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Teaching Mathematics and Computer Science

Metacognition – necessities and possibilities in teaching and learning mathematics

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Abstract. This article focuses on the design of mathematics lessons as well as on the research in mathematics didactics from the perspective that metacognition is necessary and possible.

Humans are able to self-reflect on their thoughts and actions. They are able to make themselves the subject of their thoughts and reflections. In particular, it is possible to become aware of one's own cognition, which means the way in which one thinks about something, and thus regulate and control it. This is what the term metacognition, thinking about one's own thinking, stands for.

Human thinking tends to biases and faults. Both are often caused by fast thinking. Certain biases occur in mathematical thinking. Overall, this makes it necessary to think slow and to reflect on one's own thinking in a targeted manner.

The cognitive processes of thinking, learning and understanding in mathematics become more effective and successful when they are supplemented and extended by metacognitive processes. However, it depends on a specific design of the mathematics lessons and the corresponding tasks in mathematics.

Key words and phrases: metacognition, discursivity, cognitive biases, mathematics lessons, mathematical tasks.

MSC Subject Classification: 97C30, 97C70, 97D40, 97D50, 97D70.

Introduction

This paper explores the concept of metacognition, focusing on its necessities and possibilities in educational contexts, particularly in mathematics education.

Firstly, the theoretically and empirically proven necessity of recognising, observing and evaluating one's own thinking is presented. The characteristics of fast and slow thinking as well as the peculiarities of cognitive biases are discussed.

A brief historical overview follows, tracing the genesis of metacognition from its roots in philosophy, cognitive psychology and mathematics didactics, and its development, as a vital area of research for understanding how individuals monitor and regulate their own thinking.

Next, the article provides an explanation of key metacognitive terms, such as metacognitive knowledge and metacognitive regulation. These concepts are differentiated to clarify their roles in teaching and learning.

The focus then shifts to mathematics education. The paper examines the role of metacognition and discursivity in teaching and learning of mathematics and demonstrates how classroom lessons can promote metacognitive thinking by encouraging students to articulate and reflect their thought processes. Mathematics didactics research has developed a system of categories for recording metacognitive and discursive activities in mathematics lessons, which is presented here.

Additionally, the importance of metacognition is explored in task design. Effective mathematical tasks which foster metacognitive activities stimulate students to control and reflect on their thinking, learning and understanding. There is also a category system for analysing task processing given.

This is followed by the results of empirical studies that emphasise the importance of metacognition for effective learning. These studies demonstrate the possibility of developing metacognitive skills through targeted teaching.

To summarise, metacognitive activities – especially in mathematics – are necessary, but also possible for effective learning. They can be promoted through specific task design and teaching methods.

On the necessity of metacognition

Human thinking is prone to biases, errors and mistakes. They are often caused by fast thinking. Certain biases occur in mathematical thinking.

Overall, this makes it necessary to reflect on one's own thinking in a targeted manner.

Fast thinking and slow thinking

The bat-and-ball task (Kahneman, 2011) is quite famous in cognitive psychology.

> ▷ *A bat and a ball cost $ 1.10 in total. The bat costs $ 1.00 more than the ball. – How much does the ball cost? _____ cents.*

Almost everyone reports an initial tendency to answer 10 cents because the sum $ 1.10 separates naturally into $ 1.00 and 10 cents, and 10 cents is about the right magnitude. Many people yield to this immediate impulse. But: 5 cents is the correct answer.

The surprisingly high rate of errors in this easy problem illustrates the difference between fast and slow thinking.

The related cognitive psychology theory is associated with the name of Daniel Kahneman (*1934, †2024, Nobel Prize in 2002) and his research (cf. Tversky & Kahneman, 1974; Kahneman, 2011). In *Thinking, Fast and Slow* (Kahneman, 2011), the bat-and-ball problem is used as an introduction to the major theme of the book: the distinction between two systems of mental activities, the fast System 1 and the slow System 2. System 1 operates intuitively, System 2 deliberatively. Fast thinking is fluent and spontaneous, slow thinking is effortful and reflective. Kahneman says System 1 is gullible and biased, whereas System 2 is doubting and questioning.

