12/2 (2014), 167–184
tmcs@math.klte.hu
http://tmcs.math.klte.hu
Teaching Mathematics and Computer Science

Heuristic arguments and rigorous proofs in secondary school education

Zsolt Fülöp

Abstract. In this paper we are going to discuss some possible applications of the mechanical method, especially the lever principle, in order to formulate heuristic conjectures related to the volume of three-dimensional solids. In the secondary school educational processes the heuristic arguments are no less important than the rigorous mathematical proofs. Between the ancient Greek mathematicians Archimedes was the first who made heuristic conjectures with the methods of Mechanics and proved them with the rigorous rules of Mathematics, in a period, when the methods of integration were not known. For a present day mathematician (or a secondary school mathematics teacher) the tools of the definite integral calculus are available in order to calculate the volume of three dimensional bodies, such as paraboloids, ellipsoids, segments of a sphere or segments of an ellipsoid. But in the secondary school educational process, it is also interesting to make heuristic conjectures by the use of the Archimedean method. It can be understood easily, but it is beyond the normal secondary school curriculum, so we recommend it only to the most talented students or to the secondary schools with advanced mathematical teaching programme.

Key words and phrases: heuristic arguments, deductive proof, lever principle.

ZDM Subject Classification: A30, D20, D50.

1. Introduction

The study of various curves and calculating the volume of solids obtained by their revolution is one of the most important applications of the definite integral calculus. Mechanics, as a science, was an inspiration to the development of integral calculus, as it raised very important issues, for example, in construction or shipbuilding, and problems which can be solved using definite integral. In order to examine the stability of bodies, it is necessary to determine their mass, volume or centre of gravity, and these problems can be handled easily by integral calculus. Mechanics opened new perspectives towards the practical applicability of these tools. In other words, the discovery of a special field of Mathematics, which today is called integral calculus, was guided by physical intuition. Conversely, we can say that before the appearance of integral calculus, the calculations related to the volume of three-dimensional solids or the area of geometrical figures were solved with the tools of Mechanics. So Mechanics, as science, helped the mathematical discoveries. We mainly want to refer to the works of Archimedes, who has developed two procedures in order to determine the area of figures and the volume of bodies: these are called geometrical and mechanical methods, respectively. Archimedes determined the area of the orthotome (parabola), the volume of the sphere, the volume of the spheroid, the centre of gravity of any segment cut off from an orthoconoid (paraboloid), and obtained several further results by the use of the so-called *lever principle*.

Archimedes in his work *Method* presented his mechanical method based on the application of the lever principle. This work was not known until the beginning of the 20th century. This is the main reason why most historians of mathematics devoted less attention to the heuristic methods of the Archimedean mathematics. In 1906 J. L. Heiberg found a report on a palimpset with originally mathematical content in the library of the monastery of the Holy Sepulchre in Jerusalem. He examined this manuscript, and it proved to contain an Archimedes text written on parchment, which had been erased in the 12-14th centuries, in order to right a Euchologium in its stead. It contains fragments of some works of Archimedes known from other sources, such as *On the Sphere and the Cylinder*, *Measurement of the Circle*, *On the Equilibrium of Planes*, and an almost complete text of an as yet unknown, very important work of Archimedes, what is referred as *Method*.

If we want to put the Archimedean work in perspective we have to know the circumstances in which he lived and created, and also we have to know something about the work of his ancestors. It is well-known that the mathematics of the ancient river valley cultures was purely empirical. The mathematics of the Greek classical period is totally different. If we study the works of mathematicians of that period, we can find sequences of postulates, axioms, definitions and theoretical propositions followed by rigorous proofs based on the postulates and previous propositions (A brief, but eloquent summary can be found in [5] and [6]). In fact, Greek classical mathematics is characterized by a care of the form of the mathematical argument, which, superficially viewed, seems almost exaggerated. However, we have to mention that we do not find any calculations or any heuristic argument. The mathematics of this time is purely deductive without any empiricism. We know nothing about the way of discovering the propositions or the background of the definitions and postulates. Archimedes was the most influential mathematician of the forthcoming Hellenistic time. In this era, we can find calculations besides the deductively proved theorems and the mathematicians not only established propositions with rigorous proofs but sometimes explained the way of finding their results. The mathematicians have not only dealt with abstract problems and questions, but the need of the heuristic argument appeared. Archimedes was the pioneer of this approach. In his work we can find an excellent alloying of heuristic conjectures and rigorous mathematical proofs.

