13/2 (2015), 249–266
DOI: 10.5485/TMCS.2015.0401
tmcs@science.unideb.hu
http://tmcs.math.klte.hu

Teaching
Mathematics and
Computer Science

The investigation of students' skills in the process of function concept creation

Gyöngyi Szanyi

Abstract. Function is a basic concept of mathematics, in particular, mathematical analysis. After an analysis of the function concept development process, I propose a model of rule following and rule recognition skills development that combines features of the van Hiele levels and the levels of language about function [11]. Using this model I investigate students' rule following and rule recognition skills from the viewpoint of the preparation for the function concept of sixth grade students (12-13 years old) in the Ukrainian and Hungarian education system.

Key words and phrases: function concept, Ukrainian and Hungarian secondary education, features of van Hiele levels, rule following and rule recognition skills.

ZDM Subject Classification: I23.

Introduction

The function concept interweaves the whole teaching of mathematics. Functions are incorporated in the concepts of numbers, equations, inequalities, ratio, proportionality, geometrical transformations, etc. Through the teaching of functions, it is also possible for students to develop creativity, functional thinking, and other cognitive strategies [8].

This research was supported in the framework of TÁMOP-4.2.2B-15/1/KONV-2015-0001 program.

The notion of functions evolved from dependence relationships of real life phenomena to an abstract correspondence that is usually best describe in symbolic terms ([9], [3]). Freudenthal [6] in his study notes that the concept function can be developed in a natural way building from the learner's intuitive notions of the concept.

In her study, Sierpinska [3] sets out the conditions of understanding the notion of function. These conditions illustrate that it takes time to reach a thorough understanding of the function concept. There is a long journey between beginning to develop an understanding of the links between the elements of sets to the robust function concept. Dreyfus and Vinner [16] point out that this robust concept can be defined as a rule. According to Kwari [13] the rule is an element of the function concept.

Taking into account these research regarding the skills that are necessary in the formation of the function concept, possessing rule-recognition and rule-following skills (hereafter referred to as RR and RF) is exceptionally important in the period before providing the definition of function (preparation period) in order to be able to recognize function-like relations. These skills are needed in the construction of value tables, which help children to figure out the relationship between quantities, as well [10]. So, in this study I investigated the RF and RR skills of two classes of sixth grade (12-13 years old) students. One class being the part of the Ukrainian education system and one the Hungarian education system. The examination was based on the analyses of the above described period that happens during the fifth - sixth grade, and on the analyses of the Ukrainian and Hungarian curriculum framework and textbooks. The study revealed that the development of RR and RF skills are missing from the Ukrainian curriculum, unlike the curriculum in Hungary, where the development of these skills are stressed.

Theoretical background

Definition plays an important role in mathematics. According to Skemp [14], definitions have their specific places in mathematical concept development, and teaching concepts should be based on two principles: "concepts of a higher order than those which people already have cannot be communicated to them by a definition, but only by arranging for them to a suitable collection of examples. Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner" (as cited in [14], p. 18).

The modern definition of function that frames this study is the Dirichlet-Bourbaki definition, which is "a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)"(as cited in [16], p. 357). Looking at Skemp's principles we can conclude that in order to define this concept, the knowledge of other concepts is needed.

So, we have to take a long journey until we get from the study of the links between the elements of the sets to the exact function concept. This process includes content expansion and the exploration of links between several content elements [8]. As a result of the process, the function notion is created in the students. This is supported by the study of Vinner and Dreyfus [16]. They asked secondary school students to define function. The authors, drawing on Vinner [15], categorized students' definitions of function into six categories: (A) correspondence (the Dirichlet-Bourbaki definition); (B) dependence relation (dependence between two variables); (C) rule (a function is a rule; a rule is expected to have some regularity, whereas a correspondence may be "arbitrary"); (D) operation (a function is an operation or manipulation); (E) formula (a function is a formula, an algebric expression, or an equation); and, (F) representation (graphical or symbolic representation) (as cited in [16], p. 360).

Taking into account these categories, it can be highlighted that the function can be defined in various ways. One of these define the function as a rule.