Fast thinking is constantly active and susceptible to biases. Errors and mistakes can occur. Slow thinking is generally passive. It has to be set in motion in order to check and revise the inadequate and incomplete thinking that has arisen carelessly and fleetingly.

The bat-and-ball task raises questions: Why are so many people satisfied with a superficially plausible answer that occurs to them spontaneously? Why do not they scrutinise a result that only seems to fit? Why are they so uncritical of themselves, so unwilling to be thorough and accurate?

Obviously, people tend to use as little energy as possible to accomplish a task. System 1 is used, System 2 is not. However, this has consequences. On the one hand, the fast thinking in System 1 requires little mental effort, and misdirection and inaccuracies occur easily. The slow thinking in System 2, on the other hand, requires a lot of mental effort and corrects biases and errors. The bat-and-ball task shows exactly this: By not using System 2, the checking and corrective thinking has been omitted. An important part of the cognitive resources has remained unused.

The consequence is obvious: The beneficial interaction of System 1 and System 2 requires metacognitive support with planning, monitoring and reflection of one's own thinking. Metacognition is systemically relevant. This is important for thinking, learning and understanding in schools and lessons, in particular.

Fast thinking causes cognitive biases, slow thinking is able to correct them. It is important to consciously correct what one has unconsciously been tempted to do.

The explicit moral is that people are too willing to rely on System 1, and that gets them into trouble.

Cognitive biases in mathematical thinking

Three cognitive biases relevant to school mathematics (Sjuts, 2021) are presented and illustrated below.

Confirmation bias

An example task (Sjuts, 2021) may illustrate the first bias.

▷ *You overtake the third one. – What place do you have then? The _____ place.*

The hasty answer is to be in second place. This is what the overtaking process suggests. You feel confirmed in your performance and are led to believe that you have achieved second place by overtaking the third in line. In reality, only third place has been achieved. The mental imagination differs depending on whether the person overtaking – quickly – tends to have an illusion about the new place or – thoughtfully – realises that the person has moved up from fourth to third place.

People tend to select and absorb information that corresponds to their previously formed opinion; they tend to interpret information in such a way that it confirms their own expectations. This is known as confirmation bias.

Anchoring bias

An example task (Sjuts, 2021) may also illustrate the second bias.

▷ *You are the 10th from the front and from the back. – How many are you in total? _____ persons.*

21 or 20 or 19? The (wrong) answers 20 and 21 are somewhat unconcerned. However, these two results can be explained by referring to the number 10, which has to be taken into account twice. $10 + 10 = 20$ and $10 + 1 + 10 = 21$ are the corresponding calculations.

The calculations are anchored by the (double) number 10 in the text. However, if you become aware of the carelessness, a control consideration sets in. The mental organisation can take place in different ways. One consideration, that you have counted yourself twice, involves the calculation $2 \cdot 10 - 1 = 19$, the other consideration, that there are 9 in front of you and 9 behind you, involves the calculation $9 + 1 + 9 = 19$. The so-called anchoring bias can be seen here.

Framing bias

Finally, an example task (Sjuts, 2021) may also illustrate the third bias.

▷ *If it takes 5 machines 5 minutes to make 5 widgets, how long would it take 100 machines to make 100 widgets? _____ minutes.*

5 or 100? If we assume that the number of machines producing, the number of minutes for the production time and the number of widgets produced change evenly with each other, we get the wrong answer 100 minutes. In fact, in the given situation, 100 machines only need 5 minutes to produce 100 widgets.

If the text of a task leads to a certain assumption that a calculation is appropriate, one can speak of a framing bias related to this assumption.

To summarise: Confirmation bias, anchoring bias and framing bias are three cognitive biases that occur in mathematics (Sjuts, 2021). The organisation of teaching and learning processes must take this into account.

