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Number theory vs. Hungarian highschool textbooks: The fundamental theorem of arithmetic

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Abstract. We investigate how Hungarian highschool textbooks handle basic notions and terms of number theory. We concentrate on the presentation of the fundamental theorem of arithmetic, the least common multiple and greatest common divisor. Eight families of textbooks is analyzed. We made interviews with the authors of four of them. We conclude that a slightly more precise introduction would not be harmful for pupils and could bring basic number theory closer to them.

Key words and phrases: textbook, number theory, greatest common divisor.

ZDM Subject Classification: F60, U20.

1. Introduction

Number theory is widely acknowledged to be one of the most basic topics of mathematics. In the Hungarian school system at early age in kindergarten kids meet the notion of odd and even numbers. In grade 6, when defining the rational numbers they jump into the middle of number theory. For the addition of fractions they need to find the least common denominator and it is the least common multiple of the denominators of the fractions. When understanding the notion of the equivalence class of fractions of equal value they are taught to simplify these fractions. Later, in grade 9 they meet (without proof) the fundamental

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theorem of arithmetic, and as an application they repeat the methods of finding the greatest common divisor and the least common multiple of two integers. At the same time, the notion of prime is defined, and in a couple of books, sometimes as a supplementary material, it is shown that there are infinitely many of them. Essentially, at this point the study of integers arrives at a stop in high school, and until grade 12, when pupils go for the final, maturity exam, number theory shows up only once in the curriculum, when the irrationality of $\sqrt{2}$ is proved. Hence we can conclude, that Hungarian mathematical education does not handle number theory very generously. A recent research showed that Hungary is not unique in this matter, as, for example Canadian preteachers do not even learn the proof of the fundamental theorem of arithmetic [16, 17].

In our opinion, this negligence of number theory in higher classes of highschool contradicts its role in developing ideas, developing problem solving skills and developing proof techniques. Not a single math competition can be imagined without a problem involving some ideas from number theory. We mention a few obvious examples where different techniques and different ideas are used to solve easy problems: induction is frequently introduced by a number theory example. Think of the problem $3|n^3 - n$. Here, if one factors $n^3 - n = n(n+1)(n-1)$, then divisibility properties of consecutive numbers imply the statement. When solving a problem like $5|2^{12345} + 8^{12221}$ it is required to observe that the last digits of the powers of 2 and 3 are periodic, etc. Other research projects have shown the importance of number theory in this manner, as well. For example, the role of the notion of uniqueness in mathematics in general [4] and the proof of the infinitude of the number of primes [14] are analyzed as examples helping pupils to understand the notion of indirect and constructive proofs.

The results of this paper are parts of a large survey on the status of Number Theory in Hungarian elementary- and highschool education. The research originated from an observation that Hungarian university students do not remember the contents of number theory from the high school curriculum.

In this paper we consider the fundamental theorem of arithmetic, the greatest common divisor and the least common multiple.

2. Motivation

In the scope of a larger project two tests were done by 1231 Hungarian highschool pupils and first year undergraduate students. The first one was a questionnaire about what students were taught at highschool. The goal was to make a survey about what topics of number theory are covered in different types of schools. An initial version of the questionnaire was done by 541 pupils. After the evaluation of these tests we saw that a few questions were too easy (for example: Did you hear about the divisibility rule by 2?), some of them too hard and some of them confusing. We reformulated the confusing questions, and replaced some easy and hard ones by more appropriate ones. The test was done by pupils of grade 8-12 in standard secondary grammar schools (969 pupils) (most pupils in Hungary belong to this group including vocational/technical highschools (198)), and pupils studying in secondary grammar schools with advanced math classes (262), as well. In the highschools we were concentrating on pupils of grade 10 (just learned number theory) and grade 12 (last year in highschool). For curiosity we asked a few pupils of grades 8,9 and 11 to do the tests. The distribution of students was the following: 24% grade 10, 21% grade 12, 48% first year university and 7% the rest. Our research was concentrating on pupils who aim to continue their studies at higher education, so mostly on secondary grammar school pupils. These participants covered the two ranges of highschool education approximately proportionally to their presence in the Hungarian schoolsystem. We did not take into account a regional coverage of the country, mostly we cared about the relationship between curricula and schooltype. The regional distribution would be hard to describe, because 48 percent of the students were first year university students from all over the whole country, and the questionnaire did not ask about the school or city they came from. The survey has been started in May 2014, and the majority of the tests have been done in October 2014, shortly after the start of the schoolyear.

