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Teaching Mathematics and Computer Science

# Number theory vs. Hungarian highschool textbooks: $\sqrt{2}$ is irrational

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*Abstract.* According to the Hungarian National Curriculum the proof of the irrationality of $\sqrt{2}$ is considered in grade 10. We analyze the standard proofs from the textbooks and give some mathematical arguments that those reasonings are neither appropriate nor sufficient. We suggest that the proof should involve the fundamental theorem of arithmetic.

*Key words and phrases:* proof, key steps, analogy, proof-theory.

*ZDM Subject Classification:* E50, F10, F40.

## 1. Introduction

This paper is a sequel to [5], where the presentation of basic notions of number theory in Hungarian textboooks, like fundamental theorem of artihmetic, greatest common divisor (gcd), least common multiple were analyzed. In this paper we investigate into the proofs concerning number theory appearing in highschool textbooks. The almost obligatory one is the proof of the irrationality of $\sqrt{2}$. The number $\sqrt{2}$ exists and irrational. In the inaguaral issue of this journal a paper was published about the existence of $\sqrt{2}$ [7]. In this paper we make a point that although $\sqrt{2}$ is irrational, the proof of this fact is not appropriate to introduce the notion of proof in highschool or at least not the proof that is presented in

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most books. Most people when asked about the proof of the irrationality of $\sqrt{2}$ start by saying "we prove by way of contradiction". We will argue that the first sentence should be: "The proof is based on the fundamental theorem of arithemtic (FTA)". The role of this proof is multipurpose. This is not only one of the few proofs covered in highschool, but also the first place where pupils meet with irrational numbers and the first time when they meet the strategy of proving by contradiction. The difficulties to understand this latter one is well analyzed in [15].

The role of proofs in mathematical education was always in the center of interest of researchers of the area. Among others, argumentation and proof was the topic of the working group 2 on the Sixth Congress of the European Society for Research in Mathematics Education, January 28th - February 1st 2009, Lyon (France). Also, ZDM, Mathematics Education devoted a full issue (Volume 40, Issue 3, 2008) on Didactical and Epistemological Perspectives on Mathematical Proof. Many other articles deal with the role of proofs in school mathematics. For a general account and latter references consult [11], where for example, it is stated that proofs convey important elements of mathematics such as strategies and methods, that it is proofs rather than theorems that are the bearers of mathematical knowledge. Thus proofs should be the primary focus of mathematical interest. The significance of profs for mathematics education in general and the teaching of proof is analyzed, as well. Meanwhile, constantly present, there are two competing views about mathematics, namely that it is a system of mathematical names, rules, procedures, justifications to learn, and a useful thing to explore that helps to the development of creative thinking [17]. The role of proofs and arguments is formulated in the Hungarian National Core Curriculum (NAT, 1995) [20] in a similar flavour.

We find the following in the shortened version of the Hungarian National Core Curriculum (NAT, 1995):

"Necessary knowledge, skills and attitudes

Essential knowledge in mathematics include the progressive knowledge of numeracy, measures and structures, basic operations and fundamental mathematical presentations, mathematical notions, correlations and concepts and understanding the questions to which mathematics can give answers.

Having acquired mathematical competence, the individual has the skills to apply basic mathematical principles and processes in the context of knowledge acquisition and problem solving in everyday situations, at home and work. *An individual should be able to follow and interpret a chain of arguments, to explain* *results with the means of mathematics, to understand mathematical reasoning, to communicate in the language of mathematics and to use appropriate resources.*

A positive attitude in the field of mathematics rests on the respect for truth and the disposition to seek logical reasons and their validity."

One third of this short version is about proofs and about understanding proofs. Looking at the detailed description, one can see that understanding certain concepts and skills dependent on them reappear in the objectives at the end of the grades 4, 6, 8 and 10, adjusted to fit the appropriate level of students' age characteristics. Students build a system of mathematical arguments and knowledge (concepts, procedures, theorems, justifications) during a mathematical activity. Mathematics is a useful thing to explore that helps to develope creative thinking. It is pointed out in [17] and [1] that there are thoughts among the precedents of all proofs at school that rely on authority, repetition, or acceptence based on concrete examples, and as such, are only locally arranged. The students usually do not have the background system for the studied statement. So the learnability, the details, the structure and arguments of proofs must fit into the responsibility of the textbook author and the teacher. Incomplete, misleading or even flawed reasonings just work against achieving the overall objective of teaching mathematics.

