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**Teaching**  
Mathematics and  
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# Teaching meaningful mathematics with the Computer Algebra System MAXIMA using the example of inequalities

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*Abstract.* The paper was originally motivated by the request to accentuate the meaningful contribution of inequalities in Mathematics Education. Additionally nationwide approved competences such as estimating come to the fore when organizing mathematical contents along some central Big Ideas. Not least the integration of computers enriches the reasonable discussion of inequalities by modern well accepted methodological principles. The freeware MAXIMA is used as Computer Algebra System (CAS) representatively.

*Key words and phrases:* computer algebra systems, MAXIMA, Fundamental Idea, estimating, inequalities, taxonomy, skilled defining and extrapolating, numerically based assuming and analytically verifying.

*ZDM Subject Classification:* B40, D40, D60, E50.

## 1. Introduction

It is well known that the occupation with inequalities is very unpopular with students (Blanco & Garrote 2007; Tsamir, Almog & Tirosh 1998; Tsamir & Bazzini 2002). On one hand this fact might be seen as sorrowful reality but on the other hand it has been an activator for the cooperation with colleagues in Austria and partners in Germany and the U.K. in the *ABCmaths – Project*<sup>1</sup>. A main goal of the research has consisted in refashioning fundamental mathematical topics

<sup>1</sup>ABCmaths (2010)

and developing and discussing materials for contemporary teaching and learning scenarios. Additionally the discussion on inequalities of different complexity with the help of new Media originates the variously shaped mathematical technique of *estimating*.

## 2. Estimating – a Fundamental Idea

For justifying *estimating* as a Fundamental Idea the concept of Andreas Schwill (1993) is used firstly. In the explanation of his model he formulates four criteria:

*The Horizontal Criterion* means a Fundamental Idea should infiltrate a multiplicity of (application) areas.

*The Vertical Criterion* addresses the complexity of the strategies, in detail the grade of detailing or formalization.

*The Time Criterion* underscores the importance of historical developments and finally the

*Criterion of Meaning* establishes a relationship to the living environment, to everyday speech and thinking.

Let us pick up the technique of *estimating* and quantify it following Schwill’s criteria. When speaking about inequalities one may probably assign it to manipulations such as forming or solving firstly, hence to the topic Algebra. However the students are occupied with *estimating* tasks of different complexity expressed as inequalities in Real and Complex Analysis, in Stochastics or Discrete Mathematics (i.e. the Inequalities of Bernoulli and Chebyshev or the Estimation of Erdős (Kurz 2009)).

However, students in school meet inequalities of increasing complexity  $n + 75 > 92$ , where  $n$  is an element of  $\mathbb{N}$  (level of education: grade 5) (Reichel et al. 2007) or  $\frac{x+1}{x+3} > \frac{x+2}{x+5}$ , where  $x$  is an element of  $\mathbb{Q}$  (level of education: grade 10) (Reichel, Götz, Hanisch & Müller 2007).

The fact that a multiplicity of mathematical statements formulated as inequalities has been connected inseparably with the names of famous mathematicians indicates the historical relevance of *estimation*.

Undoubtedly *estimating* has a strong relation to everyday speech and thinking.

### 3. The worthwhile contribution of inequalities

In a second step of justification the teaching (or more general: educational) goals will be taken into account. Hence the discussion refers to the Taxonomy of Anderson and Krathwohl (2000) which concentrates cognition and skills in six categories.

*Remembering* addresses reproduction which means reproducing definitions and retrieving facts.

*Understanding* outlines the process of construction, for example interpreting or explaining.

*Applying* describes the use of procedures, for example implementing models.

*Analysing* divides the process of breaking concepts into parts including the significance of the relations between the parts.

*Evaluating* addresses the ability to make judgments based on argumentation.

*Creating* describes the reorganization process, for example synthesizing parts.

Manipulating inequalities is retrieving facts firstly. For example think of the order of numbers as a fact when changing the relation doing the following manipulation  $(x + 2 < 3 \cdot x + 5) \cdot (-2)$ . But even this example clears the way for going beyond. It should be in teachers' interests to invite their students to look behind the ostensible simple manipulations. Higher categories of Anderson's and Krathwohl's concept would then find the way into the classrooms. Actually modeling has been an up-to-date theme in Mathematics Education and it has demanded a lot of abilities of different complexity related to inequalities (Fuchs & Blum 2008).