A few of these questions were related to the fundamental theorem of arithmetic (FTA) and the greatest common divisor (GCD) and the least common multiple (LCM).

Here, we list only those questions from the survey that are relevant to this paper.

In all 1231 questionnaires the following questions appeared:

(Q1) Did you know that every positive integer has a prime factorization?

(Q2) Have you heard about the notion of GCD?

(Q3) Have you heard about the notion of LCM?

There were four possible answers to these questions

(1) I did not hear for sure

(2) I might have heard

(3) It was in class, but I do not remember

(4) It was in class, and I know it

This kind of answers had two goals: the first one was to check whether or not pupils remember that they have learned about the subject, the second one if they remember what exactly the topics covered. The answers (1) and (2) said that they did not (remember) learn(ing) the topic and (3) and (4) that they did. Answer (4) suggested that they could work with those notions. In this survey it was not a question if they were really aware of the knowledge. The results of the questionnaires can be seen in Figure 1.

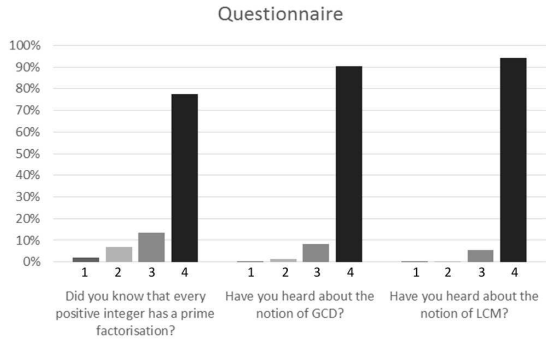

Figure 1. Results of the questionnaires

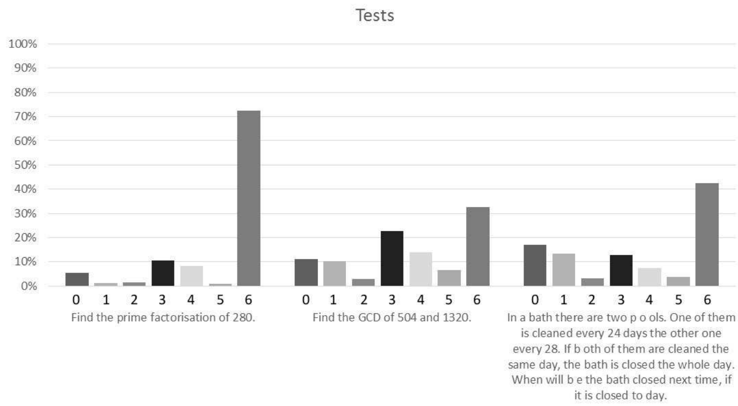

Figure 2. Results of the tests

Then we wanted to check if the pupils know what they stated to remember. We have put together two tests. The tests involved elementary questions from number theory. The technical details of the tests were very similar to those of the questionnaires. After the evaluation of the first approximately 500 tests, we made some changes the same way as we did for the questionnaires.

The tests, among others, involved the following elementary questions.

FTA Find the prime factorization of 280.

GCD Find the GCD of 504 and 1320.

LCM In a bath there are two pools. One of them is cleaned every 24 days, the other one every 28. If both of them are cleaned the same day, the bath is closed the whole day. When will be the bath closed next time, if it is closed today?

The students had to solve these problems in the classical sense. The grading was not the classical grading, we made 7 categories.

(0) Did not even start

(1) Started, but no achievement

(2) Started and made a wrong solution

(3) Good start, no solution

(4) Good solution with some (minor) miscalculation

(5) Good start, not finished

(6) Perfect solution

After grading and evaluating the tests we arrived at the results in Figure 2. For the convenience of the reader, Table 1 gives the results of pupils from standard math classes categorized according to their class. The table contains percentage that makes the answers 3 and 4 from the questionnaires and answer 6 from the tests, out of the total number of students from standard math classes categorized according to their class. Hence, the table reflects students who claimed to learn about the corresponding notions and those who have given complete solutions to those problems.