In highschool the NAT requires from pupils to recognize arguments in proofs and find and understand the key steps of the proofs.

We examine the highschool textbook proofs from these aspects. Missing arguments, incomplete reasonings will be found at several places. The leading question of our research is whether or not the proofs of the irrationailty of $\sqrt{2}$ are satisfiyng those conditions described above. We claim that ALL these proofs lack the minimum appropriate mathematical precision. We made a case study to see if people can recall or reconstruct this proof. We asked people who are supposed to remember the proof: math students at the university and high school math teachers. Some details and conclusions of this case study will be mentioned in Section 3. We do not present the whole study, it is not the main topic of our paper. The main goal is to show that these proofs are incomplete. Mostly because an average pupil is not able to fill in the gaps.

Concerning missing arguments or incomplete proofs, according to Lakatos [14] a reasoning left to the reader is far too delicate instructional element for approval. Often very careful deliberation, deep consideration are necessary in order to make students benefit from the use of statements like "obviously", "the same way", "similarly", "analogously" in making independent reasonings and not simply train them to cover up the uncomfortable parts of a proof. To achieve our goal the phylosophical concept of Lakatos [14] will be applied: no theorem of informal mathematics is final or perfect. This means that one should not think that a theorem is ultimately true, only that no counterexample has yet been found. Once a counterexample, that is, an entity contradicting/not explained by the theorem is found, we adjust the theorem, possibly extending the domain of its validity. This is a continuous way knowledge accumulates, through the logic and process of proofs and refutations. We extend this idea by the steps presented in Appendix 1 of [14]:

(1) We consider the statement as a primitve conjecture.

(2) Next we present the proof.

(3) Then, we give global counterexamples.

(4) Then, we give local counterexamples to the steps of the proof

(5) We find the "guilty" steps

Then we correct the proof. Following Lakatos' spirit, we do not claim that we finalize all possibilities of handling $\sqrt{2}$. There will be asked an unasked questions left open. We are sure that the reader himself will find such questions and develop further ideas towards this simple proof. Our counterexamples and "guilty steps" are of mathematical flavour. We concentrate on the number theoretical–ring theoretical side of the proof. We shall show structures where analogous statements do not hold or analogous arguments do not work. This points out that in those cases extra reasonings are required to avoid confusion or incompleteness.

## 2. Proofs in print

Now, we turn our attention to the proof of the irrationality of $\sqrt{2}$. We start by sampling textbooks. Investigaton of text book proof is not a new idea, for example a Sweedish analysis can be found in [16]. We looked up all textbooks in use in Hungarian high schools since 2001. All nine textbooks contained a proof of the irrationality of $\sqrt{2}$ except [4]. We shall refer to the other books later, when the proofs are presented. Although textbooks are goverment approved in Hungary, the NAT 95 gives some freedom on the order of topics covered. This is why the proof is contained sometimes in the grade 9 and sometimes in the grade 10 books.

PROOF 2.1. [13] We prove by way of contradiction: Let us assume that $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$ and they are relatively prime: $(p; q) = 1$

$$\begin{aligned} \sqrt{2} =& \frac{p}{q}, \\ 2 =& \frac{p^2}{q^2}, \\ 2q^2 =& p^2. \\ & \Downarrow \\ 2|p^2 \Rightarrow 2|p \Rightarrow 4|p^2&, \\ \text{if} \quad 4|p^2 \Rightarrow 2|q^2 \Rightarrow 2|q&. \end{aligned}$$

We obtained that that $p$ and $q$ are both even. This contradicts the assumption that they are relatively prime, so the statement is true.

This proof is lacking arguments. The setup, the context and the initial condition are explained in words, and after that you only have a series of symbols. At the second part of the verification words are replaced by the $\Rightarrow$ symbol. This symbol (by the agreement of the mathematical community) means that the sentence before implies the sentence after. First, we do not use this symbol in written, formal proofs. Secondly, even if we use, we mostly use it for (obvious) deductions.

EXAMPLE 2.2. For example consider the following sequence of arguments:

$$x > 10 \Rightarrow x > 0 \Rightarrow x \text{ has a square root.}$$

In this a sequence the first implication comes from $10 \geq 0$ and the transitivity of $>$. The second implication comes from the definition of the square root function. The first argument is so basic, well-known and widely accepted, that it does not require additional explanation. The second one is less trivial for those who have just defined the notion of squareroot, but $x > 0 \Rightarrow x$ has a square root is a part of its definition. Moreover, both implications are true if we replace 10 by any number greater than 0. Hence the arguments are true in a very general sense.