On the possibility of metacognition

What can we say about cognitive and metacognitive activities after looking at the tasks and their hasty and incorrect answers?

At a basic level, we think about positions and numbers in a certain arrangement. And that means: We organise something in our heads. At a higher level, we think about whether what we organise in our heads is adequate, how we can monitor and reflect it and whether we need to correct it.

In short form: *Cognition* is the mental activity associated with the construction, organisation and use of knowledge. *Metacognition* means knowing, controlling and feeling one's own cognition (Sjuts, 2003).

Imagine a person asking: Am I aware of what I do, what I say, what I think? Do I know what I am doing, saying or thinking? In case the answer is "yes", the person has been successful in metacognition. Metacognition encompasses knowledge and thinking about one's own cognitive system as well as the ability to manage and check this system. An essential prerequisite for this is metacognitive experience.

On the one hand, learning, storing, keeping in mind, remembering, understanding, thinking and knowing become subjects of reflection, mental activities are consciously planned, monitored and evaluated, on the other hand.

Therefore, cognition of one's own cognition and regulation of one's own cognition are the two main components of metacognition.

A short genesis of metacognition

Metacognition has a two and a half thousand years history in philosophy: "Know thyself." This ancient Greco-Roman demand is well known. One example is the Greek philosopher Heraclitus from Ephesus (*around 520 BC; †around 460 BC). His phrase is famous: "It is given to all humans to recognise themselves and to think consciously." This is the ancient pre-history of metacognition.

A more recent history goes back to George Pólya (*1887, †1985). In his works on mathematical thinking and problem solving, self-observation and reflection are often mentioned in the same sense the term metacognition is used today (Pólya, 1945). Note, that the term metacognition did not yet exist at that time.

It was the American psychologist John H. Flavell (*1928), who first introduced the term metacognition (Flavell, 1976). He is the founder of metacognition theory.

In addition, Alan H. Schoenfeld (*1947) made the concept of metacognition fruitful in mathematics didactics (Schoenfeld, 1985, 1992).

It is particularly important to link research in mathematics didactics and the design of mathematics lessons. In this sense, Elmar Cohors-Fresenborg (*1945) has inspired the cognitive-theoretical orientation of mathematics didactics with his ideas and works, together with his colleagues in cognitive mathematics (cf. Cohors-Fresenborg & Kaune, 2001, 2007; Kaune & Cohors-Fresenborg, 2010; Nowińska, 2016).

That shows: The genesis of metacognition is a history of its pioneers.

Metacognition is now a huge field of research. Although very promising, its implementation in school lessons is extremely rare.

Explanations of terms

The term c*onditio humana* expresses the following: Humans are able to self-reflect on their thoughts and actions. They are able to make themselves the subject of their thoughts and reflections. In particular, it is possible to become aware of one's own cognition, which means the way in which one thinks about something, and thus regulate and control it. This is what the term metacognition, thinking about one's own thinking, stands for. And: Knowing, controlling and feeling one's own cognition is possible.

The knowing component is the declarative metacognition, the controlling component is the procedural (or executive) metacognition, the feeling component is the motivational (or sensitive) metacognition. All these words and terms are in common use.

An overview based on Scott and Levy (2013, p. 123) is provided in Table 1:

Term	Component	Description	Process phase
Declarative Metacognition	Use of knowledge	Knowledge about own thinking, about tasks and about strategies	before, during, after
Procedural Metacognition	Planning	Recognising the existence and nature of the challenge and deciding on a strategy to overcome it	before
	Monitoring	Regulation in the approach to a specific task Control on the basis of the current progress of one's own thinking	during
	Reflecting	Recapitulation and evaluation of the procedure and the result	after
Motivational Metacognition	Securing self-efficacy	Maintaining attentiveness and the willpower to think about oneself	before, during, after

Table 1. Metacognition and its components

Although some concepts about metacognition are fuzzy, there are important questions about metacognition, as follows:

- How can metacognition be made operational in learning and teaching processes?
- How can metacognitive activities be validly recorded?