The results show that students were fairly confident about their knowledge of all three notions examined: FTA, GCD and LCM but they were far less successful when it came to actual presentation of their knowledge. This raises the question: what makes the fake impression in students about their knowledge? Did they have some knowledge, and if they did, when and how was it forgotten?

Table 1. Test and questionnaire results of standard math classes

	prime factorisation				greatest common divisor				least common multiple			
	questionnaire			test	questionnaire			test	questionnaire			test
	3	4	∑	FTA	3	4	∑	GCD	3	4	∑	LCM
class 10	18%	71%	89%	80%	13%	85%	98%	24%	4%	95%	99%	21%
class 12	18%	67%	85%	79%	15%	82%	98%	41%	10%	88%	98%	59%
university	4%	92%	96%	91%	1%	99%	100%	56%	4%	96%	100%	70%

Starting at the basis, we examine the printed sources available for pupils: the textbooks. By a recent survey 80 percent of Hungarian teachers regularly use textbooks and prepare for their classes from those [13]. The research does not specify this proportion by subjects, hence we do not know the exact percentage for math teachers, but the 80 percent average is not negligible. For pupils high school textbooks stand for an absolute source of knowledge. Hence, as a first step it looks a good strategy to initiate the investigation of Hungarian mathematical textbooks. We do not claim that this is the reason or this is the only reason resulting the problems. Very likely the situation is more complex and complicated. However, we have to start somewhere, and improving the quality of the presentation of the mentioned topics in the textbooks, can only do good anyhow.

3. Textbooks

There are several textbooks for official use in the country. Naturally, only few of them have leading places on the market. We examined the following nine (standard and widely used) books: [1, 2, 3, 5, 6, 8, 10, 11, 12] and two other publications that are summaries and abstracts for the highschool final exams [7, 9]. In all these books we looked up the introduction of primes and the fundamental theorem of arithmetic.

The fundamental theorem of arithmetic states the following:

Theorem 3.1. *Every integer distinct from 0, 1 and -1 can be written as a product of primes. This decomposition is unique up to sign and the order of the primes.*

Now, we follow the lines of the above mentioned standard textbooks. It is worth noting that almost all books cover the topic in this manner. When talking about primes, the definition of a prime in elementary- and highschool is as follows [1, 2, 3, 5, 6, 8, 10, 11, 12]:

Definition 3.2. A positive integer greater than 1 is a prime if it has exactly two divisors.

Definition 3.3. A positive integer greater than 1 which is not a prime is called composite.

Then, they come up with their version of fundamental theorem of arithmetic [1, 5, 6, 8, 10, 11, 12, 7, 9]:

Theorem 3.4. *Every composite number can be decomposed as a product of primes. This decomposition is unique up to the order of the primes. This decomposition is called the prime decomposition of the number.*

There are two main characteristics of this "theorem":

- The first one is that the statement is true.
- The second one is that this is not the fundamental theorem of arithmetic, (or shortly FTA).

Theorem 3.4 might confuse the pupils: what happens to the primes? And what happens to 1?

The fundamental theorem of arithmetic, Theorem 3.1, clearly states that 1 is distinguished as 1 has no prime divisors. The textbook version leaves a gap. By Theorem 3.1 primes are also discussed, while Theorem 3.4 does not mention them. This is another gap.

Now, we try to analyze the possible reasons, why authors do not use the original version of the theorem. These are all fictive, but can be taken as true answers to the question: Why do they state the FTA only for composite numbers?

`Possible reason 1.  Inductive strategy`: This was the presentation of the fundamental theorem of arithmetic in the past 30 years in most textbooks. The description is just simply inherited from those. □

`Possible reason 2.  The ''primes are primes''` strategy: I know, that the statement is true for every integer distinct from 0, 1 and -1. I am also aware that it is enough (and simpler) to state it for positive numbers. They will be never asked to factor a negative number. It would be confusing to tell the pupils why I mention primes twice. And, nevertheless, the whole explanation is much longer if I wrote the complete description. Further, by the exact formulation a prime, for example 7, is a one term product. The students might ask how 7 can be a product of primes. So I would have to explain that 7 is a product with one single term. It can be confusing. Until now the notion of product was always followed by plural. □

Although the above arguments are all purely speculative, there must be some reason, why authors do not state the theorem in its full form. This negligence leads to several further mistakes.

