

Regula falsi in lower secondary school education

ZSOLT FÜLÖP

Abstract. The aim of this paper is to offer some possible ways of solving word problems in lower secondary school education. Many studies have shown that pupils in lower secondary school education (age 13-14) encounter difficulties with learning algebra. Therefore they mainly use arithmetical and numerical checking methods to solve word problems. By numerical checking methods we mean guess-and-check and trial-and-error. We will give a detailed presentation of the false position method. In our opinion this method is useful in the lower secondary school educational processes, especially to reduce the great number of random trial-and-error problem solving attempts among the primary school pupils. We will also show the results of some problem solving activities among 19 grade 8 pupils at our school. We analysed their problem solving strategies and compared our findings with the results of other research works.

Key words and phrases: false position method, regula falsi, guess-and-check, trial-and-error, arithmetical procedures, algebraic methods.

ZDM Subject Classification: D20, D50, D70, H10.

1. Introduction

Research in teaching and learning algebra has detected a number of serious cognitive difficulties and obstacles especially to novice students. An important thread in international research and thinking on Mathematics education curriculum is to consider ways in which the transition from arithmetic to algebra can be made more smooth. In particular, the “early algebra” movement has examined how to teach arithmetic in a way that prepares students for algebra, and which emphasises the thinking processes which underlie algebra. The intention is not

to introduce algebraic symbols at an earlier age, but to change the emphasis of arithmetic teaching. It is no longer appropriate to have an arithmetic curriculum which focuses exclusively on computation, so that there is opportunity to include experiences of generalisation, mathematical structure and properties of operations that underpin algebra. Detori et al (see [5]) suggested that the transition from arithmetic to algebra requires a change in the nature of problem resolution and a change in the nature of the objects of study (i.e. from numbers to symbols, variables, expressions, equations, etc). Beginning students need to make both of these transitions. Algebra requires a stronger understanding of the properties of operations than does arithmetic. Many students have difficulty learning about algebra because they are unsure of the arithmetic properties which algebra generalises (see [2]). According to Warren (see [30]), the shift from arithmetic to algebra involves a move from knowledge required to solve “arithmetic equations” operating with numbers to knowledge required to solve “algebraic equations” operating with unknowns. Tall and Thomas [see 27] analyse some of the obstacles to understanding the notion of a variable, they also used a computer software to offer examples and non-examples which may be seen as typical, or generic examples of the algebraic processes. Empirical evidence from several related studies shows that such an approach significantly improves the understanding of higher order concepts in algebra [see 6].

One of the primary reasons children have trouble with problem solving is that there is no single procedure that works all the time each problem is slightly different. Also, problem solving requires practical knowledge about the specific situation. If you misunderstand either the problem or the underlying situation you may make mistakes or incorrect assumptions. According to Newman (see [18]) a person confronted with a one-step written problem has to read the problem, then comprehend what he has read, then carry out the transformation from the words to the selection of an appropriate mathematical “model”, then apply the necessary process skills and then encode the answer. Research have shown that students’ errors in algebra can be ascribed to fundamental differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from the arithmetical conventions and need to learn to deal with algebraic symbolism. The areas where major error types found were: the transformation of word problems into algebraic language, parenthesis omitted and wrong operations in solving equations. Clements studied error causes such as reading comprehension difficulty, the failure during the transformation from the written problem to an acceptable ordered set of mathematical procedures, the

form of the question, the weakness in process skills, encoding error, careless error and lack of motivation (see [4]), concluding that many errors made by children on written mathematical tasks are due to reading comprehension and transformation difficulties and that often means a child uses inappropriate process skills in an attempt to find a solution.

In the research done by Stacey and MacGregor on strategies used by students in solving mathematical word problems involving equations, students were found to apply the following different routes while they solve algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known number at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing an equation and solving it with the balance principle, and (e) algebraic route: solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning (see [24]). Stacey underlines that historically, and in the education of nearly all children, algebra grows out of arithmetic (see [23] and [25]).

Farmaki et al have studied the introduction of algebraic thinking to 13 year-old students by the use of the functional approach to algebra which widens the meaning of algebraic thinking (see [7]). They examined the students' solution processes by the two approaches, functional and letter symbolic, through problems which are expressed by equations of the form $a \cdot x + b = c \cdot x + d$. The functional orientation enabled them to connect various problem situations to graphs, tables and letter-symbolic representations. So, problems which traditionally could be answered only by the solution of an equation were now treated in many ways. They concluded that the functional approach has its own value because it enables the students to develop problem solving abilities such as the heuristic "trial-and-error", "draw a diagram" and, for some students to "solve an equation". Filloy and Rojano (see [8]) focused on problems of the form $x + a = b$, $a \cdot x = b$ and $a \cdot x + b = c$. They found that these kind of problems can easily be solved using arithmetic, mainly by inverse operations. Johanning chose problems with $a \cdot x + b \cdot x + c \cdot x = d$ and $x + (x + a) + (x + b) = c$ structures (see [13]). She found that students used many informal strategies for solving the problems, the method of systematic guess and check being the most common approach. Another important fact is that while assigning a value to a variable and verifying its accuracy, students are developing functional reasoning as it entails recognising

a relation between variables even if such relation is not always expressed in the formal language of algebra. I. Osta and S. Laban analysed the grade 7 students' problem solving strategies and found that it is very difficult to switch from arithmetic and numerical checking approaches to algebraic procedures, the students mostly use trial-and-error or guess-and-check method to solve word problems [see 15]. Amado et al studied the role of the representation in problem solving, and underlined that it is important to encourage students to represent their mathematical ideas in a way that makes sense for them, even if these representations are not conventional. Those students' answers who used properly informal representation and strategies or even trial-and-error revealed the usefulness of such approaches [see 1].

Many other research studies have shown that most of the students-when they have to solve word problems-do not appeal to the learned arithmetical and algebraic methods, they handle the problem by numerical checking methods, as follows:

- (1) Estimation or guess-and-check: "Guess-and-check" is a problem-solving strategy that students can use to solve mathematical problems by guessing the answer and then checking whether the guess fits the conditions of the problem. In fact, it consists in the estimation of unknown quantity, by perceptively comparing them to other known quantities, then we verify that the estimated values satisfy the conditions of the problem. All research mathematicians use guess-and-check, and it is one of the most powerful methods of solving differential equations, which are equations involving an unknown function and its derivatives. A mathematician's guess is called a "conjecture" and looking back to check the answer and prove that it is valid, is called a "proof." The main difference between problem solving in the classroom and mathematical research is that in school, there is usually a known solution to the problem. In research the solution is often unknown, so checking solutions is a critical part of the process. When students use the strategy of guess-and-check, they should keep a record of what they have done. It might be helpful to have them use a chart or table. Guess-and-check is often one of the first strategies that students learn in case of solving problems. This is a flexible strategy that is often used as a starting point when solving a problem, and can be used as a safety net, when no other strategy is immediately obvious.
- (2) Trial-and-error: "Trial-and-error" means a repeating process using forward arithmetic operations inherent to the problem situation, testing different

numbers in the statement of the problem. According to Stacey and McGregor we can distinguish two types of trial-and-error: random trial-and-error and sequential trial-and-error. There are a number of important factors that makes trial-and-error a good tool to use for solving problems. The purpose of trial-and-error is not to find out why a problem was solved. It is primarily used to solve the problem. While this may be good in some fields, it may not work so well in others. For example, while trial-and-error may be excellent in finding solutions to mechanical or engineering problems, it may not be good for certain fields which ask “why” a solution works. Trial-and-error is primarily good for fields where the solution is the most important factor.

