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Teaching Mathematics and Computer Science

Transition from arithmetic to algebra in primary school education

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Abstract. The main aim of this paper is to report a study that explores the thinking strategies and the most frequent errors of Hungarian grade 5-8 students in solving some problems involving arithmetical first-degree equations. The present study also aims at identifying the main arithmetical strategies attempted to solve a problem that can be solved algebraically. The analysis focuses on the shifts from arithmetic computations to algebraic thinking and procedures. Our second aim was to identify the main difficulties which students face when they have to deal with mathematical word problems. The errors made by students were categorized by stages in the problem solving process. The students' written works were analyzed seeking for patterns and regularities concerning both of the methods used by the students and the errors which occured in the problem solving process. In this paper, three prominent error types and their causes are discussed.

Key words and phrases: arithmetic computations, strategy implementation, early algebra teaching, algebraic methods.

ZDM Subject Classification: C10, C30, D70, E40, F10, F30, H10, H30.

1. Introduction

The shift from arithmetic to algebra is considered to be a difficult but an essential step for mathematical progress (see [18]). According to Warren (see[22]), this shift involves a move from knowledge required to solve "arithmetic equations" operating with numbers to knowledge required to solve "algebraic equations" operating with unknowns.

In the research done by Stacey and MacGregor on strategies used by students in solving mathematical word problems involving equations, students were found to apply the following different routes while they solve algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known number at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing an equation and solving it with the balance principle, and (e) algebraic route: solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning (see [19]). Stacey underlines that historically, and in the education of nearly all children, algebra grows out of arithmetic (see [20]). An important thread in international research and thinking on mathematics education curriculum is to consider ways in which the transition from arithmetic to algebra can be made more smooth. In particular, the "early algebra" movement has examined how to teach arithmetic in a way that prepares students for algebra, and which emphasises the thinking processes which underlie algebra. The intention is not to introduce algebraic symbols at an earlier age, but to change the emphasis of arithmetic teaching. It is no longer appropiate to have an arithmetic curriculum which focuses exclusively on computation, so that there is opportunity to include experiences of generalisation, mathematical structure and properties of operations that underpin algebra. Detori et al (see [5]) suggested that the transition from arithmetic to algebra requires a change in the nature of problem resolution and a change in the nature of the objects of study (i.e. from numbers to symbols, variables, expressions, equations, etc). Beginning students need to make both of these transitions. Algebra requires a stronger understanding of the properties of operations than does arithmetic.Many students have difficulty learning about algebra because they are unsure of the arithmetic properties which algebra generalises (see [2]).

In this work we also focused on analyzing errors made by children attempting to solve verbal arithmetic problems, which can be solved using both algebraic and arithmetic methods. One of the primary reasons children have trouble with problem solving is that there is no single procedure that works all the time each problem is slightly different. Also, problem solving requires practical knowledge about the specific situation. If you misunderstand either the problem or the underlying situation you may make mistakes or incorrect assumptions. George

Pólya in his book "How to Solve It" (see [15]) identified four basic principles when solving a problem. In brief, these four principles are:

(1) Polya's first principle: *Understand the problem*
Studens are often stymied in their efforts to solve problems simply because they don't understand it fully, or even in part.

(2) Polya's second principle: *Devise a plan*
Polya mentions that there are many reasonable ways to solve problems. The skill at choosing an appropriate strategy is best learned by solving many problems. A partial list of strategies is the following: guess and check; consider special cases; use direct reasoning; work backwards; draw a picture; use a model; use a formula; solve a simpler problem; look for a pattern; eliminate possibilities; use symmetry.

(3) Polya's third principle: *Carry out the plan*
This step is usually easier than devising the plan. In general, all the students need is care and patience, given that they have the necessary skills. They have to persist with the plan that they have chosen. If it continues not to work they must discard it and choose another.

(4) Polya's fourth principle: Polya mentions that it is necessary to look back at what you have done,what worked and what didn't. Our main goal was to survey how the students examine the solution obtained, can they check the result or can they check the argument.

According to Newman (see [13]) a person confronted with a one-step written problem has to read the problem, then comprehend what he has read, then carry out the transformation from the words to the selection of an appropiate mathematical "model", then apply the necessary process skills and then encode the answer. In recent years, many research projects on mathematics education have focused on learning difficulties of students related to algebra. Research have shown that students errors in algebra can be ascribed to fundamental differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from the arithmetical conventions and need to learn to deal with algebraic symbolism. Egodawatte (see [6]) in a study categorized the errors made by students according to the stages of the problem solving process in the Newman model and found that the greatest number of errors occured during the processing stage (57.8 %) followed by the comprehension error (21.9 %), encoding error (15.6 %) and verification error (4.7 %), so we can see that nearly 80 % of the errors had occured during the comprehension and processing stages, and the balance 20 % had accounted during encoding and verification. The areas where major error types found were: the transformation of word problems into algebraic language, parenthesis omitted and wrong operations in solving equations. Clements studied error causes such as reading comprehension difficulty, the failure during the transformation from the written problem to an acceptable ordered set of mathematical procedures, the form of the question, the weakness in process skills, encoding error, careless error and lack of motivation (see [4]), concluding that many errors made by children on written mathematical tasks are due to reading comprehension and transformation difficulties and that often means a child uses inappropiate process skills in an attempt to find a solution. The frequency and type of errors a child makes when attempting a verbal problem in mathematics depends on the interaction between "question variables" (such as the vocabulary and syntax used in the question, the complexity of the ideas in the question, and the level of mathematics needed to solve it), and "person variables" (such as intelligence, reading ability, mathematical knowledge and ability, persistence), so it is inevitable that children will make errors on written mathematical tasks for a variety of reasons. The German mathematics educator, Hendrik Radatz concluded that error analysis research in different countries has been characterized by very different starting points and interests. Radatz himself proposed an information-processing classification of errors, and delineated five main categories, consisting of errors due to students' language difficulties, difficulties in obtaining spatial information, deficient mastery of prerequsite facts and concepts, incorrect associations or rigidity of thinking, application of irrelevant rules or strategies (see [17]). According to Radatz it is often difficult to make a sharp separation among the possible causes of a given error because there is a close interaction among clauses.