The first very important consequence of this kind of statement is that **primes have no canonical forms**, or in "school-language" primes have no prime decompositions. None of the above mentioned books talk about canonical forms of primes. In most of these books [1, 2, 3, 5, 6, 8, 10, 11, 12, 7, 9]: the GCD, the greatest common divisor of two or more numbers is defined as the largest divisor among all common divisors of these numbers. Then the following strategy is presented in order to find the greatest common divisor of two positive integer numbers a and b:

A procedure [1, 2, 3, 5, 6, 8, 10, 11, 12, 7, 9]:

Consider the prime factorizations of a and b. Take the primes occurring in the factorizations of both a and b. Multiply these primes on the smaller power occurring, and this product will be the GCD of a and b.

For example, the GCD of 63 and 108 is 9, because $63 = 3^2 \cdot 7$ and $108 = 3^3 \cdot 2^2$. How would someone find the GCD of 100 and 21? Let us apply the recipe: $100 = 2^2 \cdot 5^2$ and $21 = 3 \cdot 7$. They do not have a prime factor in common, hence the recipe of the GCD does not work in this case.

Another example is when we look for the GCD of 21 and 7. The number 7 is a prime. By "our version" primes have no prime factorizations, hence the recipe cannot be applied for 21 and 7.

The third possible example is when you look for the GCD of 1 and say 15. No need to say, that the recipe is not applicable by any means.

We try to understand, again, purely fictionally, what happens in the heads of authors, referees and editors.

`Possible reason 1. The inductive` strategy. This technique has been working for over fifty years. Why would it need a change, now? □

`Possible reason 2. ''Primes are primes''` strategy.

I know that this explanation is not complete. But in most cases the recipe works. Additional mentioning of the exceptions would confuse the students and they would even omit the learning of these easy and important notions. □

What can be the solution? The solution is to change the recipe. At first one may try to change the description of the GCD.

The next item showing up in the textbooks is the LCM, the least common multiple, of numbers. The definition is the dual of the GCD: the smallest number that is a multiple in common.

Then the following strategy is presented in order to find the least common multiple of two positive integer numbers a and b [1, 2, 3, 5, 6, 8, 10, 11, 12, 7, 9].

The procedure

Consider the prime factorizations of a and b. Take the primes occurring in the factorizations of both a and b. Multiply these primes on the larger power occurring, and this product will be the LCM of a and b.

The questions arising about the LCM are similar to those about the GCD. As primes have no factorization, the recipe tells nothing about the LCM of a prime and any other number. Similarly, 1 has no LCM with any other number by the recipe.

What can be the reason to exclude primes from the statement of the fundamental theorem of arithmetic? Does it result a problem that the algorithms presented about the GCD and LCM are not precise? Would it not be better to present those algorithms precisely? We approached the authors of several Hungarian highschool math books with these questions. Six of them have answered our emails, four of them gave a meaningful answer. We had longer correspondence and interviews with three of them. Dr. László Gerőcs, who is an experienced teacher with 41 years behind him in highschool wrote three longer letters before we interviewed him. We summarize those thoughts of him that strictly belongs to this paper. In our first letter we asked him about the fundamental theorem of arithmetic:

Well, this question is very thoughtful. If you look at textbooks, you will find both formulations: For every integer $n > 1$ ...and For every composite number... I myself learned it in this second way at the University. What can be the reason? The essence of the theorem is about UNIQUENESS and the notion of multiplication. This latter thing is an operation for a pupil, involving two real numbers with a dot between them. A one term product would result confusion in most pupils' head. Furthermore, what the theorem says, is that any way we start to factor a number, we obtain the same result, the same primes as factors. Starting with a single prime would be totally uninteresting for the pupils. This is probably why we state the fundamental theorem of arithmetic this way in our book. Moreover, even if we played with the idea that we alter the statement in a next edition of our book, we should seriously consider that is it worth doing just because a few "overprecise" people think this way. We need to convince ourselves that the change will not result any harm for the pupils weaker in mathematics.