Some of the math teachers place an emphasis on using trial-and-error to find a solution to problems, and they do not spend a lot of time explaining “why” a solution works. A good aspect of the trial and error method is that it does not try to use a solution as a way of solving more than one problem. Trial and error is primarily used to find a single solution to a single problem. Trial-and-error is not a method of finding the best solution, nor it is a method of finding all solutions. It is a problem solving technique that is simply used to find a solution. One of the most powerful advantages of this technique is that it does not require you to have a lot of knowledge. However, in some cases, it may require a large amount of calculus to find a solution.

- (3) False position method: the false position method or regula falsi method is a term for problem-solving methods in arithmetic, algebra, and calculus. In problems involving arithmetic or algebra, the false position method or regula falsi is confused with basic trial-and-error methods of solving problems by substituting test values for the unknown quantities. This is sometimes also referred to as guess-and-check. Versions of this method predate the advent of algebra and the use of equations. This is a specific arithmetical problem solving method used to solve word problems with two or three unknowns. In simple terms, this method begin by attempting to evaluate a problem using test values for the variables (which numbers, for the most part, happen to be false) and we try to compare the situation created in this way with the data and conditions of the question. Taking this difference into account, we can conclude how to change the values of the variables to obtain the right answer in few steps. The importance of this method increases because research has revealed that students prefer to use arithmetic methods in solving algebraic word problems and show difficulties in setting up and using equations to solve

such problems. There is also evidence that the most frequently used methods are guess-and-check or trial-and-error among the students of 13-14.

2. False position method

Some of the problems in the Rhind Papyrus translate for us into the form “Find x if $x + a \cdot x = b$.” In four of those problems, a is a unit fraction, and in four others it is a sum of two or three unit fractions. The equations where $a = 1/n$ are solved by Ahmes by a method which has become known as *Regula Falsi* (in Latin), or **FALSE POSITION** (or *False Supposition*). In this method, an incorrect but convenient **GUESS** of the value of the unknown is made, the left side of the equation is evaluated at the guessed value, and then the guess is adjusted by multiplying by a suitable amount. For example problem 25 from Rhind Papyrus is the following: “A quantity and its $1/2$ added together become 16. What is the quantity?” Translated into the language of modern mathematics this becomes: find the value of x when $x + \frac{x}{2} = 16$. Using the method of false position the Egyptian mathematician begins with an initial try, by supposing that the unknown quantity is two. Since two added to its half gives three, the mathematician knows that two is to the unknown quantity as three is to sixteen. More exactly, the mathematician knows that if he find a number which when multiplied by three gives sixteen, then that same number when multiplied by two will give the unknown quantity. Armed with this knowledge, the mathematician first performs a division (sixteen divided by three gives five-and-one-third) and then a multiplication (after making this calculation the mathematician concludes that the unknown quantity is ten-and-two thirds). The foregoing method is called *simple false position* (or *single false position*). The method of solving what we would now write as $a \cdot x = b$ begins by using a test input value x' , and finding the corresponding output value b' by multiplication: $a \cdot x' = b'$. The correct answer is then found by proportional adjustment, $x = x' \cdot \frac{b}{b'}$. The method of double false position arose in late antiquity as a purely arithmetical algorithm. In the ancient Chinese mathematical text called “The Nine Chapters on the Mathematical Art”, dated from 200 BC to AC 100, most of Chapter 7 was devoted to the algorithm [see 14]. The procedure was justified by concrete arithmetical arguments, then applied creatively to a wide variety of story problems, for example:

PROBLEM 2.1. *Now chickens are purchased jointly; everyone contributes 9, the excess is 11; everyone contributes 6, the deficit is 16. Tell: The number of people, the chicken price, what is each?*

In the terms of algebra, we have to solve the system of equations

$$9 \cdot x - 11 = y$$

$$6 \cdot x + 16 = y$$

where x denotes the number of people and y denotes the price of the chickens. The double false position method is the following:

1. First position $x_1 = 5 \Rightarrow y_1 = 9 \cdot 5 - 11 = 34$ and $y_2 = 6 \cdot 5 + 16 = 46$, so the first error is $k_1 = y_2 - y_1 = 46 - 34 = 12$
2. Second position $x_2 = 6 \Rightarrow y'_1 = 9 \cdot 6 - 11 = 43$ and $y'_2 = 6 \cdot 6 + 16 = 52$, so the second error is $k_2 = y'_2 - y'_1 = 52 - 43 = 9$

The number of people is given by the formula:

$$x = \frac{k_1 \cdot x_2 - k_2 \cdot x_1}{k_1 - k_2} \quad (1)$$

so $x = \frac{12 \cdot 6 - 9 \cdot 5}{12 - 9} = 9$ and $y = 70$ follows.

How the ancient Chinese mathematicians found the formula (1) remains a mystery, but they used it, as we can see in Chapter 7 of “The Nine Chapters on the Mathematical Art”. We can prove the formula using the tools of algebra in the following way. We consider, more generally, the following system of equations:

$$a_1 \cdot x + b_1 \cdot y = c_1$$

$$a_2 \cdot x + b_2 \cdot y = c_2$$

1. First position $x = x_1 \Rightarrow y_1 = \frac{c_1 - a_1 \cdot x_1}{b_1}$ and $y_2 = \frac{c_2 - a_2 \cdot x_1}{b_2}$, so the first error is

$$k_1 = y_1 - y_2 = \frac{c_1}{b_1} - \frac{c_2}{b_2} - \left(\frac{a_1}{b_1} - \frac{a_2}{b_2} \right) \cdot x_1 \quad (2)$$

2. Second position $x = x_2 \Rightarrow y'_1 = \frac{c_1 - a_1 \cdot x_2}{b_1}$ and $y'_2 = \frac{c_2 - a_2 \cdot x_2}{b_2}$, so the second error is

$$k_2 = y'_1 - y'_2 = \frac{c_1}{b_1} - \frac{c_2}{b_2} - \left(\frac{a_1}{b_1} - \frac{a_2}{b_2} \right) \cdot x_2 \quad (3)$$

If we use (2) and (3), the equality (1) yields

$$x = \frac{c_1 \cdot b_2 - c_2 \cdot b_1}{a_1 \cdot b_2 - a_2 \cdot b_1} \quad (4)$$

and this is the result that we can find by the use of Cramer's Rule of Determinants or the method of Simple Elimination. An interesting geometrical interpretation is given by Szalay in [26].