Sierpinska [3] described the "worlds"that the study of functions should focus on: the world of changes or changing objects; the world of relationships; and, the world of rules, patterns, and laws. Rivera [7], for example, discuss linear functions as instances of numerical patterns that can be naturally described and expressed in several different representational formats (verbal, graphical, symbolic, etc.). According to Sierpinska [3] the change can be described as a transformation. The difference between the rule and relationship is subtle because the rules, patterns and laws are simply well defined relationships. Relationships can be expressed verbally or using diagrams, tables, graphs or in symbols. A rule can be a verbal statement, a formula or an equation. It is possible for one to detect a relationship but fail to explicity state the rule. Finding rules, patterns and laws can be used as an entry point to the development of the function concept.

Among the skills that could be linked to the above listed "words", possession of the RR and RF skills are significant in order to recognise and express function-like relations.

The *skill* (as cited in [4], p. 196) is considered to be the psychic feature of an individual, that evolves by the practice of some kind of activity, and is manifested in the doing of that activity, then the mentioned skills can also be developed by cognitive operations. The recognition of a rule (regularity), the following of the rule, and in some cases, the appropriate application of the rule, presumes the execution of a series of cognitive operations (categorisation, selection, and link-recognition).

The information acquisition process is strongly influenced by the development of students' cognitive operations. Pierre van Hiele and Dina van Hiele-Geldof developed a pedagogical theory in 1957 for the understanding of the process of geometric thinking, which differentiates between five levels of geometric thinking: visualization; analysis; informal deduction; deduction; and, rigor (as cited in [5], p. 51). The features of van Hiele levels are the following: (1) *Language hierarchy*. Each level has its own language and the levels are hierarchical; (2) *The existence of un-translatable concepts*. The corresponding contents of different levels sometimes conflict; (3) *Duality of object and method*. The thinking of each level has its own inquiring object (subject matter) and inquiring method (the way of learning); (4) *Mathematical language and student thinking in context*. While the levels are distinguished as sets of mathematical language, the actual thinking of each student varies depending on the teaching and learning context [11].

Freudenthal [6], Hoffer [2] and Isoda [11] extend the van Hiele levels from geometry to other areas. Van Hiele, himself, has written about levels in arithmetic and algebra [12]. He observed 'a change in level'from the act of counting to the concept of number. Freudenthal viewed progressive mathematization as the main goal of school mathematics. For this ongoing task, he provided a framework by recursively defined levels: the activity of the lower level, that is the organizing activity by the means of this level, becomes an object of analysis on the higher level. Freudenthal's theoretical approach rests on the Van Hiele levels.

Isoda [11] first discusses the levels of function from the point of view of language, using the features of van Hiele levels. He point out that they are also characteristics of the proposed levels of language about function. He shows the duality between object and method in van Hiele's levels (the levels of geometry) and in the levels of function. These levels of language are: *Level 1. Level of everyday language* (students describe relation in phenomena using everyday language obscurely: students explore phenomena (object) using obscure relations or variation (method)); *Level 2. Level of arithmetic* (students describe the rules of relations using tables. They make and explore tables with arithmetic: students explore the relations using rules); *Level 3. Level of algebra and geometry* (students describe function using equations and graphs: students explore the rules using notations of function); *Level 4. Level of calculus* (students describe function using calculus); *Level 5. Level of analysis* (an example of language for description is functional analysis which is a metatheory of calculus).

Using features (1) and (3) of van Hiele levels and the first three of the five levels of function described by Isoda [11] in the present study I set out the levels of the cognitive operations that are crucial for the possession of RF and RR skills and the criteria for categorising activity forms into levels. Noticing an analogy between these levels and the van Hiele levels, I used the names of the van Hiele levels for the marking of the discussed levels. The levels which I created by joining the features of van Hiele levels and Isoda's levels and using them to develop a deeper understanding in (sixth grade) students' development of the function concept, are the following:

Level 1 (visualization): Students recognise some kind of rule (method) between the element pairs (object) and follow the recognised rule (*level of everyday language*).

Level 2 (analysis): Students are able to phrase the recognised rule with words (they can argue in favor of the recognised links between the cohesive element pairs) and follow the rule which is given by words or by simple formulas (*level of everyday language and level of arithmetic*).