- How can mechanisms of metacognitive activities be identified? What does this mean for mathematics and mathematics didactics, in particular?
- How can mathematics lessons and mathematics tasks be designed to stimulate learners' metacognitive activities?
- To what extent can metacognitive processes be identified in classroom discussions and when working on tasks?
- What influence do certain metacognitive activities have on learning?

However, the research into metacognition and the practice of metacognition have so far raised questions rather than provided answers.

Metacognition and task design in mathematics

The success of metacognition depends on the extent to which the individually formed mental models are externalised. Thus, the question arises in mathematics educational research: How can you formulate tasks that stimulate reflective thinking on your own thinking? The basic idea is the following: Explaining can make the invisible visible. Writing makes actions and thoughts conscious. Explaining clarifies the way of thought.

The following table shows possibilities for effective task design (Sjuts, 2022):

Extension of tasks	**Cognitive activities**	**Examples of formulations**
Explicating	explaining – scripting – reasoning	*„Explain your thoughts."*
Varying	noting – drawing – transferring	*„Write down a calculation for it." or „Create a drawing and label it."*
Formalising	symbolising – formulating – representing	*„Find a suitable algebraic formalisation."*
Analysing	comparing – commenting – reviewing	*„You are asked how you ensure the correctness. How do you answer?"*
Synthesising	complementing – continuing – producing	*„Complete the thought process presented here."*

Table 2. Task design in mathematics

An example (Sjuts, 2022) illustrates what an extended task can look like. The basic version is:

▷ *Calculate:* (1 – 2) – (3 – 4) – (5 – 6) – ... – (99 – 100) =

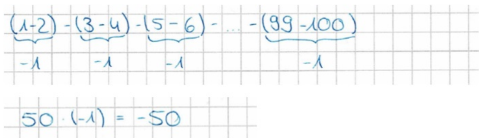
The task is not difficult. The result of the calculation is

$$(-1) - (-1) - (-1) - \ldots - (-1) = -1 + 1 + 1 + \ldots + 1 = -1 + 49 = 48.$$

However, the answers 49 and 50 also occur in everyday school life. Therefore, there should be a modified task that stimulates reflective thinking (Figure 1):

▷ *Calculate*: (1–2) – (3 – 4) – (5 – 6) – ... – (99 – 100) =

a) Judy wrote down her answer like this:



Comment on the procedure and the result.

b) Create your own solution and justify your result.

Figure 1. A modified task

The modification of the task leads to metacognitive activities taking place (Sjuts, 2022). By analysing another person's thinking (Figure 1), one becomes aware of one's own thinking. There are several more tasks (like this) that have been proven being suitable for causing the pupils to reveal their own cognition (Sjuts, 2022).

Analyses of metacognitive activities

Mathematics didactics with a focus on cognitive theory has developed respective category systems for analysing metacognitive activities in task processing and in lesson scenes.

A category system of existing and missing metacognition in task processing

A category system has been developed to reconstruct metacognitive processes in written task processing (Sjuts, 2022). It contains the three usual components of procedural metacognition – looking ahead, considering inward and looking back – as well as additional deficits in task processing due to a lack of metacognitive activities and subject-related or language-related errors.

Planning the task processing	Monitoring during the task processing	Reflecting on the task structure or processing	Deficits in task processing due a lack of metacognitive activities
P1: Planning the metacognitive activities	M1: Monitoring the subject-specific activities	R1: Reflecting on the subject-specific structure and activities	D1: Inadequate performance of subject-specific activities
	M2: Monitoring choice and use of words and terms	R2: Reflecting on choice and use of words and terms	D2: Inadequate choice and use of words and terms
	M3: Monitoring the notation and representation	R3: Reflecting on the subject-specific representation with changing and modifying	D3: Incorrect handling of subject-specific representations
	M4: Monitoring the permissibility or adequacy of use of tools or methods, in particular in relation to the planning approach or modelling approach	R4: Reflecting on the mode of action or application of subject-specific tools or methods, in particular specification of a tool with which a given result is to be produced (effect and application analysis)	D4: Non-permitted use of tools or methods
	M5: Monitoring the consistency of an argumentation or statement	R5: Reflecting on content-related or structural aspects of an argumentation (argumentation analysis) R5a: Specification of an argumentation R5b: Analysing an existing argumentation	D5: Incomplete or inconsistent argumentation
	M6: Monitoring the fit between result and question	R6: Reflecting on the intention of the question	D6: Lack of fit between question and result
	M7: Monitoring the adequacy of individual conceptions	R7: Reflecting on the interaction between representation and imagination	**Subject-related or language-related errors**
	M8: Monitoring the personal assessment	R8: Reflective evaluation	F1: Misconception or technical mistake
			F2: Incomprehensible presentation due to linguistic mistakes