### 4. The significance of graphical representations

When talking about Computer Algebra Systems one firstly thinks of the primary innovative contribution in particular its symbolic features (Davenport 1994). However the following example will focus on the combination of symbolical and graphical facilities.

The unit starts by solving the inequation  $|3^{2x} - 1| > 0.8$  in a Black-Box-Mode<sup>2</sup>:

```
solve(abs(3^(2*9x)-1)>0.8,x);
```

<sup>2</sup>Well accepted Didactical Principles (see Buchberger 1989)

MAXIMA<sup>3</sup> will lapidarily answer with *Cannot solve inequalities*.

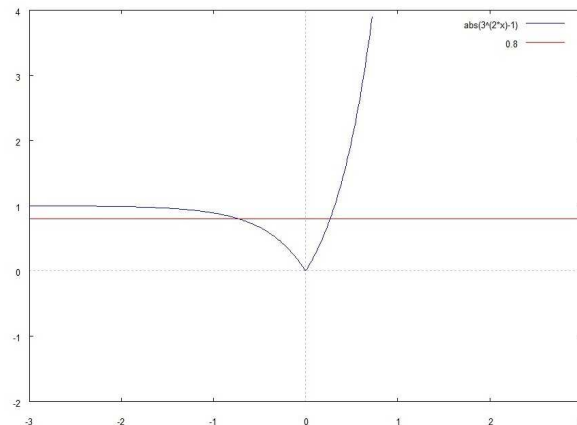
The reader might be surprised that we use the CAS MAXIMA in this problem context furthermore although the mathematical software cannot handle inequalities. Briefly the following statements should highlight the dominance of the advantages compared to the disadvantage of the system expressed in *Cannot solve inequalities*.

- MAXIMA is a powerful but free software based on the CAS MACSYMA which was developed at the Massachusetts Institute of Technology in the late sixties (Winkler1990).
- The source code can be compiled on many operating systems such as Windows, Linux, Mac OS X or Android<sup>4</sup> and therewith
- in time CAS MAXIMA provokes a situation from a methodological meaningful point of view in this context where the students get the chance to bring in mathematical competences in a subsequent White-Box-Phase.

Hence we go on with solving the problem. It will be advantageous for explaining to first separate the left and the right side of the inequality. It is indicated to plot the graphs of the two functions  $f_1(x) = \text{abs}(3^{(2*x)} - 1)$  and  $f_2(x) = 0.8$ .

```
plot2d([abs(3^(2*x)-1),0.8],[x,-3,3],[y,-2,4]);
```

yields



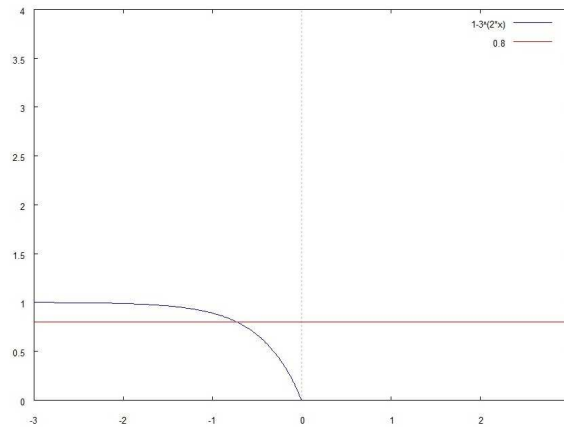
<sup>3</sup>The latest version for the Source Code of wxMAXIMA is 12.09.0 retrieved January, 30, 2013, from the homepage of the CAS <http://sourceforge.net/projects/wxmaxima/files/wxmaxima/12.09.0/wxmaxima-12.09.0.tar.gz/download>

<sup>4</sup>Homepages manufacturer: <http://www.microsoft.com/de-at/default.aspx/>; <http://linuxhomepage.com/>; <http://www.apple.com/de/osx/>; <http://www.android.com/>

The knowledge about the prototypical behaviour of the abs – function which is expressed by the branching of the graph in two parts depending on whether the argument is less than or greater than or equal to zero is activated when *analyzing* the figure. The distinction will lead to  $3^{2x} - 1 < 0 \rightarrow |3^{2x} - 1| = -3^{2x} + 1$  or  $3^{2x} - 1 \geq 0 \rightarrow |3^{2x} - 1| = 3^{2x} - 1$ .