In the time of Archimedes the mathematics of the Greeks achieved the highest results. From his ancestors Democritus was the first to state that the cone is one-third of the cylinder having the same base and equal height (even though without proof), and Eudoxus was the first to discover the proofs to this theorem. In a sense, the Greeks knew the coordinate geometry. For example, Archimedes used the equation of conic sections in the so called "two-abscissas" form [9]. This cannot be regarded as analytic geometry in a modern sense, but we will see further that the main results might also be achieved by using these calculations.

The Greeks had a low level Mechanics knowledge, compared with their Mathematics. In fact, we can say that Archimedes was the founder of the Greek Mechanics. He discovered, for example, the laws of the floating bodies, the laws of the lever and the main properties of the centre of gravity.

2. The heuristic argument and the rigorous mathematical proof

The *Method* starts with a number of lemmas on centre of gravity, some of which can be found as postulates or propositions in *On the Equilibrium of Planes*. This part is followed by the propositions. These propositions contain all the heuristic conjectures formulated using mechanical principles, with regard, for example, to the area of an orthotome (parabola) or to the volume and the centre of gravity of three-dimensional bodies. Proposition 1 refers to the area of the orthotome, as follows [2]:

Proposition 1. *Let the segment $\alpha\beta\gamma$ be given, comprehended by the straight line $\alpha\gamma$ and the orthotome $\alpha\beta\gamma$; let $\alpha\gamma$ be bisected in δ , let $\delta\beta\epsilon$ be* *drawn parallel to the diameter, and let* $\beta\alpha$ *and* $\beta\gamma$ *be joined. I say that the segment* $\alpha\beta\gamma$ *is larger by one-third than the triangle* $\alpha\beta\gamma$*. (See Figure 1)*

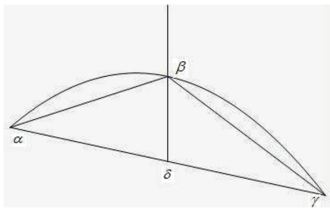

Figure 1

In order to formulate his heuristic conjecture Archimedes makes use of considerations taken from mechanics. The method consists in geometrical figures to be attached to a lever in such a way that the lever remains in equilibrium, and then draws up conditions for such equilibrium. This method is also based on the view that the area of a plane figure is to be looked upon as the sum of the lengths of all the line segments drawn therein in a given direction and of which the figure is imagined to be made up; similarly, a three-dimensional solid is conceived to be made up of all the intersections determined therein by a plane of fixed inclination that is displaced, and the volume of the solid is looked upon as the sum of the areas of those intersections. The importance of this method becomes clear from a letter to Erathostenes, where Archimedes states "it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge".

We have to mention that Archimedes does not recognize the results obtained with the above method as actually proved conclusions. This point of view is evident from the fact that in his treatise *Quadrature of the Parabola* he proves the results gained in Proposition 1 by a deductive way of thinking, satisfying all requirements of exactness. Both of the methods, namely the heuristic conjecture and the deductive proof, with regard to the area of the segment of the orthotome (parabola) are contained in several works, such as [2], [5], [6] and [9].

3. The Volume of an paraboloid (orthoconoid) segment

In the *Method* the Proposition 4 is related to the volume of an orthoconoid (paraboloid) segment in the following way [3]:

PROPOSITION 4. *Any segment of a right-angled conoid (i.e. a paraboloid of revolution) cut off by a plane at right angles to the axis is one and a half times the cone which has the same base and the same axis as the segment.*

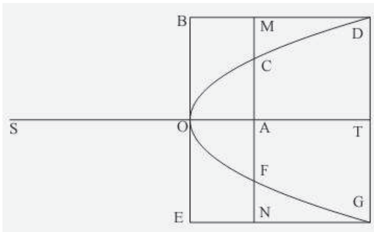

Figure 2

3.1. The heuristic conjecture

In Figure 2 let the orthotome DOG generate, by revolution about the diameter OT, a paraboloid, from which a segment is cut off by a plane through T at right angles to OT. A variable plane MN at right angles to OT intersects the paraboloid in a circle on CF as diameter, and the cylinder which has its base and height in common with the segment of paraboloid in a circle on MN as diameter. We denote the common height of the paraboloid segment and the cylinder by $OT = h$ and the common radius of their bases by $DT = R$. Summarising, in Figure 2:

DOG - the section of the paraboloid segment,
$BEGD$ - the section of the cylinder,
A - a variable point on the line OT.

We compare the volume of the paraboloid segment and the volume of the cylinder using the following terms:

The weight of a line segment: the length of it.

The area of a figure: the sum of weighted (parallel) line segments covering the figure.

The volume of a body: the sum of weighted circles filling up the body.

The volume of a paraboloid segment: the sum of weighted circles on CF as diameter.