Table 3. Category system of existing and missing metacognition in task processing

An example task processing under consideration may illustrate the importance of the category system for analysing of metacognitive activities (Sjuts, 2022). The task "Four-digit numbers with additional conditions" is:

▷ *You are looking for all four-digit numbers where the sum of the digits is 6 and the digit 3 is in the second position. – Think up a plan. Explain your thoughts.*

The answer is shown in Figure 2:

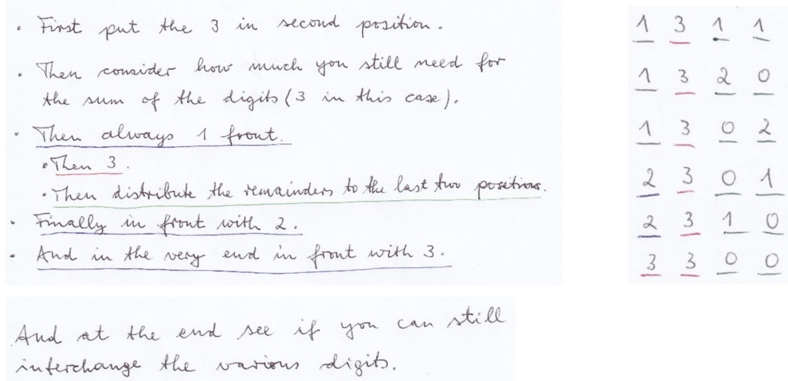


Figure 2. Solution of the task "Four-digit numbers with additional conditions"

The task processing shows a comprehensible procedure that follows a formulated plan (P1, R1). A conceptual reflection (R2) is indirectly recognisable. The solution elements are notated in accordance with the planning considerations. The notation is systematic (R3) and is checked (M4, R4).

The task has been solved correctly. The explanations are elaborate. The high intensity of the metacognitive activities with regard to planning, monitoring and reflecting ensures a visibly secure thought process and presentation of the task working (Sjuts, 2022).

A category system of metacognition and discursivity in lesson scenes

Empirical teaching research focusses on the effectiveness of classroom lessons. The distinction between surface structure and depth structure (Nowińska, 2011) is crucial in this regard. It is not the surface structure but the depth structure that is important for the effectiveness of teaching-learning processes.

The following overview (Table 4) summarises the key features of the surface and depth structure (Nowińska & Sjuts, 2019).

Surface structure (visual structure)	Depth structure (analysis structure)
• Organisational, social and working forms such as plenary, group, partner and individual work • Design according to expository, instructive or open, but also course- or project-related teaching approaches	• Classroom climate • Interaction in the teaching-learning process • Intensity of content processing • Logical and conceptual clarity and comprehensibility • Adaptivity • Feedback • Self-regulation support and activation with challenge and demand

Table 4. Surface structure and depth structure of lessons

Empirical teaching research has consistently demonstrated low effect sizes for surface structure characteristics (Hattie, 2009). However, this does not mean that the surface structure is unimportant; the planning and design of lessons must be well-founded. Well-considered organisation is an essential condition for the sustainability of lessons. This is because it depends on the extent to which the organisation of lessons actually enables effective learning.

