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Teaching
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The use of different representations in teaching Algebra, 9th grade (14-15 years old)

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Abstract. Learning Algebra causes many difficulties for students. For most of them Algebra means rote memorizing and applying several rules without understanding them which is a great danger in teaching Algebra. Using only symbolic representations and neglecting the enactive and iconic ones is a great danger in teaching Algebra, too. The latter two have a primary importance for average students.

In our study, we report about an action research carried out in a grade 9 class in a secondary school in Hungary.The results show that the use of enactive and iconic representations in algebra teaching develops the students' applicable knowledge, their problem solving knowledge and their problem solving ability.

Key words and phrases: Binomial formulas, algebra teaching, representations, problem solving.

ZDM Subject Classification: H20, C30, C50, U60, U80.

Introduction

Algebra is an important part of secondary Mathematics curriculum. It is the fundamental base for solving equations, inequalities, systems of equations. It is also important for functions, differential and integral calculus and for analytic geometry and because it is essential in applying different area and volume formulas it plays a dominant role in geometry too. The main power of Algebra lies in its symbolic nature. In this situation where many child hates Algebra, they memorize the different algebraic rules and their applications in exercises by rote. This statement is true mainly for average and low achiever students.

In the following study, I report an action research carried out in a Grade 9 class in a secondary school in Paks.

The main aim of this research was to demonstrate, that the use of concrete and visual representations in Algebra teaching is an effective teaching and learning tool for average pupils. The investigated topic was teaching special products. These formulas play an important role in higher grades in different topics such as analytic geometry, combinatorics, statistics and probability, differential and integral calculus. Although the algebraic identities are important in mathematics, the curriculum consists of only five lessons traditionally. The algebraic approach dominates these lessons. In our experiment we taught the identities in 13 lessons with algebraic and geometric methods. I am convinced that the invested work has paid off and students will be able to recall these formulas successfully in the future.

The mathematics textbooks formulate the algebraic identities in one direction by textual words. However, they do not examine them in reverse direction, for instance, with regard to their relationship with the presentation of the identities. In our study, we focus on the teaching and learning experiences of a square of a two-term sum and two-term difference, the cube of a two-term sum and two-term difference, analyzing not only mathematical but also organizational, motivational and emotional factors too.

Theoretical background

We considered various scientific theories in our research.

The main aim of mathematics teaching is to build mathematical proficiency. So I kept in mind Schoenfeld's theory. Schoenfeld formulated it on the basis of the following five aspects:

Conceptual understanding: comprehension of mathematical concepts, operations and relations

Procedural fluency: skill in carrying out procedure flexibly, accurately, efficiently and appropriately

Strategic competence: ability to formulate, represent and solve mathematical problems

Adaptive reasoning: capacity for logical thought, reflection, explanation and justification

Productive disposition: habitual inclination to see mathematics as a sensible, useful, worthwhile belief in diligence and one's own efficacy. (Schoenfeld, 2007 [1])

In our teaching we applied the three-component model about algebraic activity that was defined by C. Kieran.

Generational activity: generating expression and equations

Transformational activity: factorizing, manipulating and simplifying algebraic expressions and solving equations. These activities are predominantly concerned with equivalence, form and preservation of essence.

Global/meta-level activity: awareness of mathematical structure, awareness of constrains of problem situations, justifying, proving and predicting problem solving. These activities are not exclusive to algebra. (Kieran, 1996 [2])

The essence of our research was the use of enactive and iconic representations together with symbolic ones. (Bruner, 1966 [3]) According to Bruner, there are different levels of representations of any problems (within a domain of knowledge).

Enactive (material) level: means representations by a set of actions appropriate for achieving a certain result.

Iconic (visual) level: is a set of summary images or graphics that stand for a concept without defining it fully.

Symbolic level: is a set of symbolic manipulations on logical propositions drawn from a symbolic system that is governed by rules and laws for forming and transforming propositions.