Level 3 (informal deduction): At this level the harmony of the simple rule-making and its description with formula develops (*level of arithmetic and level of algebra*).

Methodology

Sample

Participants were 26 sixth grade students (12-13 years old), with moderate abilities, in a school with Hungarian as the language of instruction in Ukraine and 23 students from the education system of Hungary (12-13 years old). When choosing our sample, we tried to balance between the two groups in a way that none of them are specialised classes in Mathematics. They study the subject in 4 hours per week and by the end of the sixth grade they acquire the same material. Based on their grades the students are on the same level of knowledge. The students had four classes of mathematics a week, according to the state curriculum framework. In both countries they use the textbook supported by the Ministry of Education of the given country (in a school with Hungarian as the language of instruction in Ukraine the Hungarian version of the mathematics textbook is used at this level).

As the research was carried out in March, during the second semester of the sixth grade. Students of both countries were already familiar with the natural numbers, fractions (common fractions and decimals), and had learned arithmetic operations with rational numbers. The introduction of proportional amounts and direct proprotionality occurred during this period, with the practical application in the initial phase.

Background

In the Ukrainian and Hungarian education system, function as a mathematical concept is defined at the seventh grade of the secondary school. Before the introduction of the concept both countries use the same material, according to the curriculum. In the lower classes, students are prepared with the use of different ways for introduction of the function concept. I analysed the Hungarian and Ukrainian curriculum and the textbooks for the fifth and sixth grade from the point of view of topics and their content that are supposed to support the development of the function concept. In Table 1, I summarized the Ukrainian and Hungarian textbook and curriculum themes that could support the preparation of the function concept. As the result shows (Table 1) major deficiencies come to the surface in the requirements for developing RF and RR skills (in the lower classes in the Ukrainian education it does not exist at all). In the development requirements of the themes of the Ukrainian curriculum, RR and RF skills are not mentioned definitly, unlike the curriculum in Hungary, where the development of these skills are more stressed. Prior research (cf., studies cited above), however, suggest that they are necessary for the development of the function concept. The Hungarian curriculum contains more materials which, together with the aforementioned skills target on those skills that are necessary for the preparation of the function concept (highlighted in the table). The numbers in brackets under the themes (5^{th} or 6^{th} form) indicate the grades in which the theme is taught.

Based on these aspects, in this study I am looking for the answers to the following questions:

(1) *At the end of the* 6^{th} *grade, what level do Ukrainian and Hungarian students (hereafter referred to as UA and HU students) reach in their RF and RR skills?*

Table 1. Themes preparing the function concept in the Ukrainian and Hungarian textbooks[1] and curricula[2]

Textbook and curriculum, Ukraine		**Textbook and curriculum, Hungary**	
Themes	**Development requirements in the curriculum**	**Themes**	**Development requirements in the curriculum**
Letter expressions (5^{th} form)	The recognition of number- and letter expressions and the illustration with examples	*Sum, difference, multiplication, ratio changes (5^{th} form)*	Discipline, consistency, *development of rule following behaviour.* Improving algorithmic thinking
Linear relationship (6^{th} form)	Illustration of proportional amounts with examples, the definition of the concept of linear relationship, finding the unknown element of the proportion, *defining the proportion between amounts*	*Proportional conclusions (5^{th} form) Functions, inverse and linear relations (6^{th} form)*	Developing the inferential skills. Observing the changes of one quantity caused by the changes of the *adherent quantity.* Developing the sense of proportion, the evaluation of real relations based on the maps of settlements
Diagrams (6^{th} form)	Editing column- and circle diagrams	*Interpreting simple figures, preparing and reading tables ($5^{th} - 6^{th}$ form)*	Developing analysing skills using figures and tables from daily press and other newspapers
Cartesian coordinate system (6^{th} form)	Finding the coordinate of the point in the coordinate plane and representing the given coordinate point	*Cartesian coordinate system ($5^{th} - 6^{th}$ form)*	Reading the coordinates of a given point, and depicting points of given coordinates in the Descartes coordinate system
Graphs (6^{th} form)	*Representing correlations between quantities by graphs* and analysing these graphs. The student is able to read the data from the graphs	*Correlations, graphs, series, tables ($5^{th} - 6^{th}$ form)*	Recognising correlations. *Creating experimental functions, series. Grounding the right function perspective.* Observation skills, *recognising correlations*, developing organisation skills. *Developing the skill rule following and rule recognition*

(2) *Is there any difference between the students of the two countries on each level, and if yes, in which activities are they manifested?*

(3) *What are the typical mistakes students make when carrying out activities at each level and what might explain these errors?*

The Questionnaire

A written test was used in order to investigate the RF and RR skills of students.