2. Context and purpose of the study

In the Hungarian mathematics curriculum, it is only until grade 7 that solving equation is formally taught as an independent set of mathematical algebraic procedures. Prior to this formal teaching very little attention is given to the prealgebraic preparation. Even though children are faced with word problems, but these types of exercises are usually treated in a purely arithmetic approach, with no attention to set foundations for algebraic thinking. The purpose of this study is to explore the thinking strategies of Hungarian grade 5-8 students in solving word problems involving both arithmetic and algebraic methods. It is expected that even before students are taught the formal approaches of solving algebraic equations, they might develop some aspects of those procedures which are required to solve word problems, by building on their elementary school experience and prior knowledge. The grade 6 Hungarian Mathematics curriculum includes problem solving methods such as making a graph, working backwards or using a balance. In the international literature a few previous studies explored students' pre-instructional informal algebraic structures. Filloy and Rojano (see [7]) focused on problems of the form $x + a = b$, $a \cdot x = b$ and $a \cdot x + b = c$. They found that these kind of problems can easily be solved using arithmetic, mainly by inverse operations. Johanning chose problems with $a \cdot x + b \cdot x + c \cdot x = d$ and $x + (x + a) + (x + b) = c$ structures (see [10]). She found that students used many informal strategies for solving the problems, the method of systematic guess and check being the most common approach. Another important fact is that while assigning a value to a variable and verifying its accuracy, students are developing functional reasoning as it entails recognising a relation between variables even if such relation is not always expressed in the formal language of algebra. I. Osta and S. Labban attempted to extend the understanding of grade 7 students' thinking strategies to solve a problem whose algebraic structure is a first-degree equation with the unknown occuring on both sides of the equality sign, namely equations with $a + b + c + x = m \cdot x$ structure, or, when reduced $A + x = m \cdot x$ (see [12]). Most participants in their study solved the given problem using intuitive, non-algebraic methods. Numerical cheking methods (trial-and-error and estimation) were the most used. Very few students used algebraic symbolism or presented the problem by a first-degree equation. Another conclusion was that students might transform their prior knowledge in arithmetic into building algebraic equations but they return and proceed arithmetically to solve them.

The present study attempts to survey the grade 5-8 students thinking when they have to deal with word problems whose algebraic structures are described by the following first-degree systems of equations:

System 1:

$$\begin{aligned} a \cdot x + b \cdot y &= c \\ x + y &= d \end{aligned}$$

System 2:

$$\begin{aligned} x + y &= c \\ y &= m \cdot x + n \end{aligned}$$

System 3:

$$x + y = a$$
$$x + z = b$$
$$y + z = c$$

We also tried to categorize the errors that have occured. We took into consideration the Polya's four basic principles of problem solving and the stages of the problem solving process in the Newman model. Several researchers concluded that the greatest number of errors occur in the comprehension stage and processing stage. So we focused on the errors that occur in three stages of the problem solving process, as follows:

First stage: *Errors related to poor understanding of the problem.* Many students had difficulty in understanding the information existing in the text of the problem, the relationships between unknown quantities. This fact was revealed by examining students' work, especially the errors concerning the arithmetical or algebrical interpretation of the relationships between unknowns. There also were some typical calculational mistakes which reflect poor understanding of the problem.

Second stage: *Errors in choosing the appropriate strategy.* Some works reveal that the students understood the problem, they wrote properly the relationships between unknowns, but they have no idea how to choose the right strategy to solve the problem. Many students failed when they tried to build up step by step a problem solving strategy. This mainly happened in the case of arithmetical methods where the order of the steps is very important.

Third stage: *Errors in strategy implementation.* A number of students understood the problem, chose the right strategy, but they failed in carrying out the plan. In most of the cases the computational errors led them to wrong answers.

Another goal was to survey the number of students who use the method of *false supposition*. A sample of this method we can find in section 4. In our opinion this method has its own right place in the educational processes because it is an important step from pure groping (we mean trial-and-error and guess-and-check methods) to deductive arithmetic and algebraic calculations, however, it is not appreciated by most of the teachers (similar to the method of groping). We can find only a few examples of problem solving methods based on false supposition in the Hungarian student textbooks (we can find an example in [3], page 87).

3. The method of false supposition

Let us solve the following brain teaser which may amuse intelligent youngsters.

PROBLEM 3.1. *A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?*

At first, we suppose all of the animals are hens, so there are $2 \cdot 50 = 100$ feet, this means less $140 - 100 = 40$ feet. If we replace a hen with a rabbit the number of feet increases by 2. So we have to change $40 : 2 = 20$ hens with rabbits, so there are 20 rabbits and 30 hens. This procedure is called the method of *false supposition* (see [21]) and it is worth mentioning in the primary school educational processes, because having solved a problem in this way the students aquire a precious possesion: a pattern, a model that they can immitate in solving similar problems. The importance of this method increases because research has revealed that students prefer to use arithmetic methods in solving algebraic word problems and show difficulties in setting up and using equations to solve such problems. There is also evidence that the most frequently used arithmetical processes are guess-and-check or trial-and-error among the students of 13-14.

More generally, Problem 3.1 can be treatead as a system of equations, as follows:

$$a \cdot x + b \cdot y = c$$
$$x + y = d$$

To solve such an equation it is not a great challenge for a teacher, but a student of fourteen knows only the first-degree equation with an unknown. Of course, the teacher can treat the problem with an unknown in the educational processes, and there are also arithmetic methods to solve this kind of problems. We omit the detailed presentation of these algebraic and arithmetic methods and we will focus on the method of false supposition. We take an arbitrary number $x = x_1$, so $y = y_1 = d - x_1$ and

$$a \cdot x + b \cdot y = a \cdot x_1 + b \cdot (d - x_1) = c' \ .$$

We consider $k = \dfrac{c - c'}{a - b}$ and we will prove that the solution of the system of equations is $x = x_1 + k$ and $y = d - (x_1 + k)$. Indeed,

$$a \cdot x + b \cdot y = a \cdot (x_1 + k) + b \cdot [d - (x_1 + k)] = a \cdot x_1 + b \cdot (d - x_1) + k \cdot (a - b) =$$

$$= c' + k \cdot (a - b) = c' + (c - c') = c \, .$$

In the following, we will solve the system of equations

$$a \cdot x + b \cdot y = c$$
$$y = m \cdot x$$

We take an arbitrary number $x = x_1$, so $y = y_1 = m \cdot x_1$ and $c' = a \cdot x_1 + b \cdot m \cdot x_1$. We consider $k = \frac{c}{c'}$ and we will prove that the solution of the system of equations is $x = k \cdot x_1$ and $y = m \cdot k \cdot x_1$. Indeed,

$$a \cdot x + b \cdot y = a \cdot k \cdot x_1 + b \cdot m \cdot k \cdot x_1 = k \cdot (a \cdot x_1 + b \cdot m \cdot k \cdot x_1) = k \cdot c' = c \, .$$