We shall see later in the paper that the arguments of Proof 2.1 are not of this kind.

PROOF 2.3. [10] Let us assume that $\sqrt{2}$ is rational. It means that it can be written as the ratio of two integers $p$ and $q$, $\sqrt{2} = \frac{p}{q}$ where we may assume that $p$

and $q$ have no common factors. (If there are any common factors we cancel them in the numerator and denominator.) Squaring both sides gives $2 = \frac{p^2}{q^2}$ which implies $2q^2 = p^2$ Thus $2|p^2$. *The only way this can be true* is that $2|p$. However, in this case actually $p^2$ is divisible by 4. Hence $2|q^2$ and *therefore* $2|q$. So $p$ and $q$ are both divisible by 2 which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational! $\square$

Almost the same proof shows up in a few other books. In [9, 6] *The only way this can be true* is replaced by *This implies that*

And a proof from [2, 3, 12]:

Proof 2.4. Let us assume that $\sqrt{2}$ is rational. It means that it can be written as the ratio of two integers $p$ and $q$, $\sqrt{2} = \frac{p}{q}$ where we may assume that $p$ and $q$ have no common factors. (If there are any common factors we cancel them in the numerator and denominator.) Squaring both sides gives $2 = \frac{p^2}{q^2}$ which implies $2q^2 = p^2$. $2q^2$ is even, hence $p$ itself is even, because only an even number's square can be even. But then $p^2$ is actually divisible by 4. Hence $q^2$ and *therefore* $q$ must be even. So $p$ and $q$ are both even which is a contradiction to our assumption that they have no common factors. The square root of 2 cannot be rational! $\square$

We give two evaluations, two different evaluations of Proof 2.3 and 2.4.

`Evaluation 1.` Proof 2.3. This proof is nothing else, but Proof 2.1 planted into words. Each $\Rightarrow$ is replaced by some text that has no information on the ongoing mathematics. Hence, it is equivalent to Proof 2.1. Or evenmore dangerous, because words are more convincing than arrows. The uncertain reader, who is unexperienced, who is not safe with proofs yet, might easily think that he is the (only) one who does not understand the proof and accept it without interpreting it for himself. Or, he could simply believe that this is what a proof looks like. Then, as he does not see the point, he may join the (large) group of people for whom mathematics is mystery. Being the first and one of the very few proofs in highschool, the irrationality of $\sqrt{2}$ deserves more care.

`Evaluation 2.` Proof 2.4. One could argue as follows. Being even is so wellknown, widely accepted notion as being integer. Everybody knows about even and odd numbers. Everybody sees that if $p^2$ is even then $p$ is even. So this proof is correct, and if you understand the difference between even and odd, then you understand this proof, as well.

Which of the above two evaluations is correct? If we agree, that Proof 2.1 is not a proof, then Evaluation 1. is correct, because a proof is recognized (among others) by the sequence of deductions, and the two proofs do not differ in this sense. The validity of Evaluation 2. is more questionable. The difference between Proofs 2.1 and 2.3 is that we replaced the formula $2|n$ by the expression $n$ is even. The two things are equivalent, the definition of beeing even is being divisible by 2. Does it make a difference? We leave this question open for a while.

We arrived at a set of correct proofs. The first one has an even-odd type argument, as Proof 2.4 but with more explanations. This is just an extra half sentence, but it makes the picture clear.

PROOF 2.5. Let us suppose that $\sqrt{2}$ is a rational number. Then we can write it as $\sqrt{2} = \frac{a}{b}$ where $a$, $b$ are integers, and $b$ is not zero. We additionally assume that this $\frac{a}{b}$ is simplified to lowest terms, since that can obviously be done with any fraction. Notice that in order for $\frac{a}{b}$ to be in simplest terms, both of $a$ and $b$ cannot be even. One or both must be odd. Otherwise, we could simplify $\frac{a}{b}$ further. From the equality $\sqrt{2} = \frac{a}{b}$ it follows that $2 = \frac{a^2}{b^2}$, which means that $2b^2 = a^2$. So the square of $a$ is an even number since it is two times something. From this we know that $a$ itself is also even, because it can't be odd; if a itself was odd, then $a \cdot a$ would be odd, too. Odd number times odd number is always odd. If $a$ is an even then $a$ is 2 times some other integer. In symbols, $a = 2k$, where $k$ is this other number. If we substitute $a = 2k$ into the original equation $2 = \frac{p^2}{q^2}$, this is what we get:

$$
\begin{aligned}
2 =& \frac{(2k)^2}{b^2} \\
2 =& \frac{4k^2}{b^2} \\
2b^2 =& 4k^2 \\
b^2 =& 2k^2
\end{aligned}
$$

This means that $b^2$ is even, from which follows again that $b$ itself is even. And that is a contradiction. Therefore $\sqrt{2}$ cannot be rational.