How can we succeed in adequately grasping the depth structure? This cannot be achieved sufficiently through observation. Rather, three components are necessary. First: A video recording must be made. Second: A transcript must be produced. Third: A theory-based category system is required to analyse the transcript.

A (shortened and slightly modified) part of the category system is intended to provide insights, as shown in Table 5:

Planning	Monitoring	Reflecting	Discursivity
P1: Indication of a focus of attention, in particular with regard to tools or methods to be used or (intermediate) results or representations to be achieved	M1: Controlling of a subject-specific activity	R1: Analysis of structure of a subject-specific expression	D1: Measures to improve the discussion or link a contribution
P2: Planning metacognitive activities	M2: Controlling of terminology or vocabulary used for a description or an explanation of a concept	R2: Reflection on concepts or analogies or metaphors	D2: Education for discourse

	M3: Controlling of notation or representation	R3: Result of reflection expressed by a wilful use of a (subject-specific) representation	**Negative Discursivity**
	M4: Controlling of the validity or adequacy of tools and methods used, in particular with regard to a planned approach or a modelling approach	R4: Analysis of the effectiveness and application of subject-specific tools or methods or indication of a tool needed to achieve an intended result	ND1: Superfluous contributions
	M5: Controlling of (consistency of an) argumentation or statement	R5: Analysis of argumentation or reasoning with regard to content-specific or structural aspects	ND2: Inadequate vocabulary (in a description, comment, argumentation, statement)
	M6: Controlling if the results meet the question	R6: Reflection-based assessment or evaluation	ND3: Violence of rules for a well-orchestrated discourse
	M7: Revealing a misconception	R7: Analysis of the interplay between representation and conception	ND4: No intervention taken against severe disregard of discursivity rules, in particular when discourse falls into pieces; ignoring an objection
	M8: Self-monitoring		

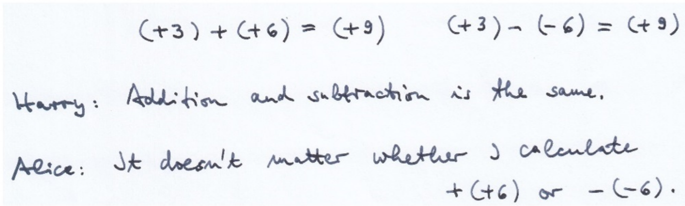
Table 5. Category system of metacognitive and discursive activities

In order to assess metacognition in mathematics lessons, this category system records metacognitive and discursive activities (Cohors-Fresenborg & Kaune, 2007; Nowińska, 2016, 2020). It makes it possible to clarify the promotion of metacognition in mathematics lessons. And it allows statements to be made about the quality of teaching. At the same time, it is a useful analytical and diagnostic research tool.

An analysis of the next teaching scene (Nowińska, 2021), illustrated in Figure 3, will show how the category system is used.

Recording the statements on the board enables a precise response to what has been said as well as to facilitate the subsequent discussion (D1). In the first contribution, the teacher ensures clarity of content (D1) by asking Eliza to state the reference point of her comment. Eliza complies with the request (D1). She evaluates the statement (R6). Simon takes Harry's statement as the basis for his own argument (D1). With the help of a counterexample, he justifies the incorrectness of the statement (R5). The teacher uses Harry's statement to stimulate further reflection on what Harry meant (D1).

On the board:



…

Teacher: *Eliza, can you first name the statement you want to say something about now?*

Eliza: *I would like to comment on Harry's statement that addition and subtraction are the same thing. I would say that addition and subtraction are not the same thing, but in this case the result is the same.*

Teacher: *Yes. Simon.*

Simon: *So, I mean, plus three plus plus six is plus nine. So that's not the same as plus three minus plus six. Because if he says 'addition and subtraction are the same', then you don't have to change the sign and you don't get plus nine. The result is minus three.*

Teacher: *Okay. So, the sentence as Harry said and I wrote on the board is not tenable.*

Simon: *Mmh.* [agreeing]

Teacher: *What could he have meant by that?*

…

Figure 3. Mathematics lesson scene

The class discussion shows how the metacognitive-discursive behaviour of the pupils and the teacher can manifest itself in their interactions with each other during the clarification of individual considerations in the analysis of mathematical laws. The pupils respond to what others say and mean in a controlling way. Discursive behaviour contributes to the clarification of learners' externalised thought processes (Nowińska, 2021). If the transcript analysis shows that, one can speak of a special metacognitive-discursive teaching quality.