The method of false position was used in Europe from the 1200s until algebra notation was firmly established in the late sixteenth century, owing largely to its popularization by Leonardo of Pisa (nowadays known as Fibonacci) as a way to solve problems of commercial arithmetic.

In reference [29] we can find the following RULE: "Multiply the suppositions by the errors alternatively, i.e. multiply the FIRST POSITION by the SECOND ERROR, and the SECOND POSITION by the FIRST ERROR: and if the errors are both of the same kind, to wit, both of them excesses, or both defects, divide the difference of the products by the difference of the errors; but if they are of different kinds, to wit, one an excess, and the other a defect, divide the sum of the products by the sum of the errors, and the quotient will give the number sought."

In the Hungarian Mathematics education the Regula Falsi method had its own upholders (see [9] and [10]).

György Maróthi in his book *Arithmetika* solved the following problem by the method of *Regula Falsi duarum Positionum* (see [16]).

PROBLEM 2.2. *A man is asked: how old are you and your wife? He answered: I am 8 years older than my wife. The sum of our ages is 92.*

We consider at first, that the wife is 20 years old and the husband is 28. In this way they are 48 years old together, so the error is 44. Secondly, the wife's age is 30, the husband's age is 38. They are 68 years together, and the error is 24. We subtract the second error from the first error ($44 - 24 = 20$) and the first position from the second position ($30 - 20 = 10$). The ratio is $20 : 10 = 2 : 1$, so we have to increase the first position by $44 : 2 = 22$. Therefore the wife is $20 + 22 = 42$ years old.

Károly Nagy in his book *Elementary arithmologia, Arithmographia* solved the following problem (problem 202 in reference [17]) by the rule of Double Regula Falsi in the following way.

PROBLEM 2.3. *Once a man was asked: how many coins do you have in your pocket? He said: The quintuple of my coins is as much more than 30 as the duplex of it is more than 6.*

First position: There are 20 gold coins.

5 times 20 = 100. It is 70 more than 30

2 times 20 = 40. It is 34 more than 6

the error is $70 - 34 = 36$

Second position: There are 19 gold coins.

5 times 19 = 95. It is 65 more than 30

2 times 19 = 38. It is 32 more than 6

the error is $65 - 32 = 33$ so the error decreases by 3 when we decrease the number of coins by 1. Therefore if we subtract $\frac{36}{3} = 12$ from 20 the error disappears. So there are 12 coins.

We have to mention that, for a present day mathematician or a mathematics teacher, the foregoing ancient methods are quite stereotyped and the problem solving involves the use of some severe algorithms. At first glance, these methods are not adequate in the present day educational processes. In the following, we will show, through some patterns and examples, how the main idea of the false position method (i.e. we give some values to the variables, we compare the situation created in this way with the data and conditions of the question and we try to change the values of the variables to obtain the right answer) can be used in the present day educational processes.

3. Patterns and examples

An important question arises, how a primary school teacher can use the method of false position in the educational processes? Tuzson mentions that the false position method can be considered as an arithmetical method and should be taught in the lower secondary school educational processes. He also offers some examples of problems (one of them with 3 or 4 unknowns) which can be handled more easily by false position method than by other problem solving strategies [see 28]. The primary school pupils have to solve word problems which, from a teachers point of view, could be treated as system of equations with two unknowns, as follows:

Example 1

$$\begin{aligned} a \cdot x + b \cdot y &= c \\ x + y &= d \end{aligned} \tag{5}$$

Let us try to solve the system of equations by the use of false position method. We take an arbitrary number $x = x_1$, so $y = y_1 = d - x_1$ and

$$a \cdot x + b \cdot y = a \cdot x_1 + b \cdot (d - x_1) = c' .$$

We consider $k = \frac{c - c'}{a - b}$ and we will prove that the solution of the system of equations is $x = x_1 + k$ and $y = d - (x_1 + k)$. Indeed,

$$\begin{aligned} a \cdot x + b \cdot y &= a \cdot (x_1 + k) + b \cdot [d - (x_1 + k)] = a \cdot x_1 + b \cdot (d - x_1) + k \cdot (a - b) = \\ &= c' + k \cdot (a - b) = c' + (c - c') = c . \end{aligned}$$

But the problem arises how a teacher can explain this kind of method to 11-14 years old pupils? We try to solve the following exercise that may amuse intelligent youngsters.

PROBLEM 3.1. *A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?*

We consider, at first, that there are 10 hens so there are $2 \cdot 10 + 4 \cdot 40 = 180$ feet and this means $180 - 140 = 40$ more feet. If we increase the number of hens by one (of course the number of rabbits decreases by one) the number of feet decreases by two. So we have to increase the number of hens by $40 : 2 = 20$. We will show all our attempts in a table, as follows:

	hens	rabbits	feet	difference
First supposition	10	40	180	40
Increase/decrease	+1	-1	-2	-2
Increase/decrease	+20	-20	-40	-40
Right answer	30	20	140	0

Tuzson in his book (see [28]) applied this method to solve a similar problem. According to his train of thought our first supposition should be that all of the animals are hens. Pólya in his book (see [19]) mentions that this problem can be solved less “empirically” (he refers to the sequential trial-and-error, random trial-and-error and guess-and-check) and more “deductively” (with fewer trials, less guesswork, and more reasoning), his solution is the following. The farmer surprises his animals in an extraordinary performance: each hen is standing on one leg, and each rabbit is standing on its hind legs. In this remarkable situation

just one half of the legs are used, that is, 70 legs. In this case the head of a hen is counted just once but the head of a rabbit is counted twice. Take away from 70 the number of all heads, which is 50, there remains the number of the rabbit heads, so there are $70 - 50 = 20$ rabbits and, of course, 30 hens. We can see that the foregoing methods are very similar.