Bruner's representations are external representations. Internal (mental) representations are their memory codes in our memory system. Most implications from brain research for mathematics teaching and learning are considered as important aspects of our research action. Teaching tactics are formulated by Bender in 10 points. (Bender, 2009 [4])

Brain-Compatible Guidelines for Math Instruction

(1) Less is more!

(2) Present informations at three levels (concrete, visual, abstract)

(3) Teach the *big ideas* in mathematics

(4) Emphasize mathematical patterns

(5) Teach mathematical facts to a high level of automaticity

(6) Use novelty to build on students' strengths

(7) Teach algorithms explicitly

(8) Teach to both brain hemispheres (left: abstract, logical, sequential; right: visual, creative, intuitive)

(9) Scaffold the students' practice

(10) Understand the fear and explore the beauty.

We learn Maths and everything with our brain so we cannot ignore the systems of memory and content of memory. Types of memory structures are:

Sensory memory (Perceptual memory): holds information only for a fraction of a second.

Working memory: holds informations for some seconds. Working memory has a limited capacity of 7 ± 2 info-units for a few minutes. There are opportunities to increase the working memory capacity by *chunking (compressing informations into greater units)*, *automaticity* and *using different brain regions to free the working memory.* This part of our brain is responsible for thinking and problem solving. It is a workbench of our brain. (Figure 1) The central executive can be regarded as a Supervisory Attention System. Its main functions are goal setting, planning, organizing, prioritizing, initiating, holding information, inhibiting irrelevant information, self-monitoring, memorizing, self-regulating, representing, problem solving. (Baddeley, 1999 [7])

Long-term memory: holds informations for hours, days, months, years.

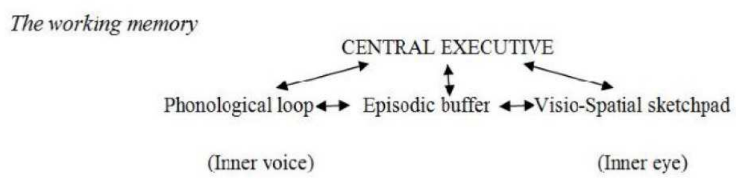

Figure 1. working memory

Content memory:

Explicit memory, Semantic memory: means our knowledge about the world. (Concepts, symbols, operations with symbols, meaning of symbols)

Episodic memory: means holding events, which happen at time and space. We are convinced that the students' actions with concrete materials positively contributed to remembering the binomial formulas after a couple of weeks, too.

Implicit memory, Procedural memory: means knowing (remembering) skills, rule based manipulations.

Research hypotheses

The material and visual representations accompanied by conscious use of textual formulations are more effective in Algebra teaching than using symbolic method only. The students are better at remembering what they learnt and they can apply the mathematical identities in both directions effectively.

Research methodology

The action research took place in October 2012, in a 9^{th} grade class (students were 14-15 years old), in a Hungarian secondary school, in Paks. At the beginning of this experiment, in October 2012, the Math group consisted of 15 members. 13 students were average ability but two of them were talented in Math. During the experiment - 13 lessons - I was teaching them 5 lessons per week. At the manipulative activities, students used colored paper and plasticine, while they worked at the interactive whiteboard as well. The students were working together in pairs, in teamwork with fixed members and individually. We chose the fixed pairs and teams based on a pretest. The methods of data collections were observation of the teacher, video recording with sound recorder, analyzing students' documents, personal communication, analyzing exercise books, notebooks with students' commentaries, mathematical pretest, posttest, and delayed test.

Description of the research

Learning trajectory

(1) The square of a binomial sum,

(2) The square of a binomial difference,

(3) The cube of a binomial sum,

(4) The cube of a binomial difference,

(5) Pascal's triangle, binomial coefficients,

(6) The square of a trinomial,

(7) The binomial difference of squares,

(8) The binomial sum of cubes,

(9) The binomial difference of cubes,

(10) Solving Math "word problems" with special products.