But the problem arises how a teacher can explain this kind of method to 11-14 years old students. A method is shown in the first part of this section. Let us solve Problem 3.1 in an other way. We consider, at first, there are 10 hens so there are $2 \cdot 10 + 4 \cdot 40 = 180$ feet and this means $180 - 140 = 40$ more feet. If we increase the number of hens by one (of course the number of rabbits decreases by one) the number of feet decreases by two. So we have to increase the number of hens by $40 : 2 = 20$. We will show all our attempts in a table, as follows:

	hens	rabbits	feet	difference
First assumption	10	40	180	40
Increase/decrease	+1	-1	-2	-2
Increase/decrease	+20	-20	-40	-40
Right answer	30	20	140	0

Problem 3.2. *Ann and Barbara together weighed 93 kg. Ann and Cathey together weighed 95 kg. Barbara and Cathey together weighed 102 kg. How much does each of the girls weigh?*

From a teacher's point of view this problem involves a system of three equations, as follows:

$$x + y = 93$$
$$x + z = 95$$
$$y + z = 102$$

There are several arithmetic procedures to solve this problem but we will show the method based on false suposition. Let us consider, for example, Ann's weight equal to 30. So from the first and second equations the solution $x = 30$; $y = 63$; $z = 65$ follows. But $y+z = 128$ contradicts the third equation, the difference being $128-102 = 26$. We can see if x increases by 1 then both of y and z decreases by 1 (this follows from the first and second equations), so $y + z$ decreases by 2. Therefore to decrease $y + z$ by 26, we have to increase x by 13. So Ann's weight is $30+13 = 43$ and the solution $x = 43$; $y = 50$; $z = 52$ follows. We summarise the foregoing calculations as follows:

	x	y	z	y+z	difference
First assumption	30	63	65	128	26
Increase/decrease	+1	-1	-1	-2	-2
Increase/decrease	+13	-13	-13	-26	-26
Right answer	43	50	52	102	0

We can conclude that the method of false supposition is a deductive method which is preceded by an initial guess. In our opinion this method has its own place in the educational processes, because according to previous researches the students mainly prefer the method of trial-and-error instead of the conventional deductive methods (the result of our survey proves the same, as we can see in the following). The method of false supposition can be considered as a transition from pure groping to the deductive methods.

4. Place of the survey, students involved in the survey

This study was carried out with a group of grade 5-8 students from 15 schools in Vác region. The schools selected were from urban and rural areas in order to get a group of mixed ability students. The students have been specially selected for this study by their teachers, mainly high and average achiever students who have a serious attitude toward mathematics (we excluded the low-achievers and the students who have a hostile attitude toward mathematics). The school teachers explained to the students that the aim of the study was not to evaluate or grade students' work, but to explore their thinking strategies while solving the given problem. All of the students solved the test paper in their school. Every student received a test-paper with four exercise on it. The exercises were choosen by the author, and the students have 60 minutes to solve them. The students were asked to write in detail their attempts, to give reasons for their actions even though they could not solve the problem entirely. During the work on the solution, students were observed by their mathematics teachers. Our test paper was returned by 380 students. The students' written works were analyzed seeking for patterns and regularities. We also analyzed the prominent error types and their possible causes. The study was conducted in the first term of the school year when the grade 7 students were already introduced to algebra topics and equations, including the use of letters to designate quantities and the use of equations to represent relations. Our objective was also to capture the emergence of students' algebraic thinking, representations and procedures.

5. Discussion of the results

In this paper we will discuss the problem solving strategies and the main errors by analyzing some of the students' works. We can not analyse all of the problems because it will exceed the size of this paper so we will focus on the most eloquent exercises and problem solving strategies.

At first, we will analyse problems which can be described by the following system of equations

$$\begin{aligned} a \cdot x + b \cdot y &= c \\ x + y &= d \end{aligned}$$

Problem 5.1. *(grade 5 and 6) A farmer has hens and rabbits. These animals have 14 heads and 36 feet. How many hens and how many rabbits does the farmer have?*

Problem 5.2. *(grade 7 and 8) A hotel has 23 rooms with 52 beds. The rooms have two or three beds. How many double bed rooms are there in the hotel?*

Tables 1 and 2 show the repartition of right and wrong answers.

We also analyzed the correct answers taking into consideration the methods used to solve the problem, the repartition is the following:

As shown in Table 2, only a few students used the methods of algebra. We can underline that only a small number of grade 7 and 8 students used equations, although this method is available for them. One grade 7 student and five grade 8 students solved the equation $2 \cdot x + 3 \cdot (23 - x) = 52$ and one grade 8 student

Table 1

	grade 5	grade 6	grade 7	grade 8
Right answer	49	40	32	36
Wrong answer	43	38	46	38
No response	14	13	20	11
Total	106	91	98	85