Firstly the discussion is concentrated on the left branch  $f_3(x) = -3^{(2*x)} + 1$ . Again plots of the graphs of  $f_2$  and  $f_3$  are indicated.

`Plot2d([-3^(2*x)+1,0.8],[x,-3,3],[y,0,4]);`



Positioning the cross at the intersection point will show  $-0.73$  for the  $x$  – value approximately.

To find out the exact  $x$  – value for the intersection point we use the CAS analytically

`solve(-3^(2*x)+1=0.8,x);`

yields two solutions (presented in a list)

$$\left[ x = \frac{\log\left(-\frac{1}{\sqrt{5}}\right)}{\log(3)}, x = -\frac{\log(5)}{2\log(3)} \right].$$

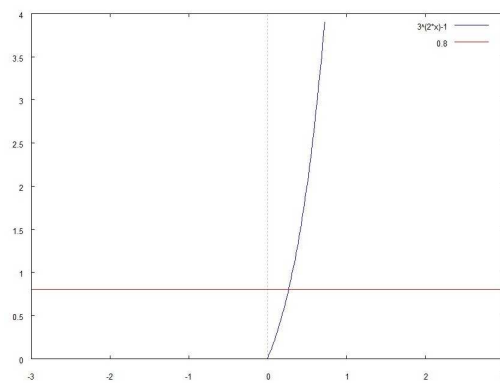
*Remembering, Argumentation and Reorganizing* come into demand now. Remembering addresses the knowledge of the domain of definition of the real log – function whereby the first element of the list can be thrown out by *argumentation*. The second element is represented exactly. Many students prefer numerical representations and *reorganize* the output.

```
-log(5)/(2*log(3)),numer;
```

translates the exact solution into -0.73248676035896. The approximate solution for the  $x$  - value of the intersection point located by positioning before is confirmed.

The analog strategy for the right branch  $f_4(x) = 3^{(2*x)} - 1$  is as follows:

```
plot2d([3^(2*x)-1,0.8],[x,-3,3],[y,0,4]);
```



```
solve(3^(2*x)-1=0.8,x);
```

$$\left[ x = \frac{\log\left(-\frac{3}{\sqrt{5}}\right)}{\log(3)}, x = \frac{\log\left(\frac{3}{\sqrt{5}}\right)}{\log(3)} \right]$$

```
log(3/sqrt(5))/log(3),numer;
```

```
0.26751323964104
```

At this point one has the opportunity to go beyond the illustrated process by provoking first insights into the idea of ‘small neighborhood’ which is fundamental for infinitesimal calculus. The exact  $x$  - values of the intersection points are *substituted* into the initial inequality:

```
subst(-log(5)/(2*log(3)),x,abs(3^(2*x)-1)>=0.8),numer;
```

```
subst(log(3/sqrt(5))/log(3),x,abs(3^(2*x)-1)>=0.8),numer;
```

MAXIMA yields  $0.8 \geq 0.8$  which is true.

After heightening (in the first case) and diminishing (in the second case) the exact  $x$  - values by 0.0000001:

```
subst(-log(5)/(2*log(3))+0.0000001,x,abs(3^(2*x)-1)>=0.8),numer;
```

```
subst(log(3/sqrt(5))/log(3)-0.0000001,x,abs(3^(2*x)-1)>=0.8),numer;
```

MAXIMA yields  $0.7999999560555 \geq 0.8$  and  $0.79999960449962 \geq 0.8$ . The students are invited to find arguments on the relevance of ‘small transformations’ in infinitesimal calculus. Although the expected argumentations will be preformal, the discussion will prefigure characteristics of infinitesimal calculus.

### 5. Skilled defining and extrapolatingn

The Geometrical – Arithmetical (GA) Inequality has become a prototype for structured constructions in mathematics. CAS can give essential support and the topic also allows deeper insights into generalizations.

Starting point of the teaching unit is the well known inequality:  $\sqrt{x_1 \cdot x_2} \leq \frac{x_1+x_2}{2}$ ;  $x_1, x_2 \in \mathbb{R}_0^+$ . The validity of this inequality can evidently be accepted by the following figure *applying* theorems of the rectangle triangle.