Introducing a geometrical method in teaching special products

In the first five lessons, students represented the binomial identities in algebraic and geometric ways. We started the work with concrete numbers. Every student had a squared exercise book.

The starting open question was the following: How can you calculate $(5+2)^2$, if you think of it like the square of a binomial sum? Students did not receive more instructions.

One student (H. Á.) used a visual representation in his exercise book. He drew a square with sides of 5-units and extended the perpendicular sides with 2-units at one vertex. He linked the $(5+2)^2 = 5^2 + 2 \times 2 \times 5 + 2^2$ calculation with area of the shapes. He presented his solution on the interactive whiteboard where he used the grid background. The other students copied his solution into their Math exercise books.

Then, I gave the students a blank coloured cardboard sheet whose shape was a 7 cm by 7 cm square so they could represent what they had seen on the interactive whiteboard. The students were working together in fixed pairs. The ones sitting next to each other formed a fixed pair. (One group consisted of three members). They cut the specific geometric forms from the coloured cardboard sheet, and wrote the area measures into the corresponding squares and rectangles. Then we checked the square of a binomial sum identity with another pair of concrete numbers. The students were asked to find out how the area of a 4 cm by 4 cm square changed if its adjacent edges at a selected vertex were extended by 6 cm. We projected the task and the geometric figures on the interactive whiteboard. The children drew a picture of the model into their exercise books and a volunteer student (M. P.) placed the geometric forms to the right place on the interactive board. We used the mathematical symbols in one direction, $(a+b)^2 = a^2+2ab+b^2$ and we reached the solution by multiplying polynomials.

Only one pair (H.Á. and D. Á.) could put the mathematical identity into textual words in this direction correctly. The rest of the students said *adding twice the sum of the first and second terms* instead of *adding twice the product of the first and second terms* in textual words although their drawings were good. So we practiced the rules in textual words together and reciting it aloud then the students wrote the correct answers in their exercise books.

After these, we studied the three-term sum $a^2 + 2ab + b^2$. The starting *open question* was the following: how can we factorize the three-term sum? (a and b are *positive real numbers*) I gave coloured sheets (three 10 cm by 10 cm squares one yellow, one blue and one red cardboard) to the students again and they were free to choose how to solve the task. Three students (V. M., H. Á., D. Á.) used the algebraic method, $a^2+2ab+b^2 = a^2+ab+ab+b^2 = a(a+b)+ba(a+b) = (a+b)(a+b) = (a+b)^2$. 12 students cut out the shapes and they wrote the area of the square $(a+b)^2$ into their exercise book. They also drew a square. After this we put the algebraic rule into textual words and wrote it down. First, when practicing algebraic identities the teacher assigned the tasks. After that, the students gave tasks to each other. During practice time, everybody checked their own work and corrected it with a colored pen. Therefore, they could focus their attention on their own typical mistakes.

As the next step working with the square of a binomial difference, students began to work with the manipulative activities in fixed groups. I described a group activity task. The groups received one 10 cm by 10 cm square of a yellow cardboard, two 10 cm by 3 cm rectangles of blue cardboards and one 3 cm by 3 cm square of red cardboards and two extra 10 cm by 10 cm squares of blue and red cardboards, a pair of scissors and a ruler.

Student A's task was:: How can you illustrate $(10-3)^2$ with areas?

Student B's task was: How can you formulate a square's area, if its sides are $a-b$ long? $(a-b)^2 =$? How can you put it into textual words?

Student C's task was: You have four shapes. One square, the area of this square is a^2, another square, the area of this square is b^2 and two rectangles, the area of these rectangles is $a \times b$. How can you represent with the help of these areas square whose area is $(a-b)^2$? How can you put it into textual words?

Student D's task was: to check the answer. Everybody wrote the results in their own exercise book. (In the three-member group, the students checked each others results.)