The volume of a cylinder: the sum of weighted circles on MN as diameter.

By the property $\frac{AC^2}{DT^2} = \frac{OA}{OT}$ of the orthotome (parabola) and the equality $DT = AM$ we obtain:

$$AC^2 \cdot OT = AM^2 \cdot OA \,. \tag{3.1}$$

We consider the point S in Figure 2, specified by $OS = OT$. Then identity (3.1) yields

$$AC^2 \cdot OS = AM^2 \cdot OA \,. \tag{3.2}$$

Now we give a mechanical analysis of identity (3.2):

(a) Consider OS and OA as arms of a lever with fulcrum at O.

(b) Place the weight of the circle on CF as diameter at S. Then A is the point where placing the weight of the circle on MN as diameter we reach the equilibrium of the lever (because of the fact that the area of a circle is proportional with the square of its radius).

(c) Consequently the sum of the weights of all circles on segments like CF as diameters placed at S will balance the sum of all circles on segments like MN as diameters whenever their weight is placed at their midpoint.

(d) The collection of all circles on segments like MN as diameter the weight of each placed at its midpoint is equivalent to the cylinder placed at its centre of gravity.

(e) The collection of all circles on segments like CF as diameter placed at S is equivalent to the paraboloid segment placed at S.

(f) The cylinder has its centre of gravity situated in the middle of the segment OT, so we have the equality

$$V_{orthoconoid} \cdot OS = V_{cylinder} \cdot \frac{OT}{2} \,, \tag{3.3}$$

and $OS = OT$ yields the result:

$$V_{orthoconoid} = V_{cylinder} \cdot \frac{1}{2} \,. \tag{3.4}$$

So the Archimedean results with regard to the volume of a paraboloid segment might be formulated as follows:

(a) the volume of a paraboloid segment is equal to the half of the volume of the cylinder which has its base and height in common with the orthoconoid-segment;

(b) the volume of a paraboloid segment is one and a half times the volume of the cone which has the same base and the same height as the paraboloid segment. (In this argumentation we used the fact proved by Eudoxos, that the volume of a cone is a third of the volume of a cylinder which has its base and height in common with the cone).

We have to mention that in our argumentation we used the Archimedean way of thinking, based on the comparison of different three-dimensional bodies (in the Hellenistic period the notion of π was not yet known). Since for a present day mathematician the volume formulas are available, we can formulate our results in a modern-way of thinking (and using our notations, h denotes the height of the paraboloid segment):

$$V_{orthoconoid} = V_{cylinder} \cdot \frac{1}{2} = \frac{\pi \cdot DT^2 \cdot OT}{2} = \frac{\pi \cdot (\sqrt{h})^2 \cdot h}{2} = \frac{\pi \cdot h^2}{2} \ . \qquad (3.5)$$

This formula can also be obtained by using definite integral. Namely, the graph of the function $f(x) = \sqrt{x}$ by revolution about the x axis generates a paraboloid segment whose volume is

$$V = \pi \cdot \int_0^h x \, dx = \pi \cdot \left[\frac{x^2}{2}\right]_0^h = \frac{\pi \cdot h^2}{2} \ . \qquad (3.6)$$

3.2. The deductive proof

Archimedes was perfectly aware of the fact that his argument is only a heuristic conjecture that should be proved geometrically. The geometrical method used by Archimedes to prove his results regarding the area of a parabola segment is contained in the treatise *Quadrature of the Parabola* (see [2], [5], [6] and [9]). The Archimedean proof is based on the method of double reductio ad absurdum.

In his work *On Conoids and Spheroids*, Archimedes determined the volumes of segments of solids formed by the revolution of a conic section (circle, ellipse, parabola, or hyperbola) about its axis (see [2] and [9]). In modern terms, these are problems of integration. In this work Archimedes gives rigorous mathematical proofs concerning the volume of the above mentioned three-dimensional bodies. In order to prove the formula regarding the volume of a paraboloid segment in a rigorous geometrical way, we have to split the paraboloid segment in pieces which have the same height (see Figure 3). Each of the pieces has its own inscribed and circumscribed cylinder. We denote by $V_1, V_2, \cdots, V_n$ the volumes of the circumscribed cylinders. Each of the cylinders has the same height, therefore, using the properties of the parabola, we obtain

$$\frac{V_k}{V_1} = \frac{R_k^2}{R_1^2} = k \ ; \qquad 1 < k \leq n \ ; \qquad k \in N \ . \qquad (3.7)$$