Example 2

$$\begin{aligned} a \cdot x + b \cdot y &= c \\ y &= m \cdot x \end{aligned} \tag{6}$$

We take an arbitrary number $x = x_1$, so $y = y_1 = m \cdot x_1$ and $c' = a \cdot x_1 + b \cdot m \cdot x_1$. We consider $k = \frac{c}{c'}$ and we will prove that the solution of the system of equations is $x = k \cdot x_1$ and $y = m \cdot k \cdot x_1$. Indeed,

$$a \cdot x + b \cdot y = a \cdot k \cdot x_1 + b \cdot m \cdot k \cdot x_1 = k \cdot (a \cdot x_1 + b \cdot m \cdot x_1) = k \cdot c' = c.$$

We can see that this method follows the rule of proportions from the simple false position method of the ancient Egyptians. Let us try the method to solve the following problem:

PROBLEM 3.2. *A meter of red cloth costs three times as much as a meter of white cloth. We have bought 2,7 meters of red cloth and 3,8 meters of white cloth and we paid 16660 Ft. What is the price of each cloth?*

Our first supposition is that a meter of white cloth is 100. Then we have to pay $2,7 \cdot 300 + 3,8 \cdot 100 = 1190$. But we paid $16660 : 1190 = 14$ times as much, so a meter of white cloth and a meter of red cloth costs 1400 and 4200, respectively.

Example 3

$$\begin{aligned} a \cdot x + b \cdot y &= c \\ y &= m \cdot x + n \end{aligned} \tag{7}$$

We'll make the transformation $y' = y - n$ so the system of equations becomes

$$\begin{aligned} a \cdot x + b \cdot y' &= c - b \cdot n \\ y' &= m \cdot x \end{aligned}$$

We take an arbitrary number $x = x_1$, so $y' = y'_1 = m \cdot x_1$ and $a \cdot x_1 + b \cdot m \cdot x_1 = c'$. We consider $k = \frac{c - b \cdot n}{c'}$ and we will prove that the solution of the system of equations is $x = \frac{c'}{k} \cdot x_1$ and $y = m \cdot k \cdot x_1 + n$. Indeed, $a \cdot x + b \cdot y = a \cdot k \cdot x_1 + b \cdot (m \cdot k \cdot x_1 + n) = k \cdot (a \cdot x_1 + b \cdot m \cdot x_1) + b \cdot n = k \cdot c' + b \cdot n = (c - b \cdot n) + b \cdot n = c$.

PROBLEM 3.3. *We have 2029 kilos of bricks. A white brick weighs 3 kilos and a red brick weighs 2 kilos. The number of red bricks is by 3 less than four times the number of the white bricks. How many red bricks and white bricks do we have, separately?*

If we purchase other 3 red bricks, the number of the red bricks is four times the number of the white bricks, and we have 2035 kilos of bricks altogether. In the following the problem solving method is very similar to Problem 3.2, so we will omit its detailed presentation.

4. Methodological suggestions

In this section we will extend the method that we used to solve Problem 3.1 in the foregoing section. Stacey and MacGregor detail an example which reveals some of the ways of thinking that students need to adopt to solve problems (see [24]). They analysed Australian years 9-11 pupils' solutions to the following problem:

PROBLEM 4.1. *Mark and Jan share \$ 47, but Mark gets \$ 5 more than Jan. How much do they each get?*

The pupils were generally more successful without algebra than with it. The authors offer some samples of the pupils' works.

Brenda

$$47 : 2 = 23.5 - 2.5$$

$$47 : 2 = 23.5 + 2.5$$

Wylie

$$(47 - 5) : 2 + 5 = 42/2 + 5 = 26$$

$$(47 - 5) : 2 = 42/2 = 21$$

Sara

$$15 + 32 = 47 \text{ difference } 17 \text{ too big}$$

$$16 + 31 = 47 \text{ difference } 15 \text{ too big}$$

...
 $21 + 26 = 47$ difference 5 solution
William
 $x + (x + 5) = 47$
 $2 \cdot x + 5 = 47$
 $2 \cdot x = 42$
 $x = 21$

According to the research data guess-check-improve solutions such as Sara's were very common and successful in this simple problem with a whole number answer. Solutions such as William's were rare. Many pupils used logical arithmetic reasoning, and the two main solutions are illustrated by Brenda and Wylie. Sara's procedure is characterized as a solution by *trial-and-error*. In fact, it consists of a series of trials, each of which attempts to correct the error committed by the preceding and, on the whole, the errors diminish as we proceed and the successive trials come closer and closer to the desired final result. She got the solution, because the given numbers, 5 and 47, were relatively small and simple. But, if the problem, proposed with the same wording, had larger or more complicated numbers, we would need more trials or more luck to solve it in this manner. It is more straightforward to proceed in the following way. In this problem there are two unknowns and two conditions (the sum is 45 and the difference is 5). Our first position is Mark gets 32 and Jan gets 15, this satisfy the first condition, but the second condition is not fulfilled, the error is $17 - 5 = 12$. If Mark gets 31 and Jan gets 16 then the error is $15 - 5 = 10$. We can see that if we decrease Mark's money by one (of course, Jan's money increases by one) the error decreases by two. We can conclude that if we decrease Mark's money by 6, the second condition will be fulfilled (the error will decrease by 12 and will be zero), so Mark gets $32 - 6 = 26$ and Jan gets 21. In this way, we can obtain the answer in a few steps and we can avoid the huge number of trials.

Let us try to apply this method in a word problem with three unknowns, as follows.

PROBLEM 4.2. *Ann and Barbara together weighed 93 kg. Ann and Cathey together weighed 95 kg. Barbara and Cathey together weighed 102 kg. How much does each of the girls weigh?*

Let us consider, for example, Ann's weight equal to 30. So from the first and second conditions the solution $x = 30$; $y = 63$; $z = 65$ follows. But this solution contradicts the third condition, the error is $128 - 102 = 26$. We can see

if Ann's weight increases by 1 then both of Barbara's weight and Cathey's weight decreases by 1 (this follows from the first and second equations), so the error decreases by 2. Therefore to decrease the error by 26, we have to increase Ann's weight by 13. So the solution is 43, 50 and 52 kilos, respectively. We summarise the calculations in the following table:

	x	y	z	y+z	difference
First supposition	30	63	65	128	26
Second supposition	31	62	64	126	24
Increase/decrease	+1	-1	-1	-2	-2
Increase/decrease	+13	-13	-13	-26	-26
Right answer	43	50	52	102	0

We can see that the main idea is the following. At first, we use test values for the variables, which satisfy the first and second conditions. The third condition, for the most part, is not fulfilled, so an error appears. In the second step, we increase or decrease the test values (these values also have to satisfy the first and the second condition) and we follow the variation of the error. Thereafter we can conclude how to change the values of the variables to obtain the right answer. We can use this method to solve word problems with four, five etc. variables (generally, these word problems imply the use of the systems of equations in the secondary school educational processes), the basic requirement is to use test values which satisfy all the conditions, except one (here the error appears). So we can find the right answer in a few steps, contrary to the method of guess-and-check or trial-and-error, where more trials and calculations are needed (especially when we have to deal with more variables and the numbers in the problem are not simple).