One group (V. M.; M.I.; K.K.) started the work with the 10 by 10 square. One member of the group (M. I.) chose folding. He selected a vertex. Then he measured 3 cm from the selected vertex and folded the cardboard 3 cm from edges along the adjacent sides of the square. Therefore, he represented a square with sides $(10-3)$. Another student from the group (V.M.) noticed that *double-folding* occurred at the selected vertex. Therefore, the 3 by 3 square appeared in the vertex twice because she folded this square's area twice, therefore she needed to add it. (Figure 2)

Figure 2. Representation of $(10-3)^2$

The other groups started working with one 10 by 10 yellow cardboard square two 10 by 3 blue cardboard rectangles and one 3 by 3 red cardboard square. One group (T.K., J.B., P.V., M. P.) thought that they could not tell how they solved the problem, they could only demonstrate it. (M. P.) explained her manipulative activities. First, she put the yellow cardboard on the table. She also selected one vertex. She put the two blue cardboards in the selected corner so that they were perpendicular to each other. After this she put the red cardboard in the corner above the square that appeared twice. She explained her manipulative activity by textual words. The rest of the group was happy about the solution. *'Yes, yes this is good!'* - The others from the group said.

This idea in 2 dimensional space is important in 3 dimensional space as well for demonstrating the cube of a binomial difference.

The students drew the geometric figure representing the special binomial sum. They proved the algebraic identity with multiplying polynomials. Thereafter they put the identity into textual words in two ways. I projected the tasks on the interactive whiteboard and I gave the groups printed tasks. First, the students talked about their opinion then we discussed the results at the interactive whiteboard. Finally, we practiced the mathematical identities. As we progressed in the curriculum, they started to remember the identities with the letters.

Teaching the $(a \pm b)^3$ identities

The students were working with cubing a binomial sum. It was a pair work. First, they used an algebraic polynomial multiplication. Finally, in cubing a binomial sum, they made use of the following equation: $(a+b)^3 = a^3 + 3a^3b + 3ab^2 + b^3$

The students could see only the "letters", the four terms (as shown above). They put this algebraic expressions into textual words *in both directions*: $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$ also. For example *one direction*: Cubing a binomial sum equals the first term raised to third power plus three times the product of the square of the first term multiplied by the second term plus three times the product of the first term multiplied by the square of the second term plus the second term raised to third power. First, only one student (H.Á) could put the identity into textual words in both directions. We practiced this a lot with them! These textual words helped the students remember the letters and the identity. It is even more useful if manipulative activities are linked with words. Therefore, after cubing a binomial sum, we also used the manipulation activities. (Figure 3)

Figure 3. The representation of both sides of the identity

Students could observe the assembly and disassembly of a large plastic cube, so they saw the representation of both sides of the identity in the 3-dimensional Euclidian space. During the manipulation activities student (T.Zs.) named the three-dimensional solids.

The following *task* was teamwork: There are eight plastic geometric solids in front of you on the teacher's desk (Figure 3). You have to make a cube out of plasticine. One side of a cube is 3 cm long. What happens to the volume of the cube when each edge is extended with 1 cm? Calculate the volume of the large cube! $(3\text{cm} + 1\text{cm})^3 =?$ Every student had to calculate the volume of the large cube in two different ways and after that, they had to write their calculations into their exercise book. A member of the group had to demonstrate on the interactive whiteboard how they solved the task. (Figure 4)

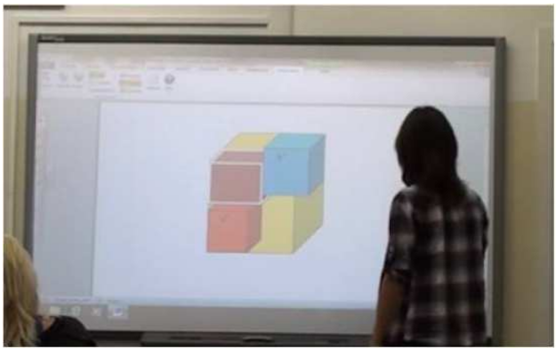

Figure 4. At the interactive whiteboard

The students received solids 2 little coloured cubes $(1 \times 1 \times 1; 3 \times 3 \times 3)$ and 3 $(1 \times 3 \times 3)$ cuboids, 3 $(1 \times 1 \times 3)$ cuboids and tried to draw what they saw. (Figure 5)