Table 2

	grade 5	grade 6	grade 7	grade 8
Algebra	-	2	1	6
Graph	4	10	1	5
Groping	45	28	26	19
False supposition	-	-	4	6
Total	49	40	32	36

solved the system of equations $x + y = 23$ and $2 \cdot x + 3 \cdot y = 52$. Several students solved the problem using graph, the grade 5 and 6 students drew circles (these mean the heads) with lines attached (these mean the legs) the grade 7 and 8 ones drew rectangles (these mean the rooms) with lines or smaller rectangles inside (these mean the beds). Two grade 6 students drew 36 dots (legs) and encircled four (a rabbit) or two (a hen) together until they came to the final result. By the groping method we mean the random trial-and-error, sequential trial-and-error and guess-and-check methods, for example a grade 8 student wrote "10 double rooms and 13 triple rooms means $2 \cdot 10 + 3 \cdot 13 = 59$ beds (too many), so we have to take a smaller number of triple rooms and this leads to fewer beds, (another try) $2 \cdot 15 + 3 \cdot 8 = 54$ (and then the final trial) $2 \cdot 17 + 3 \cdot 6 = 52$" and he gave the right answer. One grade 7 student wrote: "$2{+}3{+}2{+}3{+}\cdots{+}2{+}3$ (6 times) $= 30$ beds; 22 beds and 11 rooms remain, all of these rooms are double rooms, so in the aggregate there are 17 double rooms". Some students drew tables which contained the number of double rooms and triple rooms, respectively, in two columns (in each row there were 23 rooms in the aggregate) and then they filled the rows in this way until they came to the right answer. This kind of "groping" is usually described as a solution by *sequential trial-and-error*. In fact, it consists of a series of trials, each of which attempts to correct the errors committed by the preceding and, on the whole, the errors diminish as we proceed and the succesive trials come closer and closer to the desired final result. More straightforward is the method of *false supposition*, one of this kind of solutions is the following (a grade 8 student's work): "If all of the rooms were double rooms then there would be $2 \cdot 23 = 46$ beds, so take away 46 from the number of all beds, which is 52, and the number of the triple rooms $52 - 46 = 6$ follows". This method is less empirical and more deductive, we mean with fewer trials, less guesswork, and more deductive reasoning. We have to mention that only a few students used this method, as Table 2 shows. Many students could reach a correct result by several guesses and then they verified it to see whether it satisfied the given relationship (guess-and-check). Some of them were not convinced that their work is an acceptable solution so they tried to prove somehow their results or their conjectures. One of them gave a forced argumentation: "$36 : 4 + 5 - 4 = 9 + 1 = 10$ hens" (we can see that he found the right answer and he invented a chain of calculus to obtain this result). 3 grade 6 students argued in the following way: $36 : 2 = 18$; $18 : 2 = 9$ (there are 9 hens) $18 : 4 = 4$ (there are 4 rabbits) and 2 is the residuum of the division, this means two legs (i.e. a hen), so there are 10 hens and 4 rabbits. This bad argumentation, accompanied by a correct answer, reflects that these students found the right answer by groping but they could not give any correct deductive reasoning so they invented a forced explanation. The situation is similar to the case of two grade 8 students who wrote: "$6 \cdot 3 = 18$; $52 - 18 = 34$; $34 : 2 = 17$ is the number of double rooms".These students initially assumed that the number of triple rooms is 6 (they found this result by groping) then they calculated the number of double rooms starting from the number of triple rooms. Table 2 shows that relatively small number of students applied conventional deductive methods, such as making graph, the usage of false supposition or the methods of algebra (in grade 7 and 8). Most of the right answers were obtained by groping (this fact shows similarity with previous research works). They got the solution in this way because the given numbers are relatively small and simple. But if the problem, proposed with the same wording, had larger or more complicated numbers, they would need more trials or more luck to solve the problem in this manner.One of the grade 6 students solved the equation $2 \cdot x + 4 \cdot (14 - x) = 36$ (x denotes the number of hens) and the other solved successfully the system of equations $x + y = 14$ and $2 \cdot x + 4 \cdot y = 36$. This is strange because this type of knowledge is far ahead from the grade 6 curriculum.

We also analyzed the errors that occur in the different stages - mentioned in Section 2 - of the problem solving strategies. The result is the following:

Table 3

	grade 5	grade 6	grade 7	grade 8
First stage	24	18	31	21
Second stage	15	16	8	14
Third stage	4	4	7	3
Total	43	38	46	38

First stage errors: Table 3 shows that most of the errors occur in the first stage. Answers reveal that many students did not even understand the text of the exercise. Also, there are many works which reveal that the students have understood the text of the problem, but they did not take it at all into account when they tried to solve the problem. Some errors that occured in the first stage are the following.

Grade 5 and 6: "$14 \cdot 36 = 504$ hens and rabbits"; "14 heads and 36 feet means 50 animals"; "$36+14 = 50$ hens and $36-14 = 22$ rabbits"; "$14+36 = 50 : 2 = 25$ hens and 25 rabbits"; "$36 : 14 = 24.3$ (computational mistake) $24.3 : 2 = 12.15$ so there are 12 hens and 12 rabbits"; "$36 : 14 = 2$ hens and $50 : 14 = 3$ rabbits" "$36 : 14 = 2$ the residuum is 8, so there are 2 rabbits and 8 hens"; "$36 : 2 = 18$ rabbits and $14 : 2 = 7$ hens".

Grade 7 and 8: "$52 + 23 = 75 : 2 = 32.5$ so there are 32 double rooms and 33 triple rooms"; "$3 \cdot x + 2 \cdot x + 23 = 52 \Rightarrow x = 5.8$ (he didn't gave any answer)"; "$3 \cdot 52 + 2 \cdot (52 - x) = 23$ (then she solved this equation properly)"; "$52 : 3 = 17.33$, $52 : 2 = 26$, $26 - 17 = 9$ double rooms"; "$52 : 2 = 23 : 2 = 13$ double rooms"; "$23 \cdot 2 = 46$; $23 \cdot 3 = 69$; $46 + 69 = 115$; $115 : 2 = 57.5$ so there are 58 double rooms"; "$2 \cdot 23 = 46$ double rooms and $52 - 46 = 6$ triple rooms"; "$23 + 52 = 75$; $75 : 3 = 25$; $52 - 25 = 27$ double rooms"; "$x + (x+1) + 23 = 52$ (then she solved the equation properly) $\Rightarrow x = 14$ double rooms"; "$23 + 52 = 75$; $75 : 3 = 25$; $\Rightarrow 52 - 25 = 27$ double rooms". One grade 8 student used the percentage calculus which does not make any sense as he wrote: "$23 \cdot \frac{52}{100} = 11,96$ % of the rooms have two beds."