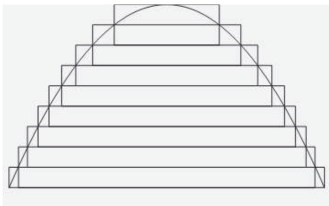

Figure 3

Thus the sum of the volumes of the circumscribed cylinders is

$$S_1 = V_1 + V_2 + \cdots + V_n = V_1 + 2 \cdot V_1 + \cdots + n \cdot V_1 = \frac{n \cdot (n+1) \cdot V_1}{2} \ . \quad (3.8)$$

The inscribed cylinders have the volumes $V_1, V_2, \cdots, V_{n-1}$, respectively. Their sum is

$$S_2 = V_1 + V_2 + \cdots + V_{n-1} = V_1 + 2 \cdot V_1 + \cdots + (n-1) \cdot V_1 = \frac{(n-1) \cdot n \cdot V_1}{2} \ . \quad (3.9)$$

and the following inequalities hold:

$$S_2 < \frac{n^2 \cdot V_1}{2} < S_1 \ . \quad (3.10)$$

The product $n^2 \cdot V_1$ is equal to the volume of congruent cylinders, each of them having the volume $V_n = n \cdot V_1$. These cylinders are the constituent parts of the circumscribed cylinder of the paraboloid segment, therefore the circumscribed cylinder has the volume $V' = n^2 \cdot V_1$. Thus formula (3.10) implies that

$$S_2 < \frac{V'}{2} < S_1 \ . \quad (3.11)$$

Our task is to show that $V = \frac{V'}{2}$, where V is the volume of the paraboloid segment. It is further assumed that the following inequality holds:

$$S_2 < V < S_1 \ . \quad (3.12)$$

The difference $S_2 - S_1 = n \cdot V_1$ is equal to the volume of the bottommost circumscribed cylinder and, by a suitable choice of n , it can be made less than any assigned positive number ε . Hence we have the inequality

$$S_1 - S_2 < \varepsilon \ . \quad (3.13)$$

To conclude the proof, Archimedes uses a double reductio ad absurdum argument. The idea is the following: if we show that both $\frac{V'}{2} < V$ and $V < \frac{V'}{2}$ are impossible, the equality $V = \frac{V'}{2}$ follows.To do this, first suppose that $\frac{V'}{2} < V$, and we find a positive integer n such that $S_1 - S_2 < V - \frac{V'}{2}$. This implies $S_1 = S_2 + (S_1 - S_2) < \frac{V'}{2} + (S_1 - S_2) < V$, contradicting relation (3.12). On the other hand, suppose that $V < \frac{V'}{2}$. Then there is a positive integer n such that $S_1 - S_2 < \frac{V'}{2} - V$. We have $S_1 = S_2 + (S_1 - S_2) < V + (S_1 - S_2) < \frac{V'}{2}$, and this contradicts inequality (3.11). Therefore the equality $V = \frac{V'}{2}$ holds. So we rigorously proved the heuristic conjecture obtained by the mechanical method.

4. Heuristic conjectures with regard to the volume of various three-dimensional bodies by the use of the lever principle

The mechanical method based on heuristic conjectures related to the volume of a sphere can be found in [1], [2], [3], [7] and [9]. The entire derivation is based on the lever principle, and the results are contained in Proposition 2 of the *Method* [3]:

Proposition 2. *Any sphere is four times the cone with base equal to a great circle of the sphere and height equal to its radius; and the cylinder with base equal to a great circle of the sphere and a height equal to the diameter is one and a half times the sphere.*

We omit the lever-principle based derivation, because its detailed presentation is contained in the above mentioned works, but in Section 5 we will give a detailed presentation of the deductive proof concerning the volume of a sphere.

4.1. The volume of a spheroid (ellipsoid)

Proposition 3 of the *Method* contains a heuristic conjecture with regard to the volume of a spheroid, as follows [3]:

Proposition 3. *A cylinder with base equal to the greatest circle in a spheroid and height equal to the axis of the spheroid is one and a half times the spheroid; when this is established it is plain that if any spheroid be cut by a plane through the centre and at right angles to the axis, the half of the spheroid* *is double of the cone which has the same base and the same axis as the segment (i.e. the half of the spheroid).*

