geometric transformation allows different types of given problems to be converted into other, more easily solvable forms. For example, a problem involving a given point and two circles can be transformed into the problem of finding the common tangents of two circles by placing the center of the inversion circle at the given point. This conversion simplifies the original problem into another one that can be solved using similarity transformations, Thales' theorem, or dilation. By experimenting with this technique, students gain not only practical experience with circular inversion but also a better understanding of geometric transformations and their properties (Nocar et al., 2022).

Use of different methods on the Apollonea.com website

Each problem on the Apollonea.com project website includes information about the primary method used to solve it. The website also features pages dedicated to individual methods, such as dilation or circular inversion. These pages contain not only a description of the method but also links to constructions where the method was applied to solve a problem. In some cases, the same problem was solved using multiple methods, allowing for a comparison of different approaches to the same problem and a deeper understanding of their advantages and disadvantages.

For example, the problem involving two points and a line, where neither point lies on the line, is solved on Apollonea.com using four different methods: (1) similarity transformation, (2) the power of a point with respect to a circle, (3) circular inversion, and (4) a non-Euclidean approach based on loci of points with given properties. This diverse approach allows users not only to compare different methods and their efficiency but also to understand the differences between Euclidean and non-Euclidean constructions. The combination of various techniques, from basic to advanced, enables an effective progression in the difficulty of the problems being solved. The ability to apply different solution methods to Apollonius' problems provides an opportunity for students to develop spatial reasoning, logical thinking, and creativity (Cibien et al., 2023).

Use of GeoGebra in solving Apollonius' problems

GeoGebra played a crucial role in the Apollonea.com project. Its ability to provide access to dynamic constructions and enable students to solve geometric problems efficiently proved to be an extremely valuable tool. This dynamic software allowed geometry students not only to construct geometric objects accurately but also to experiment and visualize more complex geometric concepts. The use of GeoGebra in the project proved beneficial both for students who had prior experience with the software, and for those who were introduced to it for the first time during the project.

Thanks to GeoGebra, students were able to dynamically manipulate constructions, adjust parameters, and explore different scenarios, which helped them gain a deeper understanding of the properties of geometric objects. They used GeoGebra to discover solutions to problems while simultaneously creating constructions that were later uploaded to the Apollonea.com website. This interactive approach significantly enhanced their ability to analyze and seek effective solutions. As one student noted during the project reflection: "Dynamic geometry, seeing how relationships between objects change through movement, was amazing."

GeoGebra was also invaluable in introducing advanced techniques such as circular inversion and dilation. The ability to visualize these methods in real time helped students grasp more complex geometric relationships and discover new mathematical connections. For example, finding the locus of centers of circles tangent to two given circles is often too abstract for students initially. It is easier for them to find a single specific solution with a given radius. By using a slider to adjust the input radius, students can explore the properties and position of the solution depending on this parameter. As a result, the final locus becomes much clearer. For better visualization of this locus, additional tools such as the trace point feature or the Locus tool can be used. Solutions based on these loci are often non-Euclidean. Such solutions can also be found on the Apollonea.com website.

The predefined tools in GeoGebra also significantly simplify the execution of constructions. For example, the Circular Inversion tool allows for the direct transformation of objects without the need for additional auxiliary constructions. Unlike drawing on paper, GeoGebra enables an easy return to previous steps, adaptation of the construction to modified input parameters, and experimentation with different scenarios. Additionally, GeoGebra's tools allow for the adjustment of visual elements such as colors, line thickness, and line styles, and the ability to hide auxiliary constructions, which improves clarity. By making constructions easier, faster, and clearer visually, the use of GeoGebra makes even more complex constructions accessible to students.

Dynamic elements, such as sliders, also allow for the step-by-step breakdown of construction processes. This approach has been implemented on the Apollonea.com website, where users can not only view the completed construction but also follow its creation step by step. As emphasized by Schreiberová & Morávková (2023), interactive visualization of mathematical concepts using dynamic software like GeoGebra enables students to experiment with different scenarios and better understand relationships between geometric objects. Dynamic elements, such as sliders, allow for the gradual display of individual construction steps and support a deeper understanding of transformation processes, which are key to solving Apollonius' problems. GeoGebra becomes a comprehensive tool that fosters students' creativity and encourages them to experiment in geometry while simultaneously connecting their theoretical knowledge with practical applications.

Development of language skills

The accompanying texts and descriptions of construction procedures on the Apollonea.com website were originally written in Czech. To make the website accessible to an international audience, all texts were translated into English. The translation process was not just a technical task but also a valuable educational opportunity for students. English teachers were invited to collaborate on the project, involving students from lower grades in the translation work. During English lessons, students worked on translating the descriptions of constructions. This task allowed them to expand their knowledge of English by learning mathematical terminology that is not commonly found in standard textbooks.

To facilitate their work, students created a shared document where they recorded translations of specific mathematical terms. This document became a valuable resource that their classmates could also use for future translations. Additionally, they kept track of which student was translating each problem in a shared project table, which helped to improve the overall organization of the process.

This interdisciplinary approach connected mathematics with language education. Working on the translations allowed students to gain a better understanding of technical terminology and learn to work effectively with mathematical texts in a foreign language. One student described it as "a helpful experience, especially if you find good tools like DeepL." Other students highlighted that working on translations taught them precision and systematic thinking when handling foreign-language materials.

