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Teaching Mathematics and Computer Science

Central axonometry in engineer training and engineering practice

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Abstract. This paper is concerned with showing a unified approach for teaching central and parallel projections of the space to the plane giving special emphasis to engineer training. The basis for unification is provided by the analogies between central axonometry and parallel axonometry. Since the concept of central axonometry is not widely known in engineering practice it is necessary to introduce it during the education phase. When teaching axonometries dynamic geometry software can also be used in an interactive way. We shall provide a method to demonstrate the basic constructions of various axonometries and use these computer applications to highlight their similarities. Our paper sheds light on the advantages of a unified approach in such areas of engineering practice as making hand drawn plans and using CAD-systems.

Key words and phrases: teaching descriptive geometry, axonometry, central axonometry, dinamic geometry.

ZDM Subject Classification: G10, G20, G40, G80, U50, U60, U70.

Introduction

Central axonometry can be considered as a projective generalization of classical axonometry. Axonometry is a surjective collinear mapping of the affine space A^3 onto the affine plane A^2 where to the orthonormal basis $(\mathsf{O}; \mathsf{E}_1; \mathsf{E}_2; \mathsf{E}_3)$ of the space, the quadruplet of non collinear points $(\mathsf{O}^p; \mathsf{E}_1^p; \mathsf{E}_2^p; \mathsf{E}_3^p)$ is assigned. If the

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plane and the space are extended with their objects at infinity to a projective plane and space we can obtain the following generalisation of axonometry. To the orthonormal basis of the space and the quadruplet of points in the plane, the vanishing points of the axes are added to obtain the following mapping:

$$P^3 \longrightarrow P^2$$

$$(O; E_1; E_2; E_3; V_1; V_2; V_3) \rightarrow (O^p; E_1^p; E_2^p; E_3^p; V_1^p; V_2^p; V_3^p)$$

where $O^p; E_i^p; V_i^p$ are collinear for every i.

This mapping is called central axonometry.

The similarity between the two mappings can be seen in the constructions. While classical axonometry can come from parallel projection, central axonometry can be considered in special cases - but not in general - to be a central projection (see Fig. 1). This is why we use the words parallel and central to distinguish the two axonometries.

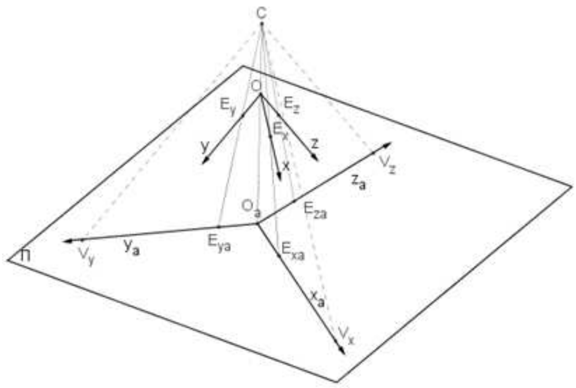

Figure 1

The two mappings are similar both in their properties and fundamental theorems as well as construction methods, so parallel and central projections can be treated with a unified approach. A generalized axonometric approach may prove useful in making correct hand drawn plans and sketches. Familiarity with central axonometry makes the use of CAD-systems easier, where modelling is done in three dimensional Cartesien coordinate systems, often with coordinates, and the image can be a parallel or central projection.

In engineering terminology axonometry mostly means parallel axonometry, central axonometry is not present even at the engineer training level. To utilize the advantages, the concept of central axonometry can be introduced to engineer training and practice using the above mentioned analogies and possibilities.

Fundamental theorems of axonometry

Ever since the concept of axonometry was developed it has been in the focus of research whether the mapping can be considered a projection of the space to the plane, and if not, when is axonometry a projection. The well-known Pohlke's theorem states that parallel axonometry can always be regarded as a parallel projection. For central axonometry there is a theorem due to Kruppa [4] which is very similar but considerably stricter than that of Pohlke.

Pohlke's theorem: The parallel axonometric image of an object is always similar to a well-defined parallel projection of that object.

First theorem of Kruppa: The central axonometric image of an object is always affine to a well-defined central projection of that object as long as the direction points of the axes are not collinear. In the case of collinearity instead of affinity the weaker projective correspondance will hold.