Of course, this method can be used in the case of word problems with a single variable. In this case we give a certain test value to the variable and we calculate the deviation from the conditions of the question (i.e. the error of the first supposition). We increase or decrease the value of the variable (second supposition) and study the increment or decrement of the error. In this way we can find how to change the variable to obtain the right answer. Let us see the following word problem with one variable.

PROBLEM 4.3. *I read $1/4$ part and 20 pages of a book, and by 8 pages less than two-third of the book remained. How many pages has the book?*

The first supposition is, for example, that the book has 60 pages. In this case I read $60 \cdot 1/4 + 20 = 35$ pages and $60 \cdot 2/3 - 8 = 32$ pages remained. This means the book has $35 + 32 = 67$ pages, so the error is $67 - 60 = 7$ pages. The second supposition is that the book has 72 pages. In this case the error is $38 + 40 - 72 = 6$ pages. So, if we increase the number of pages by 12, the error decreases by one. So, starting from the first supposition, we have to increase the number of pages by $12 \cdot 7 = 84$, so the book has $60 + 84 = 144$ pages.

As a lower secondary school mathematics teacher, I have observed an analogue train of thought in the case of some grade 5 students. The Hungarian grade 5 students' textbook contains equation solving problems, the pupils have to solve equations by the guess-and-check method. Most pupils gave the right answer after many trials and calculations. On the contrary, some pupils solved the equation very quickly. They gave a value for the variable and they calculated the difference between the right hand side and left hand side of the equation. Thereafter they increased the value of the variable by one, and studied how the difference between the two sides of the equation varied. One pupil solved the equation $5 \cdot x + 8 = 2 \cdot x + 29$ as follows: "If $x = 2$ the left-hand side is 15 bigger than the right-hand side, if $x = 3$ the left-hand side is 12 bigger than the right-hand side. So the difference decreased by 3. We have to increase x by 4, so $x = 7$ and (after he checked) here is the solution".

In his book Tuzson mentions that the method of false position is adequate to solve some simple Diophantine equations in the primary school educational processes (see [28]). In mathematics, a Diophantine equation is a polynomial equation, usually in two or more unknowns, in which only the integer solutions are thought or studied (an integer solution is a solution such that all the unknowns take integer values). Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. Tuzson gives an example, as follows.

PROBLEM 4.4. *Grandma has 35 chickens, they are 9, 10 or 12 days old. Their age is 403 days, in the aggregate. Find the number of chickens of all sort!*

Algebraic method

$$x + y + z = 35$$

$$9 \cdot x + 10 \cdot y + 12 \cdot z = 403$$

If we eliminate z from the foregoing equations then we will get $3 \cdot x + 2 \cdot y = 17$, where x and y are positive integers. So $2 \cdot y = 17 - 3 \cdot x > 0$ and $x \in \{1, 2, 3, 4, 5\}$

follows. Thereafter three right answers $(1, 7, 27)$, $(3, 4, 28)$ and $(5, 1, 29)$ follow. We have to mention that we can not deal with this algebraic method in the primary school education.

Arithmetical method (Use of false position) First position: we suppose that all of the chickens are 12 days old, so they are 420 days old, in the aggregate, this implies the error $420 - 403 = 17$. If we replace a 12 days old chicken by a 9 days old one, the error will decrease by three, so the number of 9 days old chicken is 5, at maximum. If we replace 5 of the 12 days old chickens by 9 days old ones then the error will be equal to 2. Therefore if we change one 12 days old chicken into a 10 days old one the error will be equal to zero and the right answer $(5, 1, 27)$ follows. In this way we can get all the right answers.

5. Methodological experiments

I am a Mathematics teacher in the Reformed Primary School in Veresegyház (and I also am a PhD student at PhD School for Mathematics and Computational Sciences, University of Szeged). Our school gather pupils from 15 localities, most of them have reformed religion. In all grades there are two classes, the average number of pupils is 25 per class. Our pupils are well motivated, most of them high or medium achievers. In 2014-2015 school-year I taught both of the grade 8 classes. In the autumn of 2014, following the Hungarian official curriculum for grade 8, I taught my pupils the algebraic methods to solve word problems. The Hungarian mathematics curriculum for lower secondary school pupils contains word problems concerning geometrical relationships, ages, movements, working together and other interesting problems. The algebraic structure that is inherent to these problems is an equation which represents an algebraic relationship and in many cases requires the unknown to occur on both sides of the equal sign (see [11], [12] and [21]). 19 pupils participated at extra-curricular activities, this especially means problem solving activities with high-achiever and talented pupils (one hour/week). In the spring of 2015, we dedicated four hours extra-curricular activities to solve word problems in arithmetic and algebraic way, I also show how to solve the problems by false position method (especially I showed the methods and problems from section 4 and other similar exercises). Thereafter in 5 hours extra-curricular activities I tested the pupils' way of thinking. In order to delineate the procedures used to solve the exercises and to delineate the most agreed methods (and to test especially the students' attitude towards the false position method), pupils were given a paper-and-pencil test with three different

word problems to solve completely alone in 30 minutes and thereafter we analysed the different solution processes. In this way we processed 15 exercises in 5 hours extra-curricular activities. In the following we will show the most interesting problem solving strategies used by pupils to solve the exercises.

PROBLEM 5.1. *A man has two horses and a saddle. The saddle worth 50 pounds. If the saddle be put on the back of the first horse, it will make its value double that of the second, but if it be put on the back of the second, it will make its value triple that of the first. What is the value of each horse?*

One of the expected algebraic responses is the following. The price of the first horse is x , so the second horse worth $\frac{x+50}{2}$. From the second condition the equation $\frac{x+50}{2} + 50 = 3 \cdot x$ follows.

None of the pupils solved the exercise successfully in an algebraic way.

Two pupils wrote:

“first horse + saddle = $2 \cdot x$, second horse + saddle = $3 \cdot (2 \cdot x - 50)$ ”

(we can observe they applied properly the algebraic symbolism, where x denotes the price of the second horse), but they failed when they wrote the equation $2 \cdot x = 3 \cdot (2 \cdot x - 50)$ and they get the result $x = 37,5$. Both realised that the result does not fulfill the conditions of the problem so they turned to the guess-and-check method, one of them solved the exercise successfully.

One pupil wrote properly the system of equations $x+50 = 2 \cdot y$ and $y+50 = 3 \cdot x$ but did not know how to solve it, so she also solved the problem successfully by guess-and-check method.

Eight pupils solved the problem properly by the use of numerical checking methods, most of them were paying attention to the error variation (none of them applied random trial-and-error problem solving method), so they applied consciously the false position method.