This can be recognised: The quality of understanding is related to content and subject teaching. Logical and linguistic accuracy is essential. You could also put it this way: subject-specific accuracy is a fundamental quality criterion for ensuring that cognitive activation can develop its potential to promote learning at all.

In other words: Mathematics teachers should practice and even establish this as an integral part of teaching (Sjuts, 2002, 2003).

Results of previous research on metacognition and aims of future research

There are today several studies on reconstructing, analysing and quantifying metacognition and its effectiveness (cf. Sjuts, 2002, 2003; Kramarski & Mevarech, 2003; Veenman et al., 2006; Van der Stel et al., 2010; Cohors-Fresenborg et al., 2010; Kaune & Cohors-Fresenborg, 2010; Nowińska, 2016; Vorhölter et al., 2019).

Empirical results about metacognitive processes

Along with the high expectations of metacognition and the considerable effects of metacognition confirmed in studies, it has become clear: Metacognition increases the effectiveness of thinking and learning, but the effectiveness of metacognition is conditional. The current conclusion is, at least in terms of didactics and methodology, that there is neither a content-free nor a non-binding development of metacognition. Metacognitive activities are stimulated and applied by adhering to content references and lesson agreements (Sjuts, 2003). It is also important to place the acquisition and use of metacognitive activities and strategies in a substantive context (Sjuts, 2022).

According to evidence-based meta-studies on factors for improving achievement, metacognition has an above-average effectiveness. The value of 0.69 cited for metacognitive strategies (Hattie, 2009) is in the range of very large effect sizes. A major message is that what works best for students is similar to what works best for teachers: an attention to setting challenging learning intentions, being clear about what success means, and an attention to learning strategies for developing conceptual understanding about what teachers and students know and understand.

Analyses of mechanisms in the depth structure of mathematics lessons (Cohors-Fresenborg et al., 2010) also show the beneficial influence of metacognition on students' thinking, learning and comprehension achievements.

Future research in mathematics didactics about metacognition

Research in mathematics didactics should address the depth structure of teaching-learning processes more intensively. This research can be led by the following research questions:

- How should tasks be designed that lead to effective learning?

- How should teaching processes – including instruction and explanation – be organised to ensure sustainability?
- How should curricula and textbooks be formulated that focus on the intensity of understanding?
- How can the influence of the type and intensity of metacognitive activities on learning effectiveness be measured?
- How can methods of writing and thinking aloud be combined in the study of metacognition?
- How can outward-facing metacognitive processes be used diagnostically?

Finally, the most important findings are highlighted:

- It is natural for humans to be led into cognitive biases. But humans are also able to reflect on their own thoughts and actions. Humans can therefore counteract cognitive biases (of fast thinking) with metacognitive activities (of slow thinking).
- Practising metacognition, i.e., thinking in a conscious, controlled and reflective manner, is one of the most important educational goals. It is therefore necessary – probably much more than up to now – to firmly establish this goal in the school curricula, to provide it comprehensively in the tasks in textbooks and to emphasise it continuously in lessons.
- This means that a millennia-old philosophical insight about the human ability for self-knowledge and self-reflection becomes a permanent habit in the classroom and school context, and thus for the whole life.
- The cognitive processes of thinking, learning and understanding in mathematics become more effective and successful when they are supplemented and extended by metacognitive processes.
- However, metacognitive processes depend on a specific design of the mathematics lessons and the corresponding tasks in mathematics.
- And it can be summed up: Metacognition does not usually happen by itself. Metacognition needs a stimulus. Metacognition needs a binding and substantial incentive.

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