Second stage errors: Fewer students committed errors in the second stage. They have understood the text of the problem, but they were not able to use the data properly in order to draw up a problem solving plan.

Grade 5 and 6: Many grade 5 and grade 6 students wrote the operations $36 : 4 = 9$ or $36 : 2 = 18$ (in our opinion they thought "if all of the animals were rabbits then there must be 9 animals" or "if all of the animals were hens then there must be 18 animals") and then they failed. We have to mention that it is more simple to begin this train of thought appealing to the total number of heads and to formulate the statement "if all of the animals were hens then there must be 28 legs" or "if all of the animals were rabbits then there must be 56 legs".

Grade 7 and 8: Some of them tried to reduce the mathematical problem to a problem of algebra, but they failed when they tried to translate the proposed problem into an equation. They wrote properly "there are x double rooms and $23-x$ triple rooms" then they wrote incorrect equations, such as: "$x = 23-x \Rightarrow x = 11,5$ " ; "$x + 23 - x = 52$" ; "$2 \cdot x + 23 - x = 52$ "; "$3 \cdot 52 + 2 \cdot (52 - x) = 23$ (then some of them solved the equation properly)". One grade 7 student wrote "there are eight possibilities" and then drew the following table:

double rooms	23	20	17	14	11	8	5	2
triple rooms	2	4	6	8	10	12	14	16

This student took into account only the fact that the total number of beds is 52, but he omitted that there must be 23 rooms in the aggregate. Three grade 7 students and two grade 8 students wrote "if all of the rooms had two beds then there should be 26 rooms" and then they tried to decrease the number of rooms, but they made computational errors. Another grade 7 student wrote "if all of the rooms had three beds then there must be 14 rooms, so there are $23 - 14 = 9$ double rooms".

Third stage errors: Only a few errors have occured in the third stage.

Grade 5 and 6: The students tried to solve the problem by graph, they drew 36 lines or circles (i.e. legs) and then they tried to encircle and then they failed.

Grade 7 and 8: The students drew 52 lines (i.e. beds) and then they tried to draw rectangles (i.e. rooms). In this case they knew how to solve a problem by graph but they failed because they chose the more complicated way to apply this method. One grade 8 student wrote the equation $2 \cdot x + 3 \cdot (23 - x) = 52$, but she continued $2 \cdot x + 96 - x = 52$ and then she failed.

In the following we will analyse problems which can be described by the system of equations

$$a \cdot x + b \cdot y = c$$
$$y = m \cdot x + n$$

Problem 5.3. *(Grade 5) Andrew and Paul have together 56 books. How many books has each of them if Andrew has 18 books more than Paul?*

PROBLEM 5.4. *(Grade 6) The sum of two numbers is 138, their difference is 24. What are these numbers?*

PROBLEM 5.5. *(Grade 7) A rectangle has a perimeter of 96 cm and its length is 3 cm greater than the double of its width. Find the dimensions of the rectangle!*

PROBLEM 5.6. *(Grade 8) A rectangle has a perimeter of 102 cm and the half of its length is 3 cm greater than its width. Find the dimensions of the rectangle!*

Table 4

	grade 5	grade 6	grade 7	grade 8
Right answer	42	44	21	20
Wrong answer	57	34	55	38
No response	7	13	22	27
Total	106	91	98	85

Table 5

	grade 5	grade 6	grade 7	grade 8
Algebra	-	-	5	14
Groping	10	15	6	5
Arithmetic	32	29	10	1
Total	42	44	21	20

As shown in Table 5, most of the grade 5 students obtained the right answer by working backwards (arithmetic method). 27 students solved the problem as follows "$56 - 18 = 38$; Paul has $38 : 2 = 19$ books and Andrew has $19 + 18 = 37$ books" and 5 students worked almost in the same way: "$56 : 2 = 28$; Andrew has $28 + 9 = 37$ books and Paul has $28 - 9 = 19$ books". 10 students solved the problem by groping, some of these students drew a table as they applied the sequential trial-and-error method. We can see that only a quarter of the good-solvers apealed to the method of trial-and-error, the others used the method of working backwards.

Most of the grade 6 students also gave the right answer through working backwards. 12 students' works: "the first number is $138 - 24 = 114 : 2 = 57$

and the second number is $57 + 24 = 81$ ". 17 students solved the problem in the following way: "$138 : 2 = 69$ and then they halved the difference $24 : 2 = 12$ so the numbers are $69 - 12 = 57$ and $69 + 12 = 81$". 15 students proceeded by groping, some of them making tables, as follows (one student's work):

1st number	69	70	71	72	73	74	75	76	77	78	79	80	81
2nd number	69	68	67	66	65	64	63	62	61	60	59	58	57

This student, at first, halved the sum 138, and then he increased/decreased the numbers one by one until he got the desired result. Others started with numbers whose difference is 24, and increased or decreased the numbers simultaneously. We have to mention one student's work: "$61 - 37 = 24$; $61 + 37 = 98$; $71 - 47 = 24$; $71 + 47 = 118$; $81 - 57 = 24$; $81 + 57 = 138$, here is the solution". We can see this student was aware of that the last digits of these numbers are 1 and 7 respectively, but she did not realise that increasing both of the numbers by 10, their sum increases by 20, and we can reach the answer in a single step starting with $61 - 37 = 24$; $61 + 37 = 98$ and increasing both of the numbers by 20, so their sum becomes $98 + 40 = 138$.

The grade 7 students rarely used algebraic methods, only 4 students solved the equation $2 \cdot (b+2 \cdot b+3) = 96$ and one student solved the equation $b+2 \cdot b+3 = 48$. 6 students gave the right answer by groping, one of them proceeded in an interesting way: "$96 : 4 = 24$, $a = 25$, $b = 23$" , it is not right, (and then) $a = 26$, $b = 22$ and so on until he came to the right answer $a = 33$, $b = 15$ (a is the length and b is the width of the rectangle, our unified notations). 10 students used the method of thinking backward, six of them wrote "$b = (96 - 6) : 6 = 15$ and $a = 2 \cdot 15 + 3 = 33$", the others made the chain of operations: $96 : 2 = 48 - 3 = 45 : 3 = 15 \cdot 2 = 30 + 3 = 33$ and they found $a = 33$ and $b = 15$.