In our second letter we asked him about the algorithms for finding the GCD and LCM. We summarize here the next two letters of him.

In his answer he repeated that at first we have to clarify the notion of a product: A product is a(n at least) two-variable function. Hence the fundamental theorem of arithmetic can only be stated for composite numbers. If we allow one-term products than we shall have difficulties. Although, it does not necessarily will confuse the pupils, because we use the expression "multiply an expression by a number". If we clarify that the prime decomposition of a prime is the prime itself, then no confusion will remain in the heads of the pupils. He also remarked that usually problems do not start here. The real problem will come later, when we deal with expressions, where the denominators are algebraic expressions and we need to find the common denominator. In this case what you need is the algorithm for finding the LCM and usually it works pretty well. I am sorry, but I am teaching in Trefort, and my experience is with those pupils who attain that school. (Remark: Trefort is one of the top highschools in Hungary). Concerning your tests: I find it very unfortunate if a first year math student cannot figure out the LCM of two numbers. If he does not remember exactly what the algorithm was, he is supposed to think it over and figure out, what exponents are needed for the primes in the LCM. Then Dr. Gerőcs explained that several times precision is against comprehension and we have to find the balance. He gave a few examples where the precise formulation would make statements over-complicated. Most of these notions came from geometry. Then he wrote that he has also experienced that a few pupils mix the choice of both the primes and exponents, but it is rather the result of a non-organized way of thinking. According to him this has nothing to do with the formulation of the fundamental theorem of arithmetic in the textbook.

Then we had an occasion to interview him. The interview lasted for more than an hour. We learned a lot from him during this interview. We argued, he argued. At the end he agreed that a more precise introduction to number theory would no result harm. It could be mentioned that primes have prime factorization and that the calculation of GCD and LCM works differently for 1. But, he said, he does not find it necessary. In most cases teachers explain these procedures properly in class, including the case of primes, as well. If not so, then if such a question accidentally occurs during the solution of a problem in a pupil's head, then the teacher gives the supplementary conditions to the theorem or explains the algorithms for primes, and pupils will know it. Anyway, (according to him) every pupil knows how to calculate them.

Gyula Orosz, a highschool teacher for 28 years gave a similar but shorter explanation as Dr. Gerőcs on the confusion of one-term products. Then we posed him the question about the GCD and LCM recipes. He asked for some time and on our next meeting he said that he totally agrees that the formulation of the theorem has to be made more precise, as well as the algorithms. He said, that they have no chance to alter it in the book for grade 9. The change of the statement would cause a new publishing procedure including to obtain the permission for publication by the ministry. It is lots of time and lots of money. But they will make the appropriate changes wherever it is possible in the books for higher grades. When we asked his permission to present his ideas in a paper, he said: please, mention, that independently of that I agree with you, I think that all pupils know the fundamental theorem of arithmetic and the calculations of LCM and GCD.

Dr. Katalin Fried an associate professor at the Department of Methodology of Teaching Mathematics pointed out that pupils meet the prime factorization as early as grade 6. It could be stated in its full form in grade 7-8. A full precise statement can be made in grade 9. Unfortunately it would be very difficult to change it in their book. At first she gave us all details how a book is written. It is a long procedure involving authors, referees, editors, opponents, permissions and the print. To make a change would last for 1 to 5 years, if can be done. If an author proposes a change, it is the right of the editor to decide whether or not he goes through with it. Usually if a book is printed, than it is printed in advance for 5 years. During the first two years it is an “experimental textbook” and the mistakes-misprints are collected from teachers and pupils. Then they start to distribute it. After this step to make any changes is as complicated as writing a new book. And, again, it is the decision of the editor whether or not they make the changes. Hence, if they wanted to state the fundamental theorem of arithmetic properly, it would take five years, still editor-dependent.

4. A way to treat number theory

In this section we propose two ways, with detailed explanation and discussion, to remedy the above mentioned problems To present a more fluent and correct introduction to number theory we have to start with the definition of a prime.

Definition 4.1. A positive integer distinct form 1 is a prime if its only divisors are 1 and itself.