Research by Yahya and Hashim (2021) shows that interdisciplinary projects, such as integrating mathematics and English, not only support the development of technical and language skills but also increase student motivation. These approaches provide authentic educational experiences that help students develop practical skills applicable to the real world.

Creation of the website and project coordination

The Apollonea.com project was not only a set of geometry problems for students but also an opportunity to gain experience working on a complex long-term project. Throughout the process, students learned to establish an organizational structure, clearly define responsibilities for different parts of the project, and ensure quality control and error correction.

A key tool for coordination was a shared table, which was created and managed by the students themselves. This table contained information about the different problem variants, who was working on them, and which method was being used. The records helped prevent duplicate work, ensured that no variant was overlooked, and facilitated the organization of construction reviews. The accuracy and completeness of each construction and its description were checked by another student upon completion. In addition, students systematically organized and carried out checks on the functionality and completeness of other website components. All identified errors and deficiencies were recorded in the shared table and subsequently corrected.

Beyond this, students discussed and agreed on rules for maintaining a consistent appearance of the constructions, including point types and sizes, line thickness and colors, and the naming conventions for objects. This process not only improved the quality of the constructions but also taught students effective collaboration and respect for each other's input. The project website was developed by two seminar students using the Python programming language and the Jinja templating system. A group of eight other students collaborated on animating the icons that link to different types of problems, using an SVG code editor.

The result is an intuitive and visually friendly platform that allows for easy navigation between various problem variants and their solutions. The website development required effective communication and task distribution among team members, contributing to the development of their organizational and technical skills.

Structure and functions of the Apollonea.com website

The Apollonea.com project website serves as a key output and platform that makes the project's results accessible to a broad audience. The website was designed to be clear, intuitive, and accessible for both students and teachers. The main page features a navigation panel with ten icons, each linking to one of the ten fundamental types of Apollonius' problems. Clicking on a specific icon takes the user to a page dedicated to that problem type. Here, the different problem variants are represented by icons illustrating various relative positions of the given objects. This visual system enables users to identify the desired problem. After selecting a specific problem variant, the user is redirected to a page displaying the solution. If multiple solution methods are available for the same problem, all variants are listed on the page. Each solution is linked to a slider, allowing users to view individual construction steps as they progress through the process. This interactive feature enhances understanding by presenting the entire procedure step by step.

Each construction is also accompanied by a detailed explanatory text describing the individual steps. In addition to viewing the constructions directly on the website, users have the option to download the corresponding GeoGebra (.ggb) file. This allows them to further experiment and explore various aspects of the constructions within their own GeoGebra environment.

Additional sections of the website are dedicated to the methods used in solving Apollonius' problems, such as circular inversion, dilation, and the power of a point with respect to a circle. These sections provide descriptions of the principles behind each method and offer links to problems where these methods have been practically applied. This allows users to easily connect theoretical knowledge with its practical application. Great emphasis was placed on user-friendliness and easy navigation. Icons and visual elements were designed to be intuitive, even for users with no prior experience with Apollonius' problems or GeoGebra.

Evidence of student learning outcomes

To complement qualitative observations, a short feedback questionnaire was administered to one subgroup of students ($n = 9$). Although the number of respondents was limited, the results provide indicative evidence of the project's educational impact. Most participants reported that they had only basic or no prior knowledge of GeoGebra, yet the majority rated their progress in using the software as substantial (six students selected 4/5 and three students 5/5). GeoGebra itself was perceived as a highly valuable tool for solving geometric problems, with seven students rating it as *very beneficial* and two as *beneficial*.

Regarding general geometry skills, five students rated their progress as 4/5 and four as 3/5, suggesting that the project helped them develop a deeper understanding of geometric concepts beyond their previous knowledge. The overall evaluation of the project ranged between three and four stars out of five.

These results should be interpreted with caution, as the number of respondents was small and the data therefore cannot be generalized to all participants. Nevertheless, they support the qualitative evidence that the project enhanced students' skills in geometry and GeoGebra, and provided meaningful learning experiences.

Conclusion

The Apollonea.com project demonstrates how modern technologies can be integrated with classical geometric problems to create an innovative educational tool. The web platform offers more than 150 interactive constructions of Apollonius' problems, enabling students and teachers to explore and discover geometric principles in a dynamic and practical way.

Student feedback confirms that the project had a significant impact on their understanding of geometry. They mastered advanced techniques such as circular inversion and dilation, and began to think about geometry in a more comprehensive way. One student, for example, stated: "The project made me think about geometry more as a whole rather than as a collection of individual problems."

The project was realized in an elective seminar at a high school, which made it possible to work with a small, motivated group of students over a longer period of time. Under less favorable conditions, achieving a similar scope of results would be difficult. However, the underlying approach – combining classical geometry with dynamic software, teamwork, and interdisciplinary collaboration – is transferable.

With less mathematically demanding content or a more modest project structure, the same principles could be applied in regular classes and at schools with different profiles.

The Apollonea.com platform provides both teachers and students with inspiration and tools for interactive mathematics education. Thanks to its accessibility and user-friendly design, it can reach a wide audience, including those interested in mathematics beyond the academic sphere. In the future, the project plans to expand its content and functionalities, serving as a living example of how technology and collaboration can be used to enrich the educational process and promote mathematics.

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