Students worked independently and had 30 minutes to complete the test. The test contained five tasks that were based on the recognition and application of the relationship between the cohesive elements (assignment rules), as well as on the expression of the recognised rule, including as a formula. I was interested in students' possession of the necessary skills for the preparation of the function concept. In some exercises, the cohesive element pairs did not clearly make a function, so more rules might be possible. In the direction to the test, however, I tried to make it clear that I wanted students to find only one adequate rule. When constructing the test I included tasks for Level 1, Level 2 and Level 3. When choosing the tasks, I predominately relied on the literature and used some of them without any alterations.

I indicate the level of the task, parenthetically, within the instructions.

1. *Find a rule between the first and second row of the table. Fill in the table according to the rule* (Level 1)! *Write down the recognised rule in words* (Level 2).

pék	*tér*	*ló*	*bál*	*görög*		
kép	*rét*	*ól*			*derék*	*savas*

Figure 1

2. *Find a rule for the numbers in the columns and fill in the blank places of the table according to that rule* (Level 1). *Write down the recognised rule in words* (Level 2).

40	*80*	*12*	*60*	*44*	*100*	*160*				
10	*20*	*3*					*1*	*13*	*31*	*95*

Figure 2

Both, first task (see Figure 1) and second task (see Figure 2) targeted the recognition, following of the rule (Level 1) that define the relationship between the cohesive elements (words and numbers), and verbal expression of this rule (Level 2). The filling in of the blank places of the tables assessed the following of the rule. The correct solution of both tasks assumes the same level of cognitive operations and activity forms (Level 1 and Level 2), but the difference can be found in the context of the tasks: while in the first task the cohesive element pairs are words, in the second they are numbers. Because function relationships do not only occur between numbers, it is crucial that students recognise this relationship, as well.

3. *Find a relationship between the x and y values of the columns and based on it, complete the table with the missing elements* (Level 1)! *Write down the relationship with words* (Level 2) *and as an expression* (Level 3)!

x	1	10	7	0	9	20	38
y	5	23	17				

y=.........

Figure 3

The aim of the third task was to make students recognise the rule that define the relationship between the elements, following it, and to express it with both words and symbols. In order to reach the Level 1, it is necessary to recognise some kind of relationship between the cohesive elements (x and y), but unlike in the first two tasks, the table is extended by an extra column (1. column). This column serves as a hint to record the recognised rule in the language of arithmetic (Level 2) and to express the rule with a formula (Level 3). The 'end product'(y value) should be found with the help of the given 'raw material'(x value) according to the recognised rule, while in the previous two tasks knowing the "end product" and using the recognised rule, the raw material should be found.

4. *Fill in the table according to the following rule:* $y = 2x + 3$. *Write down the rule in words* (Level 2).

x	-3	4	44	48	-20	0
y						

Figure 4

This task (see Figure 4) was aimed at the interpretation and following of a predefined rule. The same rule should be recognised in the third task. In order to solve the task, the student needed to possess the activity forms of the Level 2 in order to interpret (analyse) the given formula. A correct completion of the table indicated a correct interpretation of the rule given by formula.

5. *2 litres/second of water flows from a tap to a tank. How much water is in the tank at:*

a) 1 s,	c) 5 s,	e) 16 s, (Level 2)
b) 2 s,	d) 10 s,	f) x s (Level 3)

later if the tank was empty at the beginning?
Illustrate the relationship between the amounts in a table.