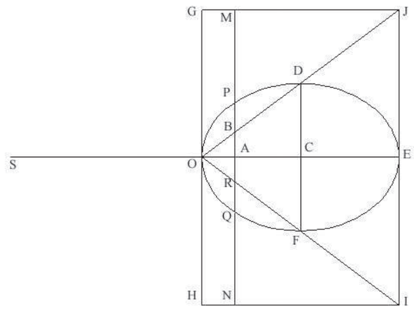

Figure 4

We try to describe in detail the heuristic arguments, namely we try to make conjectures to the volume of a spheroid with axes $OE = 2 \cdot a$ and $FD = 2 \cdot b$, respectively. In Figure 4, let *ODEF* be the greatest section of the spheroid, *OE* and *DF* its axes. Consider the cone with vertex *O*, whose base is the circle on *DF* as diameter, in the plane of *DF*, at right angles to *OE*. The extended surface of this cone intersects the plane through *E* at right angles to *OE* in a circle on *IJ* as diameter. This circle is the base of a cylinder *GHIJ* with height *OE*. We consider $OS = OE$ and *SE* as a lever with fulcrum *O*. A variable plane *MN* at right angles to *OE* intersects the spheroid in a circle on *PQ* as diameter, the cone in a circle on *BR* as diameter and the cylinder in a circle on *MN* as diameter. From the similarity of triangles $OAB\Delta$ and $OCD\Delta$, it follows that $\frac{AB}{OA} = \frac{CD}{OC}$, and the following equality holds:

$$AB = OA \cdot \frac{CD}{OC} = OA \cdot \frac{b}{a} \, . \tag{4.1}$$

In the Archimedes' works the implicit equations of the cone sections have the "two-abscissas" form, which can be imagined as follows. We consider the axis *OE* and a point *P* of the spheroid. We call $y = AP$ the ordinate of the point (AP is at right angles to OE), and the point P has two abscissas, $OA = x_1$ and $AE = x_2$. The symptom of the oxytome (ellipse) has the form $\frac{y^2}{x_1 \cdot x_2}$, where α is a parameter, which characterizes the oxytome. For example, in the case of the sphere $\alpha = 1$. Both of the points P and D are situated on the ellipse, therefore they should satisfy the symptom of the oxytome:

$$\frac{AP^2}{OA \cdot AE} = \frac{CD^2}{OC \cdot CE} = \alpha \; . \tag{4.2}$$

From this it follows that

$$AP^2 = OA \cdot (2 \cdot a - OA) \cdot \frac{b^2}{a^2} \; . \tag{4.3}$$

Moreover, using (4.1) we find that

$$AB^2 + AP^2 = 2 \cdot \frac{b^2}{a} \cdot OA \; . \tag{4.4}$$

We have to mention that (4.4) can also be obtained by the tools of modern analytic geometry, namely, if instead of (4.2) we use the equation of the ellipse in the form $\frac{AC^2}{a^2} + \frac{AP^2}{b^2} = 1$, but we opted to follow the Archimedes' mathematical tools and his logical way of thinking.

Now it is easy to see that

$$\frac{AB^2 + AP^2}{AM^2} = \frac{OA}{2 \cdot a} = \frac{OA}{OE} \; . \tag{4.5}$$

The circle on MN as diameter suo loco can therefore balance two circles on PQ and BR as diameters, respectively, both of them placed in S. Consequently there is also an equilibrium between the cylinder $GHIJ$ suo loco and the combination of the spheroid $ODEF$ and the cone OIJ, both of them placed in S (because of the fact that the three solids are filled up by the above mentioned circles if the plane MN moves from GH to IJ).

Since C is the centre of gravity of the cylinder, the following equality holds:

$$\frac{V_{spheroid} + V_{cone(OIJ)}}{V_{cylinder(GHIJ)}} = \frac{OA}{OE} \; . \tag{4.6}$$

Hence

$$V_{cylinder(GHIJ)} = 2 \cdot \left(V_{cone(OIJ)} + V_{spheroid} \right) \; . \tag{4.7}$$

The cylinder is three times the cone (this fact was firstly discovered by Democritus and proved by Eudoxus), so it follows from (4.7) that

$$V_{spheroid} = \frac{V_{cylinder(GHIJ)}}{2} - V_{cone(OIJ)} = \frac{V_{cylinder(GHIJ)}}{6} \; . \tag{4.8}$$

If we denote by $V_{cylinder}$ the volume of the circumscribed cylinder (the cylinder which has a base equal to the greatest circle and a height equal to the axis of the spheroid), we obtain

$$\frac{V_{cylinder(GHIJ)}}{V cylinder} = \frac{IJ^2}{b^2} = 4 \; . \tag{4.9}$$

From (4.8) and (4.9) we conclude the statement of the Proposition 3:

$$V_{cylinder} = \frac{3}{2} \cdot V_{spheroid} \, . \tag{4.10}$$

A present day mathematician can operate with the volume of the cylinder to obtain:

$$V_{spheroid} = \frac{2}{3} \cdot V_{cylinder} = \frac{2}{3} \cdot \pi \cdot (2 \cdot a) \cdot b^2 = \frac{4 \cdot \pi \cdot a \cdot b^2}{3} \, . \tag{4.11}$$