It has been shown that Kruppa's theorem cannot be made stronger, that is affinity cannot be replaced with similarity which we have in Pohlke's theorem for parallel axonometry. In practice this implies that the unit points on the axes of central axonometry cannot be arbitrarily chosen. After this studies were aimed at finding the conditions the object of central axonometry has to satisfy so that it should be a central projection. In the followings we shall give two important and practically useful analytical conditions by Szabó [8] and Stiefel [6].

Szabó's condition: With the notation used in the left-hand side of Fig. 2 central axonometry is a central projection if and only if

$$\left(\frac{e}{f}\right)^2 : \left(\frac{g}{h}\right)^2 : \left(\frac{i}{j}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma$$

Stiefel's condition, which was attained earlier than the one above, concerns the special case when the vanishing point V_z of the vertical axis z is infinite, that is at central projection the axis z is parallel to the image plane. It has been proved in [2] that this theorem is a limiting case of the previous condition so it can be derived from that.

Stiefel's theorem: With the notation used in the right-hand side of Fig. 2 a central axonometry where the point V_z is the point at infinity of the axis z and

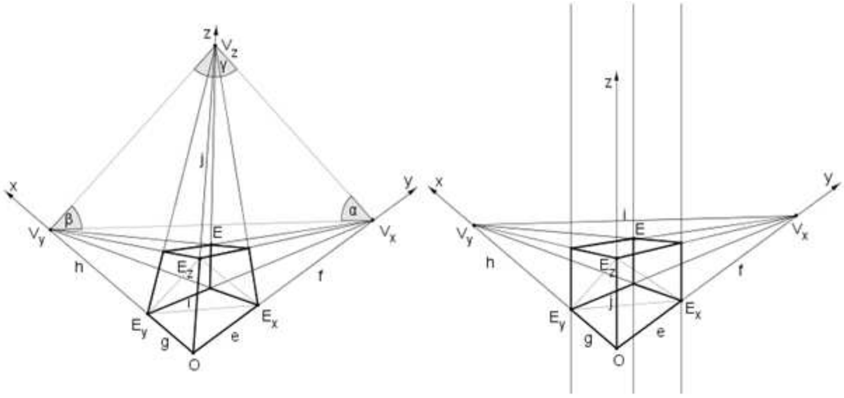

Figure 2

the axis z is perpendicular to the line V_xV_y is a central projection if and only if

$$\left(\frac{f}{e}\right)^2 + \left(\frac{h}{g}\right)^2 = \left(\frac{j}{i}\right)^2.$$

The central axonometric analogue of Gauss' theorem on the unit points of orthogonal axonometry was formulated by Dür [1]. Stachel gave a new proof of this theorem, its corollaries and its connections with the above theorems [7].

Until recently besides the analytic and synthetic conditions there was no elementary geometric theorem in this topic. Hoffman and Yiu [3] gave a condition that can be used to check with elementary geometric constructions whether a central axonometric image is a central projection. The theorem makes it possible to move the object of central axonometry and determine how the positions of the points of unity change when they are moved.

On handmade drawings of cubes

In engineering practice the problem whether axonometry can be considered as a result of a projection has a significance when making handmade drawings.

The main point in a handmade drawing or sketch is illustration. Therefore, although by Pohlke's theorem in an axonometric image of a cube any degree of foreshortening may occur, in handmade drawings or engineering sketches we see images where there are no great distortions along the axes and the angles of the axes are evenly distributed. For laymen an axonometric image with great distortions and extreme axis angles is not acceptable. Likewise, those unfamiliar with the rules of axonometry would never draw a picture like that.

When making handmade perspective drawings of cubes one cannot expect total accuracy in lengths since, as it has been shown, it can only be achieved with constructions. However it is important to avoid great mistakes in lengths or to take the rules of central axonometry into consideration. One such rule is, for instance, that the direction triangle has to be acute. Those unfamiliar with central axonometry may ignore these considerations. Surprisingly enough inexperienced people regard the perspective image of a cube unacceptable only when the foreshortening errors are huge or the triangle of the vanishing lines is very distorted. Also, they will reject - like in parallel axonometry - a geometrically correct image that is greatly distorted or has large axis angles.