Figure 5

In fifth task (see Figure 5) I examined rule recognition and its mode of illustration during the solution of a task given in context. In this case, the rule is given verbally, in context. I take students' correct responses for parts (a) through (e) (Level 2) as an indication that the student had correctly interpreted the rule. A correct response to part (f) indicated that students' had reached the Level 3, since the student was able to generalise the task, i.e. write down the relationship using a formula.

Results

Analysis of students' answers

All of the 26 UA students filled in the table in first task correctly, that is, they fulfilled the criteria of the Level 1. This indicated that the students could recognise some kind of regularity between the first and the second row of the table, and they could apply the recognised rule. This means that when the cohesive element pairs are words, students can recognise the relationship between them. Writing down the recognised rule in words, however, was difficult for 8 students. So only18 students gave the right answer for the task on the Level 2. Some students skipped this part of the task or gave a rule that was not supported by the completed table. Some examples of correct responses for recognised rules:

"Words should be read backwards."; "If we change the first and the last letters we get another meaningful word."

It didn't cause any problems for HU students to recognise the rule that define the relation between element pairs in the first task, which means they fulfilled the criteria of the Level 1. 20 students out of 23 also gave right answers to the task on the Level 2. They wrote down the rule of the cohesive element pairs with text, which was also confirmed by the filled table. The rest of the students couldn't answer this question. Some correct rules:

"I wrote the given words backwards."; "The word from the upper row goes to the lower row but backwards."

Figure 6 shows how many students solved the first tasks on each level in the two countries.

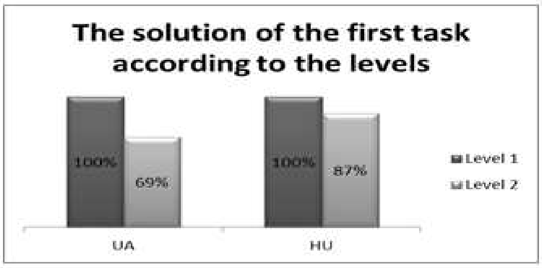

Figure 6

In the second task, where the cohesive element pairs were numbers, out of the 26 UA students only 18 students gave a correct solution. 14 students were able to give the recognised rule in words (they gave the right answer for the question corresponding to the Level 2) (Figure 7). The other students made one of the following mistakes: (1) they filled in the blank squares in the second row of the table according to a recognised rule, but in the first row they filled in the blank squares using another rule; that is, they did not apply the inverse of the recognised rule and interpreted this part of the table separately. From the point of view of the function concept, these mistakes indicated issues in recognising and differentiating between the basic set and the image set; (2) students tried to find different rules for each column and filled in the squares according to it. This could be the consequence of being unfamiliar with the table illustration of cohesive amounts. Here are some examples of correct responses for recognised rules:

"If the square in the second row is empty the number above it has to be divided by four, and where the first row square is empty the second row number has to be multiplied by four."; "Numbers of the first row are the fourfold of the lower row."

Recognising the relation between the cohesive element pairs in the second task didn't cause problems to any of the 23 HU students. 21 of them wrote down the rule as well and based on the rule filled in the table (Level 2) (Figure 7). One student didn't give answer to this part of the exercise and one student filled in the table which reflected the right thinking, but the worded rule indicates confusion with the mathematical concepts (e.g. number or digit). Some right answers:

"Each number in the upper row is four times bigger that the number in the lower row."; "$x : 4 = y$ or $y \cdot 4 = x$"; "The numbers in the lower row are only a quarter of the numbers in the upper row."

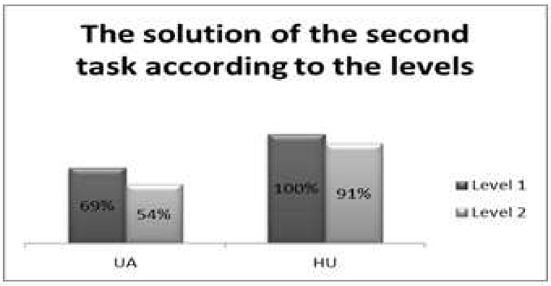

Figure 7

Only two UA students out of the 26 completed the third task (Figure 8), while other students did not give any indication of their thinking. This let me conclude that those students who possess the skill of one step rule recognition and rule creation may have difficulty with two step rule recognition.