Figure 5. The manipulative activity

Ten students drew a representation of three-dimensional cubes or cuboids in their exercise books. Five students wrote the following algebraic symbols inside the squares or rectangles $a^3, b^3, 3 \times a^2b, 3 \times ab^2$. (Figure 6)

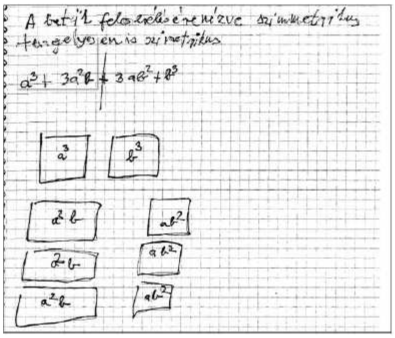

Figure 6. The algebraic symbols inside the squares and rectangles

The next task was: "Which mathematical identity is obtained if in cubing a binomial sum, like $(a + b)^3$, we replace $+b$ with $-b$? (Individual work) In this case, the children were free to choose from using the three kinds of Bruner's representations. 14 students chose to derive an algebraic polynomial multiplication. Only one student (H.Á.) used all three-representations. (Material, visual, symbolic representations)

We wanted to avoid students mixing the identities, so first they learned about the Pascal's triangle, binomial coefficients and the square of a trinomial, then they learned the binomial difference of squares and finally we taught the binomial sum and difference of cubes.

Summary of results, conclusions, plans

We are convinced that the students' manipulations with concrete materials contributed in a positive manner to remembering the binomial identities after some weeks, too.

In April, 2013 the students completed a delayed test on the binomial identities. My question was, whether the knowledge could be stored in the long-term memory this way? The questions text of the posttest and delayed test can be found in Appendix A. The results are summarized in the next table: In the first two tasks, the students had to recognize one of the binomial identities in both directions. We found that all students succeeded in both directions in tasks where the coefficient was not a fraction. Fraction coefficients caused more problems. One student put all the tasks in words (P. L). One Student (H. Á.) responded

Posttest and delayed test	Posttest results in October 2012	Delayed test results in April 2013
1. task: a, b, c, d	91.6 %	81.6 %
1. task e, f	66.6 %	56.6 %
1. task g	60 %	40 %
2. task a, b, c, e,	80 %	78 %
2. task d	53 %	40 %
3. task a, b, c	71 %	64 %
4. task	51.3 %	51.3 %
5. task	53.3 %	46.6 %
6. task	46.6 %	40 %

with words, with graphs and with algebraic identities, too. Thirteen students used the mixture of what they had learned. The students drew geometric figures for solving the fourth and the fifth tasks. Two students made a dimensional mistake, the perimeter of the shape was calculated instead of the area or the surface area instead of the volume. These results are promising, when using traditional teaching students do not remember the learned material so effectively.

At the end of the action research, we may state that our hypotheses are right:

The material and visual representations accompanied by conscious use of textual formulations are more effective in teaching Algebra than using the symbolic method only. The students are better at remembering the formulas and they can apply the mathematical identities in both directions effectively.

The analyzis of organizational, motivational and emotional factors

We reviewed the students' notebooks case-by-case. Here are the students' reflections on the action research with our comments:

We found that there was a positive change in the students' attitude towards Algebra as the consequence of the research teaching. (D. Á.) wrote in his student notebook. "So far, I didn't like Algebra, but now I understood everything, liked the lessons. I can also handle the difficult tasks with the methods we have just learned." The students practiced at home. (K. K.) wrote in his student notebook. "During lessons in school I have not been so successful in the tasks, but after practicing at home, it is better now." The students' competence gradually improved. (Sz. J.) wrote in his student notebook: "I did not understand the cubing of a binomial sum earlier, but now I'm beginning to grasp it." The students were encouraged to think independently when doing the manipulative activities.