If you compare Definition 3.2 and Definition 4.1, mathematically they are equivalent. The second definition is better, though. It essentially describes the whole notion. A prime is a number, that is a “building block”, a “brick” for factoring an integer. In other words, it cannot be factored, or in other words a prime has no nontrivial divisors.

Saying that a prime has two divisors is hiding the main property, namely that strictly speaking, we are talking about irreducibility of numbers.

The prime property, that $p|ab$ implies that $p|a$ or $p|b$ is not even mentioned in most books.

The number 1 has to be handled separately. The number 1 divides every number. When talking about primes, we have to emphasize that 1 is not a prime. However, the reason is not, that it has only one positive divisor. The reason is that 1 divides every integer, and if it was listed to be a prime, then every number had infinitely many prime decompositions, e.g.

$$6 = 3 \cdot 2 = 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 1 \cdot 1 \cdot 1 = \dots$$

If we want to make a point that primes are those numbers that cannot be factored, it is essentially the same as they have no proper divisors. The definition of composite numbers should be given in textbooks, as an expression, but not stressed too much.

We define prime factorization by Theorem 3.1. In that case primes will be products with one single term.

Now, we arrived at the most sensitive problem, the calculation of the greatest common divisor and the least common multiple of two (or more) positive integer numbers. The most fair algorithm for finding the greatest common divisor for two positive integer numbers is the following:

Procedure: Find the prime decomposition of the two numbers. If they do not have a prime divisor in common, then the greatest common divisor is 1. Otherwise, consider the prime divisors in common on the smallest power occurring in both numbers, and take the product of these prime powers. For an arbitrary positive integer a, the GCD of 1 and a is always 1.

To find the least common multiple consider the prime divisors in common on the largest power occurring in both numbers. For an arbitrary positive integer a, the LCM of 1 and a is always a. □

Another solution is to define the extended or generalized prime factorization (or canonical form). We allow to have primes on the power 0. Then, any two or more numbers have the same primes in their factorizations. For example

$$\begin{aligned} 108 &= 2^2 \cdot 3^3 & 45 &= 3^2 \cdot 5^1 \\ 108 &= 2^2 \cdot 3^3 \cdot 5^0 & 45 &= 2^0 \cdot 3^2 \cdot 5^1 \\ \text{GCD}(108, 45) &= 2^0 \cdot 3^2 \cdot 5^0 \\ \text{GCD}(108, 45) &= 2^2 \cdot 3^3 \cdot 5^1 \end{aligned}$$

This calculation of the GCD is very comfortable, there are no cases and subcases. Every prime is a prime in common. These primes are 2, 3 and 5 and the smaller exponents are 0, 2 and 0, the larger exponents are 2, 3 and 1. So where do we pay the price for it? The difficulty is to introduce the two kinds of prime factorization. The extended one, introduced for the easier calculation of the GCD varies from problem to problem. The factorization of 108 is different if we find the GCD of 108 and 63.

$$\begin{aligned} 108 &= 2^2 \cdot 3^3 & 63 &= 3^2 \cdot 7^1 \\ 108 &= 2^2 \cdot 3^3 \cdot 7^0 & 45 &= 2^0 \cdot 3^2 \cdot 7^1 \\ \text{GCD}(108, 63) &= 2^0 \cdot 3^2 \cdot 7^0 \\ \text{LCM}(108, 63) &= 2^2 \cdot 3^3 \cdot 7^1 \end{aligned}$$

Here, the primes in common are 2, 3 and 7 and the smaller exponents are 0, 2 and 0, the larger exponents are 2, 3 and 1. And, there is no factorization of 108 that works for every case, because there are infinitely many primes. Hence, in this setup of number theory we have to give a detailed explanation that for each GCD finding we have a distinct extended prime factorisation of the number, and in this problem even 1 has a canonical form. Naturally, in the extended factorisation of 1 every prime will occur on power 0.