17 out of the 23 HU students recognised the relation between the cohesive element pairs in the third task (Level 1). The recognised rule 16 students were able to give in words (Level 2) and with formula (Level 3) (Figure 8). The written rule was confirmed by the filled table too. The table filled in by one of the students represented the right logic, but the wording in the language of arithmetic caused problems. The numbers let us conclude that there are students in both groups who possess the skill of one-step RR, but the two-steps RR confuses them.

In the fourth task, the rule was given by formula. Students had to understand this rule and fill in the table accordingly. The given rule could have been familiar to the UA students, as letter expressions were from the fifth grade mathematics material, when they had to define the value of the letter expression along the certain values of the variable, but the values were not given in table form. Presumably, this new situation confused many students. Only 12 students could solve the task with only minor calculation mistakes and they worded the given rules (e.g. "We get the y value if we count the double of x and add 3") (Figure 9).

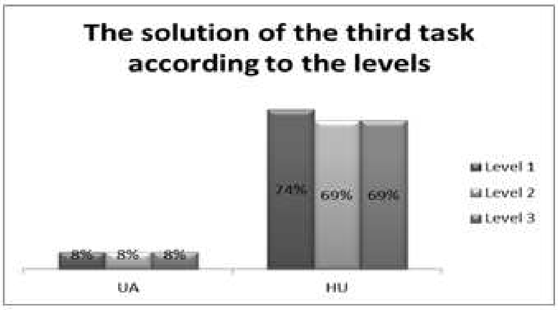

Figure 8

Contrary to these data 21 HU students solved the task: they filled in the table right which suggests they interpreted the rule well (Figure 9). This hypotheses is supported by the worded rule of the students (e.g. "I got the y that I multiplied x by two and added 3").

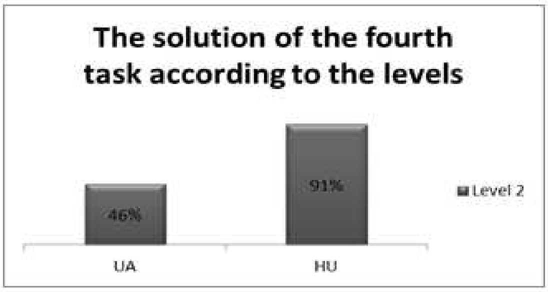

Figure 9

Comparing the data in Figure 8 and Figure 9 it can be seen that the two step rule-following was easier for the students in both countries than the recognition of the same rule.

The best results were expected from the fifth task which is connected to the direct proportionality topic. My hypothesis was that this task would not cause any problem for the students because the material was studied and tested just before the research. Contrary to the expectations, only 10 UA students out of 26 could answer all of the sub questions of the fifth task (by solving the task they fulfilled the criteria of the Level 3), including the last *(f)*. So they could generalise the rule of calculating the amount of water in the tank if the elapsed time was unknown, and they could illustrate the relationship between the results with a table. 7 students could calculate with concrete numbers (parts *a)* through *e)*), but failed to complete the *f)* question (Figure 10).

The demonstration of relations between the quantities with tables until the fifth task's a)-e) part was successfully executed by 21 HU students (Level 2). Similar to the UA students who answered this part, knowing the actual data they could recognise the rules between the cohesive element pairs but only 6 students presented the solution of the task in case of generalisation (Level 3) (Figure 10).

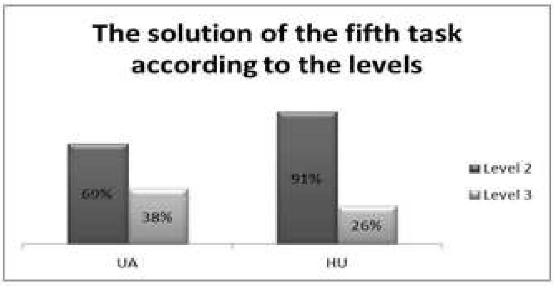

Figure 10

The number of students according to the preparation of the function concept on the examined levels