This last proof adds a single extra remark to the earlier proofs: odd times odd is odd. We agree, that this statement is true. We agree that this explanation is correct. We agree that this proof is using only elementary arguments. We agree that the notion of even-odd is supposed to be known by every student.

## 3. Case study

We made a case-study about how people relate to this proof. We asked just finishing highschool students, first, second and third year math students at the university and highschool math teachers. Each of them was asked (without any preliminary notification) to prove that $\sqrt{2}$ is irrational. We have interviewed altogether 103 people. Each interview involved writing possibilities (paper, white-board, etc). We do not give here all details of the case study. At first, it is not the topic of the paper. Secondly, it would exceed the space limit. We just mention a few interviews to demonstrate that even the mathematically educated people have difficulties with this proof.

As a first approach, over 30 students were questioned, mostly randomly selected among the university students of grade 2-4, each of them have finished a number theory course. Most of them started in the following way:

"We prove by way of contradiction. Let us assume that the statement is not true. Than $\sqrt{2}$ can be written as the quotient of two integers:

$$\sqrt{2} = \frac{p}{q}$$

Then multiplying by $q$ we get

$$q\sqrt{2} = p$$

Here 4 students has stopped. What now? Then we gave a hint that you should omit the squareroot.

Squaring both sides we obtain

$$2q^2 = p^2$$

At this point another 5 students stopped and quitted. They said something like: I do not remember from here. I am too tired, I cannot continue. Most students here went back to the first line and added that

$$(p, q) = 1,$$

saying that we may assume that the gcd of $p$ and $q$ is 1. Then they said that here something is even, I remember... I do not know, what exactly. Sorry.

Altogether 23 of the first 30 students failed to produce a proof.

At some point we have interviewed a group of five students, and we started to question them about the parity argument. They recalled that something has to be done with parity, and slowly, and not so confidently they figured out the odd times odd argument. As they have recollected their memories on divisibility, they proved that odd numbers are of the form $2k+1$ and then that really, odd times odd is odd by multiplying $(2k+1)(2l+1)$.

A former student said that she will go to the school she is teaching at and discuss it with her colleagues. We copy his letter here without any changes:

*Dear Professor,*

*I was talking to X.Y., a math teacher, who finished her studies as a math major a few years ago and we gave up the proof of the irrationality of squareroot 2 at the point, where the parity argument came. The notion of prime, irreducible came up, and the the notion of even numbers, of which we are not sure about any more. I asked her about what an even number is (as you instructed me to do so in this case), and first she said that those are the numbers ending in 0,2,4,6,8 . Then she said the divisibility by 2, end came the FTA. But in case of* $\sqrt{2}$ *if* $q = 1$ *then* $\sqrt{2} = p$ *which is an integer. So, is the squareroot of 2 even? It is scary to think about it. Y. said that we should rather stay at the proof, where we mesure the side of a square to its diagonal :-). Tomorrow I will talk to the other math teachers.*

*Best wishes:*

And finally, the 7 succesful participants of our case-study all came up with the following answer: "Let us assume that $\sqrt{2}$ is the quotient of two integers:

$$\sqrt{2} = \frac{p}{q}$$

Then multiplying by $q$ we get

$$q\sqrt{2} = p$$

Squaring both sides we obtain

$$2q^2 = p^2$$

Factor both sides into a product of primes. In a square every prime shows up on an even power (possibly 0) by FTA. Hence 2 is on an odd power on the left side and on an even power on the right side, a contradiction. Thus $\sqrt{2}$ is not rational.

And another issue is that a few books wrote and a few books imply the following:

`Remark`: It can be shown similarly that $\sqrt{a}$ is irrational for every integer $a$, where $a$ is not a perfect square.

## 4. The mathematical background

And now, let us imagine that we believe that the irrationality of $\sqrt{a}$ can be proved similarly. After copying the first few steps we arrive at

$$aq^2 = p^2$$

and we would like to conclude that

$$a|p.$$

The bad luck is that this statement is not true in general. For example,

$$12|6^2, \text{but } 12 \nmid 6.$$

And if such a statement is not true in general, then first, you need an argument why it is true for $a = 2$, secondly, you cannot claim that the proof goes similarly for other integers $a$ in general.