We have to underline two pupils' works:

“if the first horse worth 50, then the price of the second horse is $(50 + 50) : 2 = 50$ (from the first condition, a.n.) and the error is $3 \cdot 50 - 100 = 50$ (from the second condition, a.n.), if the first horse worth 40, then the price of the second horse is $(50 + 40) : 2 = 45$ and the error is $3 \cdot 40 - 94 = 25$, so if we decrease the price of the first horse by 10, the error will decrease by 25, so the right answer is first horse = 30, the second horse = 40”.

“if the first horse worth 25 (here he made the calculations in the same way as the first student) the error is 7,5, if the first horse worth 20, then the error

is 15, so if we decrease the price of the first horse by 5, the error will increase by 7, 5, therefore the right answer is $25 + 5 = 30$ (first horse) and 40 (the second horse)".

PROBLEM 5.2. Three merchants agree to buy a ship for 1200 and to be paid among them, in such proportion, that A's part of the said ship be $1/6$ of B's part, and B's part $2/3$ of C's part. How much each merchant pay?

Six pupils solved the problem successfully by algebraic method, all of them proceeded in the following way: A paid x , B paid $6 \cdot x$ and C paid $9 \cdot x$ and then they wrote the equation $x + 6 \cdot x + 9 \cdot x = 1200$, so they get the right answer.

One pupil wrote C paid x , B paid $\frac{2}{3} \cdot x$ and A paid $\frac{2}{3} \cdot x : 6$ and then wrote the equation

$$\frac{\frac{2}{3} \cdot x}{6} + \frac{2}{3} \cdot x + x = 1200$$

and she made calculation mistakes in this complicated equation with fractions.

Another pupil wrote the equation

$$\frac{x}{6} + \frac{2 \cdot x}{3} = 1200$$

this is a misunderstanding or a bad interpretation of the conditions of the problem.

Four pupils gave the right answer by the method of false position. We detail two pupils' work:

First position: A paid 1, B paid 6 and C paid 9 so they paid 16 together and the error is 1184.

Second position: A paid 2, B paid 12 and C paid 18 so they paid 32 together and the error is 1168. If we increase A's sum by one the error will decrease by 16, so we have to increase A's sum by $1184 : 16 = 74$ " and he obtained the right answer.

We have to mention it is more straightforward, in this case, to work with the rule of proportion, as Egyptians did.

First position: C paid 800, B paid 533 and A paid 88,3 so they paid 1421,3 together and the error is 221,3. (Of course, here she made some rounding errors).

Second position: C paid 750, B paid 500 and A paid 83,3 so they paid 1333,3 together and the error is 133,3. If we decrease C's sum by 50, then the error will decrease by 88, so if we decrease C's sum by 25, then the error will decrease by 44, but 221,3 is approximately 5 times 44, so C's sum is $800 - 5 \cdot 25 = 675$ (and then she checked the answer)."

There was no pupil who worked with trial-and-error method.

PROBLEM 5.3. *Andrew is 30 years older than Paul. 25 years ago, Andrew was 3 times as old as Paul. How old is Andrew? How old is Paul?*

One pupil used the method of false position in the following way:

“*First position:* 25 years ago Paul was 1 year old and Andrew was 3 years old, the difference between ages is 2 years.

“*Second position:* 25 years ago Paul was 2 years old and Andrew was 6 years old, the difference is 4 years. So the difference between ages increases by 2, therefore, 25 years ago, Paul was $1 + 14 = 15$ years old (and she gave the right answer, i.e. Andrew is 70 years old and Paul is 40 years old)”.

4 pupils solved the problem by guess-and-check method. This method is quite simple, because the ages are positive integers below 100 and after some trials the right answer follows.

Most of the pupils tried to solve the problem algebraically, they remembered the methods learned at algebraic lessons.

7 pupils solved successfully the equation $x + 5 = 3 \cdot (x - 25)$, where x denotes Paul's age today. One pupil gave the right answer after she solved the equation $x + 30 = 3 \cdot x$, where x denotes Paul's age 25 years ago. Some pupils wrote erroneous algebraic equations, as follows $x + 30 - 25 = 3 \cdot x$; $x \cdot 30 - 25 = x - 25$; $3 \cdot (x + 5) = x - 25$; $x - 25 = 3 \cdot (30 + x - 25)$, two of them (after they solved the equation) realized that the answer is not right and then they solved the problem by guess-and-check method.

PROBLEM 5.4. *Two ships departed from the same port A, travelling in the same direction at 25 km/h and 20 km/h respectively. The first ship reached port B 4 hours earlier than the second ship. Find the distance between the two ports!*

6 pupils solved the problem successfully by the method of false position, most of them gave the following solution:

“*First position:* the distance is 100 km, the time is 4 hours and 5 hours respectively, the difference is 1 hour.

“*Second position:* the distance is 200 km, the time is 8 hours and 10 hours respectively, the difference is 2 hours. So the distance increases by 100 km, the difference will increase by 1 hour. So the distance must be 400 km”.

One pupils's answer: “First position the distance is 50 km, second position the distance is 100 km”, and thereafter she draw the conclusion that if the distance increases by 50 km then the difference increases by 0,5 hours, and she gave the right answer.

Another pupil used the rule of proportions, as he wrote: “If the distance is 200 km, then the ships need 8 hours and 10 hours respectively, so the difference is 2 hours. So if we want to double the difference we have to double the distance, so the distance is 400 km.”

4 pupils gave the right answer by sequential trial-and-error, one of them made tables as follows:

First ship	13 hours = 325 km	14 hours = 350 km	15 hours = 375 km	16 hours = 400 km
Second ship	17 hours = 340 km	18 hours = 360 km	19 hours = 380 km	20 hours = 400 km

Another pupil wrote the multiples of 25 and 20 respectively in two different rows, and then she gave the right answer. The pupils who tried to use algebraic methods gave no right answer. A pupil wrote that the first ship will need x hours and the second ship will need $x + 4$ hours, but she did not set up the equation. The other pupil began in the same way, but then she wrote the erroneous equation $25 \cdot x = 20 \cdot x + 4$ (paranthesis omitted).