Coordinates of a point

If the unit points of a parallel or central axonometry have been given - either in construction or approximately in a handmade drawing - any coordinate point on the axes can be constructed or drawn. These coordinate points can be determined with division ratios for parallel projections and with cross ratios for central projections.

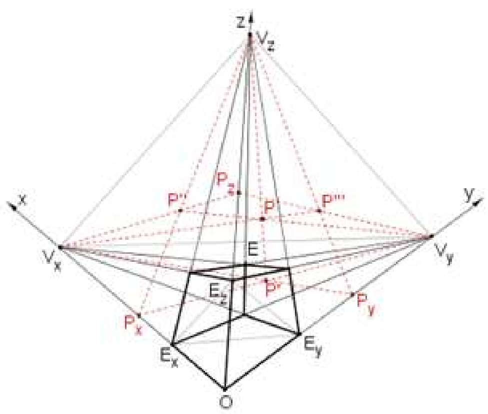

Figure 3

In practice, for parallel axonometry this means proportional division while for central axonometries the division of line segments has to be done according to the rules of perspective construction.

After determining the axonometric image of the coordinates of a point $P(P_x; P_y; P_z)$, the projection of the point can be given with its projector prism (Fig. 3).

Determining the unit points of the reference system with revolution

There are several methods known to determine the unit points of the axonometric reference system. We shall discuss the method of revolving the coordinate planes and how it can be generalized for various axonometries. We note that the method can be used to give the axonometric image of a point with any coordinates and even that of a complete object, but now, as indicated above, we shall concentrate on the unit points only.

For orthogonal axonometry it is well-known how to determine the unit points of the reference system. Orthogonal axonometry is uniquely determined by its tracing triangle (we suppose that the origin is in front of the image plane), the images of the axes are incident with the altitude lines of the tracing triangle and the image of the origin is the intersection of these lines. In the left-hand side of Fig. 4 the unit points of axes x and y were determined with the revolution of their coordinate plane and the unit point of axis z was given with the new image plane parallel to it.

In the following, for oblique and central axonometry, we shall use the fact that the position of the three dimensional Cartesien coordinate system is uniquely determined by a tracing triangle, so the revolution of the coordinate planes and the construction of the side view can be done with the reference system of the orthogonal axonometry.

For oblique axonometry it is not enough to give the tracing triangle, the projection of the origin is also needed since the tracing triangle gives the spatial position of the reference system but not the direction of the projection. The images of the axes are uniquely determined by the tracing triangle and the image of the origin. In the right-hand side of Fig. 4 the orthogonal projection of the origin O_m of the axonometric reference system belonging to the tracing triangle is shown. With this point the revolution of the xy plane and the origin can be done. The line $O(O)$ defines the direction of affinity belonging to the oblique

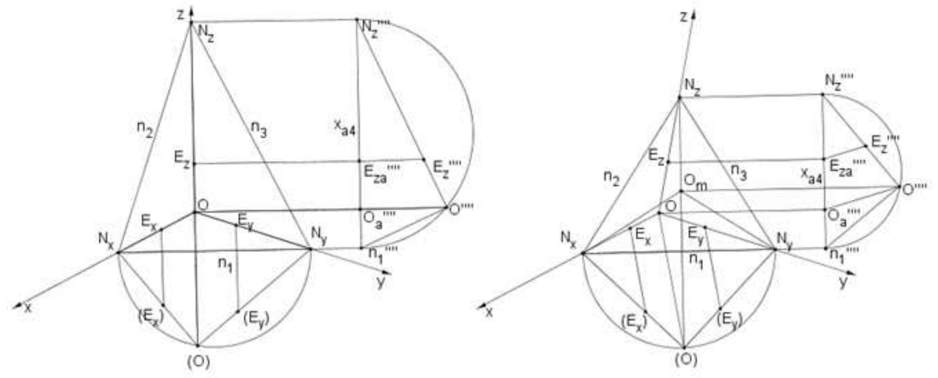

Figure 4

axonometry so the backward revolution of the unit points can be done for oblique axonometry as well.

Similarly, with the point O_m the sideview image of the axonometric reference system can be created and also the sideview image O_a^{IV} of the oblique projection of the origin can be constructed. The direction of projection is given by $O^{IV}O_a^{IV}$ in the fourth view so the projection of the unit point of axis z in the sideview image can be done and then it can be determined from the obtained point E_{za}^{IV} with the projection ray.