Let us examine, for which integers $a$ can the above conclusion be drawn:

Theorem 4.1. *Let $a$ be an integer. The conclusion*

$$a|n^2 \textit{ implies } a|n, \tag{1}$$

*holds if and only if $a$ is squarefree (a product of distinct primes).*

Proof 4.2. Let $a = p_1 p_2 \cdot \ldots \cdot p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes. Then $a|n^2$ implies $p_i|n^2$. Then, by the prime property, is $p$ is a prime, $p|n \cdot n$ implies $p|n$ or $p|n$. Hence $p_i|n$ for every $1 \le i \le k$. As the primes $p_i$ are distinct, they are pairwise relatively prime, so the product of these primes divides $n$. Thus $a|n$.

For the other direction let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot_k^{\alpha_k}$, where $\alpha_1 > 1$. Now, define $n = p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdot \ldots \cdot_k^{\alpha_k}$. We claim that $a|n^2$ and $a \nmid n$. As $a > n$, we have $a \nmid n$. Now, $n^2 = p_1^{2(\alpha_1 - 1)} p_2^{2\alpha_2} \cdot \ldots \cdot_k^{2\alpha_k}$. We need to show that the exponent of every prime in $n^2$ is greater or equal to their exponents in $a$. As $\alpha_1 > 1$, we have $\alpha_1 \ge 2$, and so $2\alpha_1 \ge \alpha_1 + 2$ and finally $2\alpha_1 - 2 \ge \alpha_1$. The desired inequality for $i > 1$ trivially holds. Hence $a|n^2$.

$\square$

Hence, the attempt for a “similar” proof could only work for squarefree numbers, and even in that case the proof is very complicated. Now, let us turn our attention to the "copiability" of Proof 2.5. There, the argument says that even times even is even and odd times odd is odd. How could it be translated for a general integer $a$, as suggested by the authors? Well, saying even means divisible by 2, and being odd means not divisible by 2. Hence, the parity argument can be reformulated in the following way:

$$\text{If } a \nmid b \text{ and } a \nmid c\text{, then } a \nmid bc.$$

Or, the contrapositive statement

$$\text{If } a \mid bc \text{ then } a \mid b \text{ or } a \mid c.$$

This is exactly the prime-property. In higher mathematics this IS the definition of a prime. This propety does nor hold for non-primes, for example,

$$6 \nmid 4 \text{ and } 6 \nmid 15\text{, but } 6|4 \cdot 15 = 60.$$

In highschool the prime property is not mentioned, or, if they mention it, then it is a consequence of the fundamental theorem of arithmetic. Although the proof is not difficult, without proving the prime property, the generalisation of Proof 2.5 cannot be complete. Pupils in general may easily have difficulties with the notion of prime or irreducible numbers [18, 19]

THEOREM 4.3. *If $p$ is a prime and $p|ab$ then either $p|a$ or $p|b$.*

PROOF 4.4. By the fundamental theorem of arithmetic $a$ and $b$ can be written as a product of primes, $a = p_1p_2 \cdot \ldots \cdot p_k$ and $b = q_1q_2 \cdot \ldots \cdot q_t$. As $p|ab$, there is a $k \in \mathbb{Z}$ such that $ab = pk$. Now, again by the fundamental theorem of arithmetic $k$ can be written as a product of primes, $k = r_1r_2 \cdot \ldots \cdot r_s$. Now,

$$ab = p_1p_2 \cdot \ldots \cdot p_kq_1q_2 \cdot \ldots \cdot q_t = pr_1r_2 \cdot \ldots \cdot r_s$$

This is two factorisation of $ab$ into a product of primes. By the uniqueness of the factorisation of $ab$, the prime $p$ occurs on the left side, as well. Hence $p$ has to be equal to $p_i$ or some $q_j$ for some $i$ or $j$. Thus either $p|a$ or $p|b$ □

## 5. To prove or not to prove

We arrived at the point, where we can look at the validity, essence, key steps and generalizability of the above proofs of Section 2. The most dangerous proof seems to be Proof 2.1, because it contains no arguments, only arrows of implications. However, Proof 2.3 is worse, because it makes an impression that explains the steps. Neither Proof 2.1 nor Proof 2.3 should be allowed to appear in print. The "*this can only happen if*" argument is not an argument in its own. It needs, it requires, it refers to, that there is a short, brief, understandable explanation. This is a typical place for the responsibility of authors and teachers. This is the uncomfortable place where you would need reasoning, an unconvinient reasoning, maybe cutting the rithm of the proof, but it is needed. But in all of these books the proof is omitted. It suggests that agitating expalantions are enough to have a statement proven. We might risk the claim that the ones who wrote these proof do not follow the spirit of proofs described in Section 1 by the National Core Curriculum.