The context of the task or the way the rule articulation is asked can influence the successful solution. So the fact, that the student cannot solve some task does not necessarily means that he/she wouldn't solve a similar task. By analysing the responses of the students in the tasks according to the criteria of the set out levels it can be said that a student reached Level 1 if he/she could complete at least one of that part of first, second, or third tasks which corresponds to the Level 1. I considered that a student had reached Level 2 when he/she correctly provided the rule in at least three tasks out of the five. The student reached Level 3 if he/she gave the correct answer to all of the questions of the third task and to the question of the fifth task which corresponds to the Level 2. In some tasks students made calculation mistakes (such as in fourth and fifth tasks), but I did not take these into consideration if the student demonstrated the correct reasoning.

The students' answers were analysed based on the levels at which the various parts of the tasks were categorised. The results are summarised in Table 2. Based on the analysis, the UA students were most successful at demonstrating a Level 1 understanding in the first task since every student correctly completed it. However, the part of the same task, which was categorised as Level 2, was completed by fewer students (18). It can be concluded, however, that in the case of each task, the highest results were reached on Level 2, as compared to the other levels. The third task was the most difficult. Only 2 students gave a complete solution.

In the case of HU students, first, second and fourth tasks were solved by most of them. The fifth task happened to be problematic for them as it is shown in the table. Only 6 students gave a complete solution, that is, fulfilled the criteria of the Level 2.

Table 2. Counts of correct solutions to the tasks according to levels

	Level 1		**Level 2**		**Level 3**	
	Number of students/number of examined students		Number of students/number of examined students		Number of students/number of examined students	
Tasks	**UA**	**HU**	**UA**	**HU**	**UA**	**HU**
1.	26/26	23/23	18/26	20/26		
2.	18/26	23/23	14/26	21/23		
3.	2/26	1/23	2/26	16/23	2/26	16/23
4.			12/26	21/23		
5.			17/26	21/23	10/26	6/23

Based on these aspects, out of 26 UA students, 14 are on the Level 1, 10 are on the Level 2, and only 2 students are on the Level 3. Out of 23 HU students, only 2 are on the Level 1, 5 are on the Level 2, and 16 are on the Level 3. So, most of the UA students can recognise some kind of rule between the element pairs and can follow it, but to write these rules down with words cause them difficulties. Unlike HU students, only 2 of them (out of the 23) remained on the Level 1. In addition, interpreting the rules given by symbols and making multistep rules also seems to be problematic among UA students. The number of students who participated in the research and reached the Level 2 is almost identical. However, there is a big difference between the numbers who reached the Level 3.

This study also confirmed the hierarchy of the levels. There was no student who could meet the requirements of Level 3, but not Level 1 or Level 2.

Conclusions

The goal of this paper was to investigate the RF and RR skills of sixth grade students studying in the Ukrainian and Hungarian education system, from the point of view of the development of the function concept. The results showed that UA students certainly reached Level 1. This indicate that they can recognise the rule that define the relationship between simple elements. In many cases, however, I could see that some students fulfilled the requirements of Level 1, but could not get to Level 2 due to possible deficiencies in the area of communication in the language of mathematics. Many could also not successfully use the table as a tool for displaying cohesive elements. I suspect that students' deficiencies are not only age-specific, but are also related to the absence of tables from the curriculum requirements and from the textbook tasks.

UA students' lack of success in correctly completing fourth and fifth tasks. This entailed the use of already known concepts (letter expressions and linear relationship) in new situations (problem solving), indicated that this was also a problematic area for the students. The part of the fifth task that belong to the Level 3 was difficult for the HU students, namely, when the rule is in a textual form hidden in the context then the generalisation of it with function is problematic for these students.

Looking at the results we also have to stress that those UA and HU students who reached the Level 2 and Level 3 represent a similar thinking in the way they solved the tasks. Do despite the fact that the Ukrainian curriculum focus less on the development of RF RR skills, students in this period of their cognitive development [1] possess the Level 1 without any support because they use their mathematical knowledge. However, in order to reach Level 2 and Level 3 targeted development would be necessary.