The method of definite integral calculus delivers us the same result. We consider the graph of the function $f(x) = b \cdot \sqrt{1 - \frac{x^2}{a^2}}$ on the interval $(-a, a)$ .This graph, by revolution about the x axis, generates the ellipsoid. Thus the volume of the ellipsoid can be obtained as follows:

$$V = \pi \cdot b^2 \cdot \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx = \pi \cdot b^2 \cdot \left[x - \frac{x^3}{3 \cdot a^2}\right]_{-a}^{a} = \frac{4 \cdot \pi \cdot b^2 \cdot a}{3} \, . \tag{4.12}$$

4.2. The volume of a segment of a sphere

Proposition 7 of the *Method* contains the statement with regard to the volume of the segment of a sphere, as follows [3]:

PROPOSITION 7. *Any segment of a sphere has to the cone with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment.*

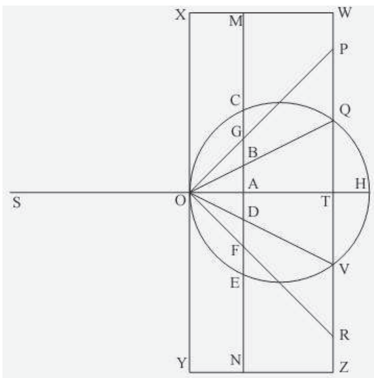

Figure 5

Primarily, our intention is to determine the volume of the segment of a sphere, its height is $OT = h$, its base is a circle on $QV = 2 \cdot \varrho$ as diameter (see *Figure 5*). We denote the radius of the sphere by R. The segment of the sphere is generated by revolution of the segment of the circle QOV about OH. We consider the cylinder $XYZW$ with height OT whose base is a circle on $WZ = 4 \cdot R$ as diameter. We consider the cone OPR with height OT whose base is a circle on PR as diameter, where $PR = 2 \cdot OT$. A variable plane MN at right angles to OH intersects the cone OPR in a circle on GF as diameter, the cylinder $XYZW$ in a circle on MN as diameter and the segment of the sphere in a circle on CE as diameter. Then $AC^2 = AO \cdot (2 \cdot R - AO)$, and we obtain:

$$\frac{AC^2 + AG^2}{AM^2} = \frac{AO \cdot (2 \cdot R - AO) + AO^2}{(2 \cdot R)^2} = \frac{AO}{2 \cdot R} . \tag{4.13}$$

Just as above, we apply the lever principle with $SO = OH$, and SH is considered as a balance with fulcrum O. From (4.13) we can see that the cylinder suo loco balances the combination of the cone OPR and the segment of the sphere, both of them placed in S. As the centre of gravity of the cylinder is located at $\frac{h}{2}$ away from the fulcrum, the equilibrum of the moments of forces yields the equality

$$\frac{h}{2} \cdot V_{cylinder(XYZW)} = 2 \cdot R \cdot \Big(V_{segment} + V_{cone(OPR)}\Big) . \tag{4.14}$$

Thus

$$V_{segment} = \frac{h}{4 \cdot R} \cdot V_{cylinder(XYZW)} - V_{cone(OPR)} . \tag{4.15}$$

Now we consider the cone OVQ which has its base the circle on $QV = 2 \cdot \varrho$ as diameter, and its height is $OT = h$. The following equalities hold:

$$\frac{V_{cone(OVQ)}}{V_{cone(OPR)}} = \frac{QT^2}{PT^2} = \frac{\varrho^2}{h^2} \qquad \frac{V_{cone(OVQ)}}{V_{cylinder(XYZW)}} = \frac{1}{3} \cdot \frac{QT^2}{WT^2} = \frac{\varrho^2}{12 \cdot R^2} . \tag{4.16}$$

From (4.15), (4.16) and the equality $\varrho^2 = h \cdot (2 \cdot R - h)$ we obtain

$$V_{segment} = \frac{h \cdot (3 \cdot R - h)}{\varrho^2} \cdot V_{cone(OVQ)} = \frac{3 \cdot R - h}{2 \cdot R - h} \cdot V_{cone(OVQ)} , \tag{4.17}$$

whence the statement

$$\frac{V_{segment}}{V_{cone(OVQ)}} = \frac{(2 \cdot R - h) + R}{(2 \cdot R - h)} . \tag{4.18}$$

of Proposition 7 follows.