In central axonometry, (Fig. 5) besides the tracing triangle and the orthogonal projection of the origin the vanishing lines of the coordinate planes must also be taken, and according to the rules of perspective constructions, must be parallel to the tracing lines. The orthocenter of the so obtained vanishing triangle is the orthogonal projection of the centre of projection. The points O, O_m, C are collinear. After the revolution of the xy coordinate plane the centre is also revolved, which can be done with the rotation of the line OC, with Thales' circle or using the fact that the lines $N_x(O), V_x(C)$ and $N_y(O), V_y(C)$ are pairwise parallel and that the points $O, (O), (C)$ are collinear. Thus we have obtained the centre of collineation emanating during the revolution, with which the axonometric images of the unit points can be constructed.

During the construction of the unit point of the axis z in the sideview image after finding O^{IV} in the earlier mentioned way we need to determine C^{IV} as well. For this, again, we can use Thales' circle, the fact that the lines $n_1^{IV}O^{IV}$, $v_1^{IV}C^{IV}$ and $N_z^{IV}O^{IV}$, $V_z^{IV}C^{IV}$ are pairwise parallel or the collinearity of the points

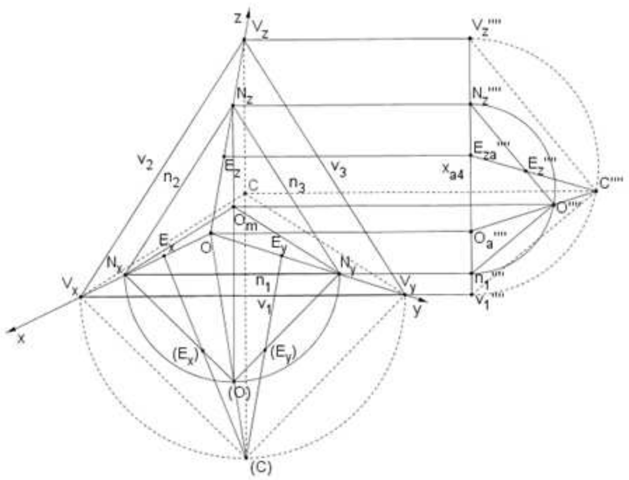

Figure 5

$\mathsf{O}_a^{\mathrm{IV}}, \mathsf{O}^{\mathrm{IV}}$ and C^{IV}. The central projection of the point E_z can be done in the fourth view, and from the obtained point $\mathsf{E}_{za}^{\mathrm{IV}}$ the point E_z can be determined with projections rays.

In these two construction steps one can recognise the method of revolution of the base plane of a practical perspective with inclined image plane and the construction method for marking off distance on vertical lines.

From the constructions the position of the origin of the three dimensional coordinate system, the direction of projection as well as the spatial position of the projection centre can be determined.

Visualisation of constructions in DGS

Dynamic Geometry Software are geometric programs that are capable of demonstrating the analogues of the above mentioned constructions on a computer. In a construction made in DGS the geometric relations between the elements are stored, the arbitrarily taken elements can be moved while the construction and the position of the dependent elements follow the movements.

The left-hand side of Fig. 6 shows the construction concerning central axonometry discussed above in DGS. As it was mentioned the axonometry was given by the vertices of the tracing triangle, the projected image of the origin and the vanishing point of axis z - for reasons mentioned at the construction method, the other vanishing points can be determined from these. So these points are freely movable points of the construction, their movement is followed in the image, the position of the unit points changes according to the geometric relations determined in the construction. For the sake of illustration we also included the image of the unit cube, on which the implications of moving the point determining the axonometry can also be followed.

If we move the point V_z to infinity, the vanishing points of the other two axes as well as the centre and its revolved image moves to infinity. Due to the moving the central projection rays in the construction become parallel projection rays, so instead of central axonometry we get parallel axonometry (right-hand side of Fig. 6). Just like one cannot mark a point at infinity on a piece of paper with a pencil, the computer mouse cannot move a point to infinity in the plane of the drawing. So the point V_z is not freely marked but is the result of a construction at the limiting case of which it goes to infinity, like the inverse of a point with respect to a circle, or the intersection of two lines. If this auxiliary construction and its elements are well defined then the position of the point V_z on the axis can be conveniently movable, and as the limiting case of the construction it can be moved to infinity. It needs to be mentioned that not all DGS programs can deal with points at infinity of the plane, so it is essential to choose the appropriate application.