As an implication for future research we can ask the following questions: a) do the students from Ukraine develop without targeted support in the aforementioned skills year by year? b) what are the differences in the skills between the students of the two countries by the end of the 7^{th} form, after the introduction of the concept of function, does it have an impact on these skills; c) using different function concept preparation processes do they reach the same level by the end of the eighth form?

Notes

[1] H.P. Bevz, V.H. Bevz, Matematika 6. osztály, Szvit, Lviv, 2006 (translation from Ukrainian to Hungarian).
A.H. Merzljak, V.B. Polonszkij, M. Sz. Jakir, Matematika 5. osztály, Szvit, Lviv, 2005 (translation from Ukrainian to Hungarian).
Hajdu Sándor, Matematika (alapszint, 5.-6.osztály), Műszaki Tankönyvkiadó, 2001, 2002 (in Hungarian).

[2] Curricula, retrieved from:

- http://mon.gov.ua/ua/activity/education/56/692/educational_programs/1349869429/ (in Ukrainian)
- http://kerettanterv.ofi.hu/02_melleklet_5-8/index_alt_isk_felso.html (in Hungarian)

References

[1] A. Ambrus, *Bevezetés a matematikadidaktikába*, ELTE Eötvös Kiadó, Budapest, 1995 (in Hungarian).

[2] A. Hoffer, Van-Hiele-based research, Acquisition of mathematical concepts and processes, (R. Lesh and M. Landau, eds.), Academic Press, New York, 1983, 205–227.

[3] A. Sierpinska, On understanding the notion of function, The concept of function: Aspects of epistemology and pedagogy, (G. Harel and E. Dubinsky, eds.), DC: Mathematical Association of America, Washington, 1992, 25–58.

[4] B. Csapó, Képesség, Pedagógiai lexikon I-III, (Z. Báthory and I. Falus, eds.), Keraban Kiadó, Budapest, 1997, II/196 (in Hungarian).

[5] E. Herendiné Kónya, A tanítójelöltek geometriai gondolkodásának jellegzetességei, *Iskolakultúra* **11**, no. 1-2 (2003), 51–61 (in Hungarian).

[6] H. Freudenthal, *Mathematics as an educational task*, Reidel, Dordrecht, 1973.

[7] F. Rivera, *Teaching and learning patterns in school mathematics: psyhological and pedagogical considerations*, Springer, Dordrecht, 2013, 82–83.

[8] I. Czeglédy, Relációk, függvények, sorozatok, Matematika tantárgypedagógia II, (I. Czeglédy, Gy. Orosz, T. Szalontai and A. Szilák, eds.), Calibra, Budapest, 1994, 93-111 (in Hungarian).

[9] J. P. Ponte, The history of the concept of function and some educational implications, *The Mathematics Educator* **3**, no. 2 (1992), 3–7.

[10] M. Blanton and J. Kaput, Functional thinking as a route into algebra in the elementary grades, Early algebraization, (J. Cai and E. Knuth, eds.), Springer, Berlin, Germany, 2011, 5-23.

[11] M. Isoda, The development of language about function: An application of van Hiele's levels, no. 3, Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, València, Spain, 1996, 105–112.

[12] P. M. van Hiele, Similarities and differences between the theory of learning and teaching of Richard Skemp and the van Hiele levels of thinking, Intelligence, learning and understanding in mathematics. A tribute to Richard Skemp, (D. Tall and M. Thomas, eds.), Post Pressed, Flaxton, Australia, 2002.

[13] R. Kwari, An investigation into the development of the function concept through a problem-centered approach by form 1 pupils in Zimbabwe (Master's thesis), 2007.

[14] R. Skemp, *The Psychology of Learning Mathematics*, Lawrence Erlbaum Associates, Hillsdale, 1987.

[15] S. Vinner, Concept definition, concept image and the notion of function, *International Journal of Mathematical Education in Science and Technology* **14**, no. 3 (1983), 293–305.

[16] T. Dreyfus and S. Vinner, Images and definitios for the concept of function, *Journal for Research in Mathematics Education* **20** (1989), 356–366.

GYÖNGYI SZANYI
UNIVERSITY OF DEBRECEN
DEBRECEN
HUNGARY

E-mail: szanyi.gyongyi@science.unideb.hu

(Received June, 2015)