PROBLEM 5.5. *In a parking place there are cars, bicycles and tricycles. Altogether there are 15 vehicles and 56 wheels. How many vehicles, bicycles and tricycles are there, separately?*

All the pupils gave the right answer, none of them tried to solve the problem algebraically or to set up equations. Previously, we solved Problem 3.1 both algebraically and by arithmetical methods, so they knew that the method of false position is more straightforward in this case. Therefore all of them applied the method of false position, most of them made the first position taking into account that there are 15 vehicles and they made assumption such as “all the vehicles are bicycles”, “all the vehicles are tricycles”, “all of the vehicles are cars” or “there are 5 cars, 5 bicycles and 5 tricycles”, but some of them made the first position more unusual, for example, “there are 8 bicycles, 4 tricycles and 3 cars”. Thereafter all the pupils changed the number of the vehicles systematically in order to increase or decrease the number of wheels as necessary (but they cared that the number of the vehicles must be 15, in the aggregate) until they get the right answer. One single pupil made the first position with taking into account that there are 56 wheels (this is more complicated), she wrote “there are 56 wheels, and we suppose all the vehicles are cars, so there are 14 cars”. Then she proceeded in an interesting way as she changed 2 cars with 3 tricycles (to adjust the number of vehicles to the condition of the problem), but the number of the wheels became 57. Thereafter

she had to change a tricycle with a bicycle to get the right answer. Most of the pupils gave the solution: 12 cars, 1 bicycle and 2 tricycles. One student gave the wrong answer: 4 cars, 5 bicycles and 6 tricycles. Another pupil considered that there are no tricycles and, in this way, the problem became analogous to Problem 3.1, then she gave the answer “12 cars, 0 tricycles and 3 bicycles”. Only one pupil realised that this problem can have more than one solution, he found two different answers, namely “12 cars, 1 bicycle and 2 tricycles” and “11 cars, 0 bicycles and 4 tricycles” (this denotes that the false position method, similarly to the other numerical checking methods, is adequate to find one answer but not all of the answers). We have to mention that the answers with “0 bicycles” or “0 tricycles” are not very right (although they satisfy the conditions of the problem concerning the aggregate number of the vehicles and the aggregate number of the wheels), because the number of the vehicles must be positive integers.

PROBLEM 5.6. The ratio of plums to apples in a basket is 5:1. If we put two apples in the basket and we remove 14 plums then the ratio of plums to apples will be 3:1. Find the number of plums and apples in the basket!

Most of the pupils solved the problem by algebraic methods. 10 pupils wrote the equation $5 \cdot x - 14 = 3 \cdot (x + 2)$, where x denotes the number of apples, and 8 of them solved properly the equation (they gave the right answer, namely there are 10 apples and 50 plums in the basket), 2 of them made calculational mistakes, when they solved the equation. One pupil wrote the equation $5 \cdot x - 14 = 3 \cdot x$, so she got $x = 7$. However she realised that this answer does not satisfy the conditions of the problem and thereafter she solved the problem properly by guess-and-check method.

4 pupils used the guess-and-check method and they gave the right answer.

None of the students used the method of false position, one of our expected responses could be, for example: first position “the number of apples is 5 and the number of plums is 25”, so the second condition delivers us the error $3 \cdot (5 + 2) - (25 - 14) = 10$; second position “there are 6 apples and 30 plums”, so the error is $3 \cdot (6 + 2) - (30 - 14) = 8$; if we increase the number of apples by one then the error will decrease by 2, so we have to increase the number of apples by 5, so the right answer (10 apples and 50 plums) follows.

In the following, we will show a geometrical problem which can be found in reference [15] and our pupils had to solve it. I. Osta and S. Labban studied the seventh grader’s problem solving strategies to solve a problem whose algebraic structure is a first-degree equation with the unknown occurring on both sides of

the equality sign, namely $a + b + c + x = m \cdot x$ or, when reduced: $A + x = m \cdot x$ (see [15]). Their study was carried out with a sample of 12 pupils. The pupils had to solve the following problem.

PROBLEM 5.7. *In the figure below, find the measure of the segment BI knowing that the perimeter of the triangle ABC is triple the measure of the segment BI .*

This problem was administered in two versions: Version *A* (corresponding to Figure 1) and Version *B* (corresponding to Figure 2) (6 pupils have to solve version *A* and 6 pupils have to solve version *B*).

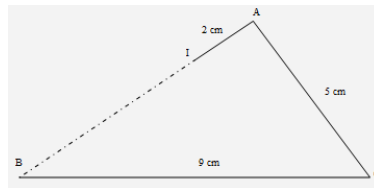


Figure 1

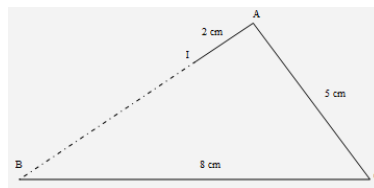


Figure 2

We can see the two versions of the problem are similar. In version *A*, the numerical answer (the measure of the segment BI) is a whole number, while in version *B*, it is a decimal number. The authors of the study analysed the problem solving strategies used by pupils. They found that ten out of the 12 participants used the numerical checking strategies (i.e. estimation/guess-and-check, random trial-and-error, sequential trial-and-error). All 6 pupils who worked on version *B* attempted numerical checking methods, but were diverted from them to other methods (such as performing various arithmetic operations on the known measures in the problem), mostly because the whole numbers they tried did not satisfy

the conditions of the problem, and when a few of them tried decimal numbers, they had to spend too long time on the calculations. Five pupils could reach a correct result (4 pupils on version *A* and one pupil on version *B*), 4 of them used numerical checking methods. As the foregoing study shows the grade 7 pupils were more successful with the numerical checking strategies, than without them.

Our attempt was to investigate how 8 grade pupils (armed with the tools of algebra and also initiated in the method of false position) can handle this kind of geometrical problem. All of the pupils had to solve both of the versions.

3 pupils solved the problem properly in algebraic way, 2 of them wrote the equations $x + 2 + 5 + 9 = 3 \cdot x$ and $x + 2 + 5 + 8 = 3 \cdot x$, one pupil wrote the equations $\frac{16+x}{3} = x$ and $\frac{15+x}{3} = x$.

6 pupils thought, that the sums $2 + 5 + 9 = 16$ (version *A*) and $2 + 5 + 8 = 15$ (version *B*), respectively, could be considered as two-third of the perimeter, so the measure of the segment *BI* is the half of the sum.

One pupil solved the problem by the rule of false position in the following way (we will show only the answer of the version *B*):

“if $BI = 3$ then the third of the perimeter is $(3 + 2 + 8 + 5) : 3 = 6$, so the error is 3; if $BI = 6$ then the third of the perimeter is 7, so the error is 1; so we have to choose $BI = 7,5$ (here he realised that if we increase the measure of *BI* by 3, then the error will decrease by 2).”

We have to underline the usefulness of the false position method contrary to the other numerical checking methods. 3 pupils tried to solve the problem by sequential trial and error, they found the right answer in the case of version *A*, but they failed in the case of version *B*, because they operated only with whole numbers. One pupil, who also used sequential trial and error method, did not find the right answer even in version *A*.