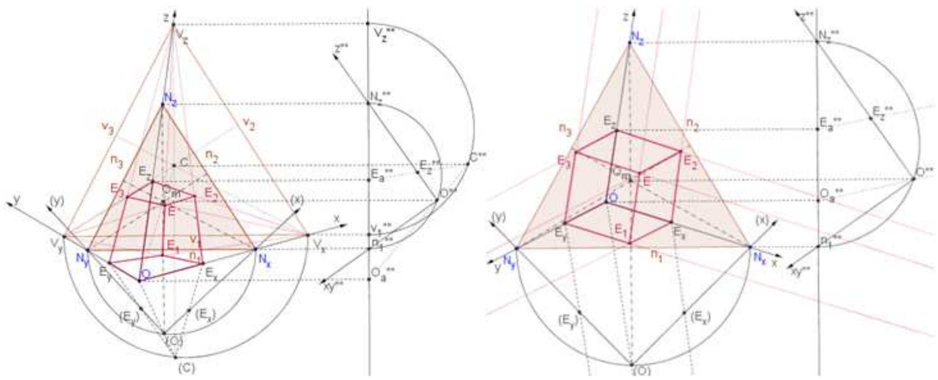

Figure 6

Moving the freely movable point O of the parallel axonometric image obtained in the above way to the point O_m we get orthogonal axonometry.

Practical application in engineer training

The development of spatial perception and hence the teaching of descriptive geometry plays an important role in engineer training. Besides Monge's projective representation great emphasis is given to (parallel) axonometric and perspective representations that give more lifelike images. The similarities in these projections and the associated construction methods are often not clear enough for the students. Frequently, axonometry and perspectivity are considered as two totally distinct mappings and the basic constructions are taught independently from one another.

The discussion of central axonometry gives a good opportunity to demonstrate the analogies between parallel and central projections. It also provides a good occasion to highlight the similarities in the construction methods of axonometry and perspectivity. The constructions made in DGS and shown above are suitable aids in introducing the concept of central axonometry in engineer training. This is partly due to the feature of DGS programs to follow the changes and also to the fact that a construction made in the program can be shown step by step.

In the previous section we showed how parallel axonometry can be shown to be the limiting case of central axonometry by moving the triangle of the vanishing lines to infinity. The similarities in the constructions - the only difference being that the central projection rays become parallel projection rays - can be shown not only in the finished images but the construction methods can in both cases be demonstrated step by step.

The constructions shown above and the illustrative nature of making them in DGS provides a good opportunity for the students to follow how the direction and the centre of projection change on moving the positions of the points of base object determining parallel and central axonometry. Observations can also be made in the reverse direction: the students can see how the image of the reference system and the unit cube, the position of the unit points, trace points and vanishing points change when we move the direction of the projection or the position of the projection centre. This experience can prove useful in making a descriptive geometric construction or a handmade drawing so that the choice of the direction or the centre of projection should be appropriate and thus result in a more illustrative picture.

The knowledge thus acquired can be useful when working with CAD-applications:

- during modelling it makes easier to choose the direction of projection and the position of the projection centre in the reference systems offered by the programs,
- the handling of rotation and movement commands used for walking around the model can become a routine
- when making visualisations it can be easier to obtain the required optical effect
- perspective anomalies can be avoided when placing the model in front of a pixelgraphical background

Further possibilities

The constructions shown for unit points above can be made for specially positioned reference systems where the axis or the coordiante plane is parallel to the image plane. The analogies of central axonometry and parallel axonometry in this case can also be demonstrated in DGS.

Another method for determining the unit points is when the method of division points known from perspectivity is adopted for central axonometry and then from there for parallel axonometry. This construction can also be nicely demonstrated in DGS.

In the constructions using the methods of division points or revolution in DGS it can be analysed and described how the positions of the unit points change when the tracing triangle, the origin or the triangle of the vanishing lines is moved.

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