Most of the grade 8 students gave the right answer by the methods of algebra, they scarcely used other methods. This was not the same in the case of Problem 5.2, so we think that the grade 8 students use the methods of algebra especially in the case of well-known geometrical formulas or other type of exercises which they have learned previously, but in the case of real-life problems or unusual exercises they persist on the use of other methods, such as "groping". This is a proof that the letter-based nature of the problem elements (geometric objects labeled with alphabetic letters) will help in focusing students' attention on solving equations through symbol-manipulation. They wrote and solved equations: $2 \cdot b + 2 \cdot (2 \cdot b + 6) = 102$ (9 students); $\frac{a}{2} - 3 + a = 51$ (2 students) and $b + 2 \cdot b + 6 = 51$ (one student). One student wrote that $b+3$ is the half of the length and solved the equation $2\cdot(b+3)+b=51$. Another student denoted the half of the length by x and solved the equation $4\cdot x+2\cdot x+6=51$. One student solved the problem thinking backwards: $a+b=51$; $51-6=45$; $45:3=15 \Rightarrow b=15$ and $a=2\cdot 15+3=33$. To solve the problem in this way is quite simple, our question is why other students did not use this method. One possible answer is that they have learned this method in sixth grade and they forgot it.

The repartition of wrong answers by stages is shown in Table 6.

Table 6

	grade 5	grade 6	grade 7	grade 8
First stage	28	15	40	27
Second stage	25	13	14	6
Third stage	4	6	1	5
Total	57	34	55	38

We can categorise the grade 5 students' errors as follows.

First stage errors: 28 students' works reveal that they did not take into account the relations $A+P=56$ and $A=P+18$. 13 of them wrote "$56-18=38 \Rightarrow A=38$ and $P=18$". Some students gave the following erroneous answers: "$56+18=74 \Rightarrow$ Paul has 74 books and $74-18=56 \Rightarrow$ Andrew has 56 books."; "$56:18=3 \Rightarrow A=3$; $P=3$" ; "$56:2=28 \Rightarrow A=28$; $P=28$" ; "$56+18=74 \Rightarrow 74:2=37 \Rightarrow A=37$; $P=37$" ; "$56-18=38$; $38-18=20 \Rightarrow P=20$ and $A=56-20=36$"; "$P=18$ and $A=18+18=36$"; "$P=56-18=38$ and $A=56+18=74$"; "$56:2=28$; $8+18=26$; $56-26=30 \Rightarrow P=30$ and $A=26$". The foregoing answer reveal that some students have not even noticed the fact that Andrew has more books than Paul.

Second stage errors: Several students seized the relations $A+P=56$ and $A=P+18$ but they could not apply the adequate arithmetic method. 17 students made the most common error: "$56:2=28$; $28-18=10$; $28+18=46 \Rightarrow A=46$ and $B=10$". Other students proceeded in the same way and they also made computational mistakes: "$56:2=18$; $A=18+18=36$ and $P=56-36=20$"; "$56:2=26$; $A=26+18=44$ and $P=56-44=8$". We can see that these students had some ideas about how to solve the problem, but they inverted the two steps. In our opinion this is the consequence of an algorithm-based learning and thinking, when the students memorize mechanically some problem solving procedures. 2 students gave the following answer: "$56 : 2 = 28$; $28 + 18 = 46 \Rightarrow A = 46$; $P = 28$ ".

Third stage errors: The errors are only due to miscalculations: "$56 - 18 = 48 : 2 = 24 + 18 = 42 \Rightarrow A = 42$; $P = 24$; $56 - 18 = 38 : 2 = 17 \Rightarrow P = 17$ and $A = 17 + 18 = 35$ "; 2 students, after some "groping" and computational mistakes gave the answer: "$A = 38$; $P = 20$ ".

The grade 6 students' most common error was the following: "$138 : 2 = 69$ and the solution is $69 + 24 = 93$ and $69 - 24 = 45$ (13 students)."

We have to mention that, among grade 7 and 8 students who gave erroneous answers, only 14 grade 7 students and 18 grade 8 students tried to use algebraic methods. Several incorrect answers contain calculations from which does not appear that the students discovered the relations $a = 2 \cdot b + 3$ (grade 7) and $a = 2 \cdot (b + 3)$ (grade 8) (first stage errors).

Some of the grade 7 students' erroneous answers are the following.

First stage errors: 10 students considered the length is 3 cm bigger than the width (i.e. $a = b + 3$) and their calculations are the following: "$96 - 6 = 90$; $b = 90 : 4 = 22.5$ and $a = 22.5 + 3 = 25.5$" (2 students); "$96 : 2 = 48$; $48 : 2 = 24$; $a = 24 + 1.5 = 25.5$ and $b = 24 - 1.5 = 22.5$" (2 students); "$96 - 2 \cdot 3 = 90$; $90 : 2 = 45 \Rightarrow b = 45$ and $a = 48$"; "$a = 96 : 2 = 48$; $b = 48 - 3 = 45$"; "$96 : 2 = 48$; $a = 48 + 1.5 = 49.5$ and $b = 48 - 1.5 = 46.5$" ; "$96 - 3 = 93$; $b = 93 : 2 = 46.5$ and $a = 46.5 + 3 = 49.5$" (3 students). One student considered the length is double of the width (i.e. $a = 2 \cdot b$) and wrote "$a = 96 : 3 = 32$ and $b = 32 : 2 = 16$". These answers show that the students have some knowledge about how to solve the problem in arithmetical way, however they misunderstood the relation between unknowns and some of them have difficulties with the perimeter formula. Other students, besides that they haven't understood the data of the problem, mixed the order of the steps in the problem solving strategy, some of their solutions are "$96 : 2 = 48$; $48 : 3 = 16$; $b = 16 - 3 = 13$ and $a = 48 - 13 = 35$ " and "$96 : 2 = 48$; $48 : 2 = 24$; $a = 24 + 3 = 27$ and $b = 24 - 3 = 21$". We can not decipher what is behind many incorrect responses. The students started from the initial conditions of the problem and they performed mathematical operations which does not make any sense, such as: "$a = 96 : 4 = 24$ and $b = 24 : 2 = 12$ " ; "$a = 96 + 3 = 99$ and $b = 99 : 2 = 49,5$ " ; "$a = 96$; $96 : 2 = 48$; $b = 48 + 3 = 51$ " ; "$96 : 3 = 32$; $32 \cdot 2 = 64$; $64 + 3 = 67$; $67 : 4 = 16,75$ all sides of the rectangle have the same length 16,75 cm" and "$96 + 3 = 99$; $99 : 2 = 49,5$, the sides of the rectangle have the same length 49,5 cm".