"Solving "word problems" was difficult for me. But thinking about it again it is not so dangerous." - (Sz. J.) said.

I will continue using manipulative activities with high school students (14-18 years old), when it is possible.

References

[1] A. H. Schoenfeld (editor), *Assessing Mathematical Proficiency*, Cambridge University Press, Cambridge, 2007, 5, 56–57.

[2] C. Kieran, The changing face of school algebra, 1996, invited lecture presented at Eighth International Congress on Mathematics Education Seville, Spain.

[3] J. S. Bruner, *Towards a Theory of Instruction*, Harvard University Press, 1966.

[4] W. N. Bender, *Differentiating Mathematics Instruction*, Corvin Press Thousand Oaks, 2009.

[5] Z. P. Dienes, *Építsük fel a matematikát*, SHL Kft, Budapest, 1999, 109.

[6] E. Jensen, *Teaching with the brain in the mind*, Association for Supervision and Curriculum Development Alexandria, Virginia USA, 1998, 104–112.

[7] A. D. Baddeley and R. H. Logie, *Working memory: The multiple component model*, in: *Model of Working Memory, Mechanism of active maintenance and executive control*, (A. Myake, P. Shah, eds.), Cambridge University Press, Cambridge, 1999, 28.

[8] A. Ambrus, *Bevezetés a matematika didaktikába*, Egyetemi jegyzet, ELTE Eötvös kiadó, Budapest, 2004, 38–39.

[9] A. Paivio, *Draft chapter for the conference on "Pathways to Literacy Achievement for High Poverty Children"*, University of Western Ontario, The University of Michigan, School of Education, September 29 - October 1 2006, 1–17.

[10] P. Bednorz and M. Schuster, *Bevezetés a tanulás lélektanába*, Medicina Könyvkiadó Rt., Budapest, 2006, 173–203.

[11] T. Johnson, *Teaching mathematics with the brain in mind: learning pure mathematics with meaning and understanding*, Lethbridge, Alberta, 2003.

[12] É. Vásárhelyi, *Terület, térfogat, felszín; Demonstráció és kísérletezés*, 2012, `mathded.elte.hu`.

Appendix A. Posttest and delayed test

(1) Which special products are the following?
   (a) $(a+b)^2$;
   (b) $(x+1)^2$;

   (c) $(1-2y)^2$;
   (d) $(x-1)\times(x+1)$;
   (e) $(x+3)^3$;
   (f) $(\frac{2}{3}\cdot x-\frac{1}{2})^2$;
   (g) $(a+b)^4$
(2) Factorize the following algebraic expressions.
   (a) $x^2-4=$
   (b) $4x^2+1-4x=$
   (c) $f^2+6f+9=$
   (d) $\frac{4}{25}x^2+\frac{4}{5}xy+y^2=$
   (e) $4+k^2+a^2+4k+4a+2ak=$
(3) There are some terms missing from the following identities. How can you complete the algebraic identities with the correct terms?
   (a) $x^2-\cdots+25=\cdots$;
   (b) $\cdots-n^2=(11\cdots$;
   (c) $n^3-\cdots+3n-1=\cdots$
(4) What is area of a square whose side is a cm long? Choose one vertex of the square. How does the area of the square change if we increase one edge by 5 cm and decrease the other edge by 5 cm at the chosen vertex? Calculate the area of the new polygon. ($a>5$ cm)
(5) Calculate the volume of a cube if the length of an edge is a. Choose one vertex of the cube. How does the volume of the cube change if you increase the edges by b? Calculate the volume of the new 3-D figure. By how much did the cube's volume change? (Help: Test it by $a=5$ and $b=2$.)
(6) The sum of two numbers is 2.5; their product is 1. Find the square of the sum of the two numbers.

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