5. Conclusion

There is a problem with pupils' knowledge in one of the most elementary topics of mathematics, in the factorisation of integers. The existing textbooks handle the fundamental theorem of arithmetic cruelly and that might result in a confusion and lack of pupils' knowledge in number theory. This is only a first step in the analysis towards the examination, when and how students forget number theory. In order to change that "might" to "may" or to a more deterministic connection, further investigation has to be done. We can avoid this "might", we can avoid the possibility that textbooks have effect to forgettability. The suggested solution to the problem is: define and introduce number theory properly. Being properly introduced according in our point of view is to state theorems exactly, introduce the definitions meaningfully and organize the material according to these principles. We showed two ways of handling basic issues of number theory; it is left to the reader and to the book-writer which one to choose. None of the two is easy, both have some complications, details, discussions. Still, they are correct.

When ruler Ptolemy I. Soter asked Euclid if there was a shorter road to learning geometry than through Euclid's Elements, Euclid' famous answer was

There is no royal road to geometry.

Then why would it be to number theory?

A sequel to this paper will investigate the irrationality of $\sqrt{2}$ and a few other proofs as the infinitude of primes.

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References

[1] G. Ábrahám, dr. E. Kosztolányiné Nagy and J. Tóth, *Matematika 9.*, Maxim Könyvkiadó, 2009, 80–96.

[2] G. Ambrus, A. Frenkel, K. Kaposiné Pataky, J. Mezei, P. Papp, dr. L. Szabadi, A. Szász, P. Székely, A. Tóth and dr. Ö. Vancsó, *Matematika 9.*, Ráció Könyvek, 2009, 47–90.

[3] G. Ambrus, A. Frenkel, J. Mezei, Á. Nagyné Szokol, P. Papp, K. Pataky, dr. L. Szabadi, A. Szász, P. Székely, Á. Szokol, A. Tóth, dr. Ö. Vancsó, *Matematika 9.*, Ráció Könyvek, 2003, 59–64.

[4] D. L. Ball, Prospective elementary and secondary teachers' understanding of division, *Journal for Research in Mathematics Education* **21** (1990), 132–144.

[5] E. Czapáry and F. Gyapjas, *Matematika 9.*, Nemzeti Tankönyvkiadó, 2001, 65–85.

[6] dr. K. Fried, dr. L. Gerőcs and L. Számadó, *Matematika 9.*, Nemzeti Kiadó, 2009, 59–70.

[7] A. Gábos and M. Halmos, *Készüljünk az érettségire matematikából*, Műszaki Kiadó, 2005, 52–61.

[8] S. Hajdú, I. Ceglédy, S. Z. Hajdú, A. Kovács and S. Róka, *Matematika 9.*, Műszaki Könyvkiadó, 2008, 231–275.

[9] dr. I. Hajnal, *Matematikai fogalmak, tételek*, Mozaik Oktatási Stúdió, 1994, 35–37.

[10] I. Hajnal, L. Számadó and Sz. Békési, *Matematika 9.*, Nemzeti Tankönyvkiadó, 2001, 44–47.

[11] I. Juhász, Gy. Orosz, J. Paróczay and J. Szászné-dr. Simon, *Matematika 9*, Nemzeti Tankönyvkiadó, 2009, 130–151.

[12] J. Kosztolányi, I. Kovács, K. Pintér, J. Urbán and I. Vincze, *Sokszínű matematika 9.*, Mozaik Kiadó, 2005, 62–70.

[13] J. Kaposi (project leader), *Tanulmány a tankönyvpolitikák vizsgálatáról [A study on textbook policies]*, TÁMOP-3.1.1. OF 2010 3.1.1.-7.1.8. (in Hungarian).

[14] U. Leron, A direct approach indirekt proofs, *Educational Studies in Mathematics* **16** (1985), 312–325.

[15] É. Vásárhelyi, *Lajtán innen, Lajtán túl*, `http://www.ofi.hu/sites/default/files/attachments/vasarhelyi_eva_lajtan_innen_lajtan_tul.pdf`.

[16] R. Zazkis and S. Campbell, Prime decomposition: Understanding uniqueness, *Journal of Mathematical Behavior* **15**, no. 2 (1996), 207–218.

[17] R. Zazkis and S. Campbell, Divisibility and Multiplicative Structure of Natural Numbers, *Preservice Teachers' Understanding Journal for Research in Mathematics Education* **27**, no. 5 (1996), 540–563.

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