One pupil considered that the triangle is right-angled at *C*, so he wrote the Pythagorean theorem as follows: $9^2 = 5^2 + BA^2$ (version *A*) and $8^2 = 5^2 + BI^2$.

We have to underline that most of the pupils used algebraic or arithmetical procedures, a few of them appealed to numerical checking methods, and only one pupil used the method of false position.

6. Summary and conclusions

Many research works have pointed out that pupils may understand a word problem in everyday terms but be unable to represent its formal aspects as required for an algebraic solution. Pupils tend to use numerical and arithmetical processes mainly because they are used to perform operations and do procedural computations. According to other research works the numerical checking methods, such as guess-and-check and trial-and-error are the most commonly used. In our opinion the method of false position is a deductive way of thinking which can reduce the huge number of “purely” numerical checking methods among lower secondary school students. If the pupils are initiated in the false position method they will adopt a more systematic way of thinking which can eliminate the use of random trial-and-error which may require, in some cases, a huge amount of calculus. In many cases, trial-and-error and guess-and-check is proved to be inefficient, especially when we have to deal with big fractions or decimal numbers. However the grade 8 pupils do not undervalue the methods of algebra because there are cases when arithmetical methods or false position do not work efficiently. This is the case of “real algebra” problems, where algebra enables an efficient solution to be found. Pupils mainly appeal to the method of false position when they are faced with unusual problems or some exercises which they did not met before. Otherwise, if they remember some rules, patterns, models or similar problems which they have studied previously then they will be diverted from the false position method to arithmetical and algebraic procedures. In our opinion, the false position method has its own right place in the lower secondary school educational processes. However, we can find only a few examples of problem solving methods based on false supposition in the Hungarian student textbooks (we can find an example in [3], page 87). Because of the limitation of the study to a small size of students we are not able to generalize our findings and results to a wider population of pupils of other schools, following other programs. However we recommend it to other colleagues, primary (or even) secondary school mathematics teachers. In the future, we will try to extend the research among grade 6 and 7 pupils.

References

- [1] N. Amado, S. Carreira, S. Nobre and J.P. Ponte, *Representations in solving a word problem: the informal development of formal methods*, <http://www.researchgate.net/publication/261176504>.
- [2] A. Bell, K. Stacey and M. MacGregor, Algebraic manipulation: actions, rules and rationales, in: *Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia*, Brisbane, 1993.
- [3] M. Csordás, L. Konfár, J. Kothencz, Á. Kozmáné Jakab, K. Pintér and I. Vincze, *Sokszínű matematika 6*, Mozaik, Szeged, 2008.
- [4] M. A. Clements, Analyzing children errors on written mathematical tasks, in: *Educational studies in mathematics*, 1980.
- [5] G. Detori, R. Garuti and E. Lemut, From arithmetic to algebraic thinking by using a spreadsheet, in: *Perspectives on school algebra*, Dordrecht, 2001.
- [6] G. Egodawatte, Is Algebra Really Difficult for All Students?, *Acta Didactica Napocensia* **2**, no. 4, Cluj Napoca (2009).
- [7] V. Farmaki, N. Klaoudatos and P. Verikios, From functions to equations: introduction of algebraic thinking to 13 year-old students, <http://www.math.uoa.gr/me/faculty/klaoudatos/klaoudatos2.pdf>.
- [8] E. Filloy and T. Rojano, From an arithmetical to an algebraic thought, in: *Proceedings of the 6th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, University of Wisconsin, Madison, 1984.
- [9] S. Kántor, Maróthi György élete ÁŠs munkássága, *A Természet Világa* (2015).
- [10] S. Kántor and T. Varga, Nagy Károly, A reformkor tankönyvírója, a tehetséggon-dozás úttörője, *Polygon XXI*, no. 1–2, Szeged (2013).
- [11] T. Jakab, J. Kosztolányi, K. Pintér and I. Vincze, *Sokszínű matematika 7*, Mozaik, Szeged, 2007.
- [12] T. Jakab, J. Kothencz, Á. Kozmáné Jakab, K. Pintér and I. Vincze, *Sokszínű matematika 8*, Mozaik, Szeged, 2009.
- [13] D. I. Johanning, Supporting the development of algebraic thinking in middle school: a closer look at students' informal strategies, *Journal of Mathematical Behavior* **23** (2004).
- [14] S. Kangshen, J. N. Crossley and A. W.-C. Lun, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford University Press, Oxford, 1999, 358.
- [15] S. Laban and I. Osta, Seventh Graders' Prealgebraic Problem Solving Strategies: Geometric, arithmetic and algebraic interplay, www.cimt.plymouth.ac.uk/journal/osta.pdf.
- [16] Gy. Maróthi, *Arithmetica*, Debrecen, 1782, <https://babel.hathitrust.org/cgi/pt?id=mdp.39015021321925>.
- [17] K. Nagy, *Elemi arithmologia*, Arithmografia, 1835.

- [18] M. A. Newman, An analysis of sixth-grade pupils errors on written mathematical tasks, in: *Research in Mathematics Education in Australia*, Melbourne, 1977.
- [19] G. Pólya, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, John Wiley and Sons. Inc., New York, 1981.
- [20] G. Pólya, *How to Solve It*, Princeton University Press, Princeton, 1945.
- [21] http://kerettanterv.ofi.hu/03_melleklet_9-12.
- [22] H. Radatz, Error analysis in mathematics education, *Journal for Research in Mathematics Education* **10** (1979).
- [23] K. Stacey and M. MacGregor, Implications for mathematics education policy of research on algebra learning, *Australian Journal of Education* (1999).
- [24] K. Stacey and M. MacGregor, Learning the algebraic methods of solving problems, *Journal of Mathematical Behavior* **18** (2000).
- [25] K. Stacey, The transition from arithmetic thinking to algebraic thinking, University of Melbourne, www.mathhouse.org/files/.../IMECstaceyALGEBRA.doc.
- [26] I. Szalay, *A kultúrfilozófia természettudományos alapjai*, Szegedi Egyetemi Kiadó, Szeged, 2006.
- [27] D. Tall and M. Thomas, Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics* **22** (1991).
- [28] Z. Tuzson, *Hogyan oldjunk meg aritmetikai feladatokat?*, Ábel Kiadó, Kolozsvár, 2011.
- [29] T. Weston, *A Treatise of Arithmetic: In Whole Numbers and Fractions*, University of Michigan, 1729, <https://archive.org/details/atreatisearithm00westgoog>.
- [30] E. Warren, The role of arithmetic structure in the transition from arithmetic to algebra, in: *Mathematics Education Research Journal*, 2003.

ZSOLT FÜLÖP
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
HUNGARY

E-mail: fulop.zs32@freemail.hu

(Received June, 2016)