The said news is that all those implications in the proofs are true. Why is it sad news? Because the implication drawn in that argument is true. So, it is not easy to point out where the mistake is in the proof. Because every step is true. On the other hand it is kind of an unwritten agreement what steps do we accept as steps of a proof. And a step is different in elementary school, different at highschool and different at the university, at college level. Thorough readers and highschool teachers may say that those arguments are not necessary to present in highschool, these notions are either not necessary, or the proofs are more intuitive in highschool, and we accept those arguments. It is hard to argue against this concept in general. Now, for thre proof of the irrationality of $\sqrt{2}$ this is not the case. The case study of Section 3 clearly shows that those proofs and those arguments are lost for the eternity.

In the Introduction we promised to argue only mathematically, hence we have to return to the 5 steps of Lakatos. Steps 4 and 5 could come to our help. Let us consider the two most well-known structures as places for counterexamples in number theory.

$$\mathcal{G} = \{a + bi \mid a, b \in \mathbb{Z}\} \text{ and } \mathcal{J} = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

The structure $\mathcal{G}$ is called the ring of Gaussian-integers. In $\mathcal{G}$ the number 2 is not a prime: $2 = (1 + i)(1 - i) = -i(1 + i)^2$. And, so, for example

$$2|(3 + i)^2 = 2(4 + 3i) \text{ but } 2 \nmid 3 + i \text{ in } \mathcal{G},$$

even worse, $2|(1 + i)^2 = 2i$. One might say that, sure, you are right, but we are talking about a prime in the proofs, and 2 is not a prime in $\mathcal{G}$. And this is the first point, where we can argue, that Proof 2.5 is not correct, because it does not seem to use the primness of 2. Before we believe that the proof works at least for primes, observe that 2 and $1 + \sqrt{-5}$ are both primes in $\mathcal{J}$. Thus

$$2 \nmid 1 + \sqrt{-5} \text{ but } 2|(1 + \sqrt{-5})^2 = -4 + 2\sqrt{-5} = 2(-2 + \sqrt{-5}) \text{ in } \mathcal{J}.$$

The number 2 divides the square of an other prime. What happened here, exactly? In higher mathematics we showed counterexamples to all those implications that are used to conclude that both the numerator and the denominator of the fraction are even.

As an ultimate attack to our reasoning could be that well, $\sqrt{2}$ is never a fraction of two elements in any ring. So, there must be a proof not using the fundamental theorem of arithmetic. A counterexample to this is the ring $\mathbb{Z}[\sqrt{8}]$. There, $\sqrt{2} = \frac{\sqrt{8}}{2}$, hence "rational". In Table 1. we have summarized our counterexamples to the several possible arguments.

*Table 1*

| **Guilty argument** | **ring** | **Counterexample** |
|---|---|---|
| $n \mid a \cdot b \Rightarrow n \mid a$ or $n \mid b$ | $\mathbb{Z}$ | $6 \mid 9 \cdot 4$, but $6 \nmid 9$ and $6 \nmid 4$ |
| $n \nmid a \cdot$ and $n \nmid b \Rightarrow n \nmid ab$ | | |
| $n \nmid a^2 \Rightarrow n \nmid a$ | $\mathbb{Z}$ | $12 \mid 6^2$, but $12 \nmid 6$ |
| $p$ prime and $p \mid a \cdot b \Rightarrow p \mid a$ or $p \mid b$ | $\mathbb{Z}[\sqrt{-5}]$ | $2 \mid 6 = (1+\sqrt{-5})(1-\sqrt{-5})$, but $2 \nmid (1 \pm \sqrt{-5})$ |
| $p$ prime and $p \mid a^2 \Rightarrow p \mid a$ | $\mathbb{Z}[\sqrt{-5}]$ | $2 \mid (1+\sqrt{-5})^2$, but $2 \nmid (1+\sqrt{-5})$ |
| $\sqrt{2}$ irrational | $\mathbb{Z}[\sqrt{8}]$ | $\frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$ |

Why is it true, then that $\sqrt{2}$ is irrational? The answer is easy: because the fundamental theorem of arithmetic holds among the integers.

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