Second stage errors: From 14 students' works appears that they have understood that the equality $a = 2 \cdot b + 3$ holds, but they could not draw up a correct plan to solve the exercise, some of these attempts are the following: "$b = x$; $a = x + 3$; $x + 2 \cdot x + 3 = 96 \Rightarrow x = 31$;" ; "$96 - 3 = 93$; $b = 93 : 3 = 31$; $a = 2 \cdot 31 + 3 = 65$ cm" and "$96 - 6 = 90 : 2 = 45$ (here is a mistake) $45 : 2 = 22,5 \Rightarrow b = 22,5$ cm and $a = 2 \cdot 22,5 + 3 = 48$ cm".

Third stage errors: One student wrote the equation $2 \cdot (x + 2 \cdot x + 3) = 96$, but she could not solve it properly.

Some of the grade 8 students' errors are the following.

First stage errors: 7 students wrote $b = \frac{a}{2} + 3$ (misunderstanding) and then they made the calculations. 3 students wrote $a = b + 3$ and then solved the equation $2 \cdot a + 2 \cdot a - 6 = 102$ properly. Some students thinking backwards gave the following answers: "$102 - 12 = 90$; $b = 90 : 4 = 22.5$ cm"; "$102 - 6 = 96$; $b = 96 : 4 = 24$ cm; $a = 24 + 3 = 27$ cm"; "$102 : 4 = 25.5$; $a = 25.5 + 3 = 28.5$ cm; $b = 25.5 - 3 = 22.5$ cm"; "$102 : 2 = 51$; $a = 51 + 3 = 54$ cm ; $b = 51 - 3 = 48$ cm"; "$102 : 2 = 51$; $a = 51 + 3 = 54$ cm; $b = 54 : 2 = 27$ cm". Some students performed mathematical operations which merely make any sense, such as "$x \cdot 2 - 3 = 102 \Rightarrow x = 52.5$; $a = 52.5 : 2 = 26.25$ cm; $b = 26.25 - 1.5 = 24.75$ cm".

Second stage errors: 6 students knew the equality $b = \frac{a}{2} - 3$ holds, some of their works are: "$\frac{a}{2} = b + 3$; $2 \cdot \frac{a}{2} + 2 \cdot b + 6 = 102 \Rightarrow a + 2 \cdot b = 96$ (she did not continue)"; "$a = x$; $b = \frac{x}{2} - 3$; $x = \frac{x}{2} - 3 \Rightarrow x = -6$ (he did not give any answer)". 4 students wrote $\frac{a}{2} - 3 = b$ or $b + 3 = \frac{a}{2}$ and they did not know how to continue. One student's answer "$102 : 6 = 17$; $17 + 1,5 = 18,5$; $a = 18,5 \cdot 2 = 37$; $102 - 2 \cdot 37 = 28$; $b = 28 : 2 = 14$" (we can assume that he understood the conditions of the exercise, but he did not know how to apply them when he solved the problem thinking backwards).

Third stage errors: In our opinion these errors are primarily attributable to inattention: "$a = 2 \cdot b + 6$; $3 \cdot b + 6 = 61$ "; "$a = x$; $b = \frac{x}{2} - 3$; $x + (\frac{x}{2} - 3) \cdot 2 = 102$ "; "$\frac{a}{2} = b + 3 \Rightarrow a = 2 \cdot b - 3$; $2 \cdot a + 2 \cdot (2 \cdot a - 3) = 102$ "; "$a = x$; $b = \frac{x}{2} - 3 \Rightarrow \frac{x}{2} - 3 + x = 102$ (wrong perimeter formula)$\Rightarrow x = 66$ ".

In the following we will analyse two exercises which can be described by the systems of equations:

$$x + y = a$$
$$x + z = b$$
$$y + z = c$$

and

$$x + y = a$$
$$y + z = b$$
$$\frac{x + z}{2} = y$$

respectively. We considered these exercises too difficult for the grade 5 and 6 students so we included only in the grade 7 and 8 students' worksheet.

PROBLEM 5.7. *(Grade 7) Ann and Barbara together weighed 93 kg. Ann and Cathey together weighed 95 kg. Barbara and Cathey together weighed 102 kg. How much does each of the girls weigh?*

PROBLEM 5.8. *(Grade 8) In a student hostel there are 86 students on the first and second floor and there are 94 students on the second and third floor. The number of students from the second floor is the average of the number of students from the first and third floor, respectively. How much students are on each floor?*

Table 7

	grade 7	grade 8
Right answer	54	25
Wrong answer	39	40
No response	5	20
Total	98	85

Table 8

	grade 7	grade 8
Groping	32	14
Algebraic method	-	8
Arithmetic method	22	3
Total	54	25