Remark. *For a present day mathematician it is more straightforward to substitute in (4.15) the formulas for the volume of the cylinder XYZW and the cone OPR to obtain*

$$V_{segment} = \pi \cdot h^2 \cdot R - \frac{\pi \cdot h^3}{3} \,. \tag{4.19}$$

Taking into account that $\varrho^2 = h \cdot (2 \cdot R - h)$, *we have* $R = \frac{\varrho^2 + h^2}{2 \cdot h}$, *and (4.19) takes the form*

$$V_{segment} = \frac{\pi \cdot h \cdot \varrho^2}{2} + \frac{\pi \cdot h^3}{6} \,. \tag{4.20}$$

We have to mention that (4.20) is the well known formula for the volume of the segment of a sphere, which can be found in any students' textbooks.

We can also deduce (4.19) with the tools of definite integral calculus. We consider the graph of the function $f(x) = \sqrt{2 \cdot x \cdot R - x^2}$ *on the interval* $(0, h)$. *This graph by revolution about the x axis generates a segment of a sphere. The volume of the segment can be determined as follows:*

$$V = \pi \cdot \int_0^h (2 \cdot x \cdot R - x^2)\, dx = \pi \cdot \left[x^2 \cdot R - \frac{x^3}{3} \right]_0^h = \pi \cdot h^2 \cdot R - \frac{\pi \cdot h^3}{3} \,. \tag{4.21}$$

4.3. The volume of a segment of a spheroid

In Proposition 8 is merely stated that by this method it is also possible to find the volume of any segment of a spheroid [2]. We try to reproduce the heuristic conjecture concerning to the volume formula using the Archimedean methods.

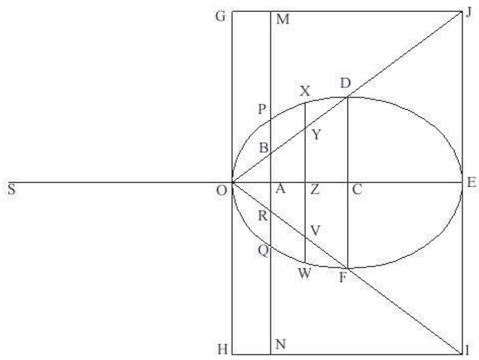

Figure 6

The segment of the spheroid is generated by revolution of the segment XOW of the ellipse about OE (the ellipse has its own axes denoted by $OE = 2 \cdot a$ and $FD = 2 \cdot b$, see *Figure 6*). This segment of spheroid has height $OZ = h$, and its base is the circle on XW as diameter. We consider a cylinder with height $OZ = h$ whose base is a circle on $GH = 4 \cdot b$ as diameter. The cone OYV has its height $OZ = h$, and its base is a circle on YV as diameter, where $YV = 2 \cdot h \cdot \frac{b}{a}$. A variable plane MN at right angles to OE intersects the segment of the spheroid, the cone and the cylinder in circles on PQ, BR and MN as diameters, respectively. The calculations are carried out in the same way as in the case of the spheroid and we get the following equalities:

$$\frac{AB^2 + AP^2}{AM^2} = \frac{OA}{OE} = \frac{OA}{OS} \ . \tag{4.22}$$

OS and OE are considered as the two arms of a lever with fulcrum O, such that $OS = OE$, and it is recognized that the cylinder suo loco balances the combination of the cone and of the segment of spheroid, both of them placed in the point S. Therefore we have

$$\left(V_{segment} + V_{cone}\right) \cdot OS = V_{cylinder} \cdot \frac{h}{2} \ , \tag{4.23}$$

whence

$$V_{segment} = \frac{h}{4 \cdot a} \cdot V_{cylinder} - V_{cone} \ . \tag{4.24}$$

A simple geometrical calculation leads to

$$V_{cone} = V_{cylinder} \cdot \frac{h^2}{12 \cdot a^2} \ . \tag{4.25}$$

From (4.24) and (4.25) we get

$$V_{segment} = \frac{h}{12 \cdot a^2} \cdot V_{cylinder} \cdot (3 \cdot a - h) \ . \tag{4.26}$$

The points X and D are situated on the oxytome, and by the symptom of the oxytome we have

$$\frac{XZ^2}{OZ \cdot ZE} = \frac{CD^2}{OC \cdot CE} = \alpha \ , \tag{4.27}$$

whence

$$XZ^2 = \frac{b^2}{a^2} \cdot h \cdot (2 \cdot a - h) \ . \tag{4.28}$$

We consider the cone which has its base and height in common with the segment of the spheroid and we denote its volume by $V_{cone(OXW)}$. Using (4.28) we obtain

$$V_{cone(OXW)} = V_{cylinder} \cdot \frac{h \cdot (2 \cdot a - h)}{12 \cdot a^2} \ . \tag{4.29}$$