Both of the problems may be solved using a system of three equations, but the students to whom it was proposed had not studied such linear systems at school yet. One thing, quite noticeable, that may facilitate the grade 7 students' work in solving the problem is that the weight measurements were made in pairs, leading to only two unknowns in each equation. In our case study groping and arithmetic strategies were the mostly used, which shows similarity with other research results. For example, Amado et al studied almost the same problem and gave some students' problem solving methods in reference [1]. In the folowing we will show some of the grade 7 students right answers obtained in typical arithmetic ways. 6 students argued in the following way: since the difference between the combined weight of Ann and Barbara and the combined weight of Ann and Cathey is $95 - 93 = 2$, we know that Cathey weighs 2 kg more than Barbara. Therefore $(102 - 2) : 2 = 50$ kg is the weight of Barbara. So we know that Cathey must weigh $50 + 2 = 52$ kg. Then Ann's weight $93 - 50 = 43$ kg follows. Another student proceed almost in the same way, but his final calculation is $102 : 2 = 51$, so Barbara weighs $51 - 1 = 50$ kg and Cathey weighs $51 + 1 = 52$ kg. 3 students analysed the combined weights and they got that Barbara weighs 7 kg more than Ann, and so Barbara weighs "$93 : 2 = 46.5 + 3.5 = 50$ kg" and Ann weighs $46.5 - 3.5 = 43$ kg. 12 students gave the following solution: the sum of the weights of the three daughters together is $(93 + 95 + 102) : 2 = 145$ kg, and from this Ann weight is $145 - 102 = 43$ kg. The random trial-and-error strategy was the most used by participants. Most of them began to write the three relations suggested by the problem but then diverged to the use of trial and error. This method was successful because the students intuited that the weights of the daughters are whole numbers situated nearby 50 and so they had to do only a few trials. Many of them at first made an assumption and then increased or decreased the number of kilos one by one until they reach the requirements of the problem. One of them begun making the following $93 + 95 + 102 = 290 : 2 = 145$ (i.e. he tried to solve the problem by arithmetic method), but then he diverged to the use of trial-and-error.

First stage errors: As Table 9 shows a large number of students could not interpret properly the relations of the problem, some of them made mistakes such as: "$A = B = 93 : 2 = 46.5$; $C = 95 - 46.5 = 48.5$" or "$93 : 2 = 46.5$; $95 : 2 = 47.5$; $102 : 2 = 51$ so Ann, Barbara and Cathey weigh 47 kg, 48 kg and 51 kg respectively".

Second stage errors: 7 students wrote the three relations, but they could not start any problem solving strategies (not even trial-and-error), some of them wrote sentences, such as "Anna has the lightest weight" or "Cathey has the heaviest weight" and they did not continue.

Third stage errors: The students who committed errors in the third stage wrote the relations properly, four of them even reached the $2 \cdot A + 2 \cdot B + 2 \cdot C = 290$ relation, but they got into a muddle when they applied the trial-and-error method.

Table 9

	grade 7	grade 8
First stage	22	31
Second stage	7	6
Third stage	10	3
Total	39	40

The grade 8 students' exercise was more difficult, maybe this is why they did not use arithmetic methods and they tried to turn toward algebra. 3 students used arithmetic methods, and they reach to the $4 \cdot II = 180$ relation, then $II = 45$; $I = 41$ and $III = 49$ followed. Most of the students who solved the problem successfully by groping initially wrote the relations of the problem using algebraic symbols (even x, y and z) but they did not know how to deal with the tools of algebra. 8 students used successfully the algebraic methods, most of them denoted the number of students from the second floor by x and solved successfully the equation $\dfrac{(86-x)+(94-x)}{2} = x$. One student solved successfully the system of three equations in the following way: "$x+y+y+z = 86+94$ so $y = 90 - \dfrac{x+z}{2}$ and $y = \dfrac{x+z}{2}$ therefore $y = 90 - y$ and $y = 45$."

First stage errors: As Table 9 shows most of the students failed in the first stage, most of them did not understand the $\dfrac{I+III}{2} = II$ relation, and they gave answers which satisfy the $I + II = 86$ and $I + III = 94$ relations, such as $I = 32$; $II = 54$ and $III = 40$. Some of the students' works reveal that they did not even understand (or they did not taken into account) the $I+II = 86$ and $I + III = 94$ relations and they gave answers such as "$86 + 94 = 180 : 3 = 60$ so $I = 86$, $II = 60$ and $III = 34$ " or "$II = III = 94 : 2 = 47$ and $I = 84 - 47 = 39$ " or "$II = 94 - 86 = 8$ $I = 86 - 8 = 78$ and $III = 86$".

Second stage errors: 6 students failed in the second stage, they wrote properly the relations suggested by the problem, but they did not find any useful strategy, although they tried to write equations, to draw lines, one of them wrote "there are 8 people more on the third floor than on the first one" and abandoned.

Third stage errors: 3 students failed when they reached the relation $I + 2 \cdot II + III = 180$ and they did not continue.

6. Summary and Conclusions

We consider that the primary school students in word problems solving mainly prefer numerical checking strategies (we refered to these as *Groping*, according to Pólya), such as estimation/guess and check (estimating the unknown measures, by perceptively comparing them to other known measures, then verifying that the estimated values satisfy the problem conditions) and trial and error (repeating process using forward arithmetic operations inherent to the problem situation, testing different numbers in the statement of the problem). The random trial-and-error was the most commonly used however a relatively large number of students used sequential trial-and-error method. This is a good evidence that students in every-day word problems mainly prefer to use intuitive, non-algebraic methods. Students tend to use numerical procedures mainly because they are used to perform procedural computations rather than to represent in an algebraic way the relations involved in a given problem. In this way the importance of the method of false supposition increases in a considerable way, because it is a pattern, a model which the students can use if they have to deal with word problems. The usage of the method of false supposition is convenient to the students who try to solve these problems by groping, because this is a deductive method in which the students should not follow severe arithmetic or algebraic rules. So we recommend a more frequent usage of the method of false supposition in solving word problems during the educational processes. In our opinion students are able to develop algebraic procedures for solving equations more easier in the case of problems with letter-based labeling, especially the geometrical exercises which involve well-known formulas, such as the perimeter of a rectangle. These problems could be useful tools in the transition from arithmetic to algebra in the case of grade 7 students. We have to mention that a large number of errors occured in the first stage. This denotes the main lacunaries, such as reading errors, reading comprehension difficulties, the weakness in understanding the relations between the data of the word problems. Many students face difficulties during the transformation from the written problem to an acceptable set of arithmetical procedures or algebraic equations. In these areas we consider serious improvements are needed.

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