From (4.26) and (4.29) it follows that

$$\frac{V_{segment}}{V_{cone(OXW)}} = \frac{3 \cdot a - h}{2 \cdot a - h} = \frac{(2 \cdot a - h) + a}{2 \cdot a - h} \ . \tag{4.30}$$

Remark. *A present day mathematician can get the following result by a substitution of the volume of the cylinder in (4.26):*

$$V_{segment} = \frac{\pi \cdot b^2 \cdot h^2}{3 \cdot a^2} \cdot (3 \cdot a - h) \ . \tag{4.31}$$

Formula (4.31) can also be obtained by integral calculus. We consider the graph of the function $f(x) = b \cdot \sqrt{1 - \frac{x^2}{a^2}}$ *on the interval* $(-a, -a + h)$*. This graph, by revolution about the x axis, generates a segment of an ellipsoid. The volume of the segment, namely formula (4.31), can be calculated as follows:*

$$V = \pi \cdot b^2 \cdot \int_{-a}^{-a+h} \left(1 - \frac{x^2}{a^2}\right) dx = \pi \cdot b^2 \cdot \left[x - \frac{x^3}{3 \cdot a^2}\right]_{-a}^{-a+h} = \frac{\pi \cdot b^2 \cdot h^2}{3 \cdot a^2} \cdot (3 \cdot a - h) \ .$$

5. Concluding methodological remark

a) Archimedes was perfectly aware of the fact that mechanically formulated conjectures are only heuristically accepted. As much as credible, without precise mathematical proofs they remain conjectures only. However, we can say these are forward-looking conjectures and the idea is worth more than the requirements of this problem. According to our present knowledge, the transition from the section to the entire body means the transition from differential calculus to the integral calculation. Archimedes' works also reveals that he knew the enclosure principle from the Riemann integral definition and he used this to determine the volume of the bodies. However, we can not say that he knew the concept of integral calculus, because his calculations are always bound to definite geometrical meaning, such as volumes and surfaces. We can not observe he recognized that a single concept forms the basis of all these geometrical interpretations.

b) The knowledge of the above-mentioned heuristic conjectures is very important for mathematics teachers. During the educational process, in many cases the presentation of heuristic conjectures is at least as important as the description of rigorous mathematical proofs. For secondary school students often there is no clear distinction between the heuristic argumentations and the rigorous proofs. We hope that these classical examples help the students clearly understand this distinction. In the educational processes, the Mathematics or Physics teachers with the method of the lever principle can stimulate their students to make their own heuristic conjectures and determine the volume of the above mentioned three-dimensional bodies, even though the students know nothing about the integral calculus. However, the examples described above are eloquent proof of the relationship between Mathematics and Physical science. It is also worth mentioning, as Archimedes draws a distinction between mechanical and mathematical ideas. Knowing his train of thought is helpful in Mathematics and Physics educational processes.

Acknowledgements

The author is grateful to Szalay István and to the anonymous reviewers for their useful suggestions and comments, which have improved the presentation of the paper.

References

[1] A. K. T. Assis and C. P. Magnaghi, *The Illustrated Method of Archimedes*, Aperion, Montreal, 2012, `www.ifi.unicamp.br/~assis`.

[2] E. J. Dijksterhuis, *Archimedes*, Princeton University Press, Princeton, 1987.

[3] T. L. Heath, *The Method of Archimedes Recently Discovered by Heiberg, A Supplement to the Works of Archimedes*, Cambridge University Press, 1912.

[4] T. L. Heath, *The Works of Archimedes with the Method of Archimedes*, Dover Publications, New York, 1953.

[5] L. Klukovits, The first clear distinction between the heuristic conjecture and the deductive proof in the ancient mathematics, *Teaching Mathematics and Computer Science* (2004).

[6] L. Klukovits, A heurisztikus sejtés és a szigorú bizonyítás együttes megjelenése az ókori görögöknél, *Polygon* **XIV**, no. 1, Szeged (2005).

[7] G. Pólya, *Mathematics and Plausible Reasoning*, Vol. I, Induction and Analogy in Mathematics, Princeton University Press, Princeton, 1968.

[8] M. Raugh, Archimedes 'Law of the Lever' and his Mysterious Mechanical Method for Finding the Volume of a Sphere,
`http://www.mikeraugh.org/Talks/UNM-2012-LawOfTheLever.pdf`.

[9] B. L. van der Waerden, *Science Awakening*, Oxford University Press, New York, 1961.

ZSOLT FÜLÖP
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
HUNGARY

E-mail: fulop.zs32@freemail.hu

(Received January, 2014)