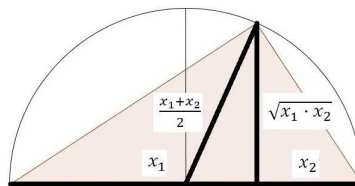


Figure 1. geometrical representation of GA – Inequality

The discussion of the inequality is rarely carried on to roots  $\sqrt[n]{x}$  with  $n > 2$  in school teaching. A short disquisition on the steps ( $n = 3, 4$ ) can be found in Schweiger (2010). Even though the essential strategy is shown in Schweiger’s book the proposed unit will go beyond when discussing the integration of CAS MAXIMA. Furthermore ‘Links to higher complexity’ will be signified.

When proving the relation  $\sqrt[4]{x_1 \cdot x_2 \cdot x_3 \cdot x_4} \leq \frac{x_1+x_2+x_3+x_4}{4}$  the expression on the left hand side  $\sqrt[4]{x_1 \cdot x_2 \cdot x_3 \cdot x_4}$  is resolved to  $\sqrt{\sqrt{x_1 \cdot x_2 \cdot x_3 \cdot x_4}}$ .

The transformation must be justified consequently. At this point it might be exciting to find out which answer MAXIMA will give.

To secure correct communication even with the CAS MAXIMA it should be noted that  $x$  is a nonnegative number in  $\sqrt[n]{x} = x^{\frac{1}{n}}$ ;  $x \in \mathbb{R}_0^+$ ,  $n > 1 \in \mathbb{N}$ . Being aware of this constraint MAXIMA will return the satisfying answer **true** when entering

```
realpart((x^(1/2))^(1/2))=realpart(x^(1/4)),pred;
```

The new created expression  $\sqrt{\sqrt{x_1 \cdot x_2 \cdot x_3 \cdot x_4}}$  is resolved into  $\sqrt{\sqrt{x_1 \cdot x_2} \cdot \sqrt{x_3 \cdot x_4}}$  afterwards. The CAS confirms the correctness of our manipulation as `rootscontract(x^(1/2)*y^(1/2))`; evaluates the input to  $\sqrt{x \cdot y}$ .

Following the relation formulated for square roots leads to the result

$$\sqrt{\sqrt{x_1 \cdot x_2} \cdot \sqrt{x_3 \cdot x_4}} \leq \frac{\sqrt{x_1 \cdot x_2} + \sqrt{x_3 \cdot x_4}}{2}.$$

Applying the rule twice for the numerator of the fraction on the right side yields

$$\frac{\sqrt{x_1 \cdot x_2} + \sqrt{x_3 \cdot x_4}}{2} \leq \frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2}.$$

Elementary simplification shows

$$\frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2} = \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

Recapitulation will proof the initial statement:  $\sqrt[4]{x_1 \cdot x_2 \cdot x_3 \cdot x_4} \leq \frac{x_1+x_2+x_3+x_4}{4}$ . Consequently the proved inequality for  $n = 4$  is picked up when substituting  $x_4 := \sqrt[3]{x_1 \cdot x_2 \cdot x_3}$  on both sides of the relation which generates the expression

$$\sqrt[4]{x_1 \cdot x_2 \cdot x_3 \cdot \sqrt[3]{x_1 \cdot x_2 \cdot x_3}} \leq \frac{x_1 + x_2 + x_3 + \sqrt[3]{x_1 \cdot x_2 \cdot x_3}}{4}.$$

The manipulations following up concentrate on the left side of the inequality.

In a first step the simplification rule  $x \cdot \sqrt[3]{x} = \sqrt[3]{x^4}; x \in \mathbb{R}_0^+$  is applied. In a second step the roots' exponents are swapped as follows:  $\sqrt[4]{\sqrt[3]{(x_1 \cdot x_2 \cdot x_3)^4}}$  to  $\sqrt[3]{\sqrt[4]{(x_1 \cdot x_2 \cdot x_3)^4}}$ . Simplification leads to  $\sqrt[3]{x_1 \cdot x_2 \cdot x_3}$  in a third step. All of the applied rules can be certified by the CAS:

First step: `x*x^(1/3)=(x^4)^(1/3),pred;`

Second step: `(x^(1/3))^(1/4)=(x^(1/4))^(1/3),pred;`

Third step: `(x^(1/4))^4=x,pred;`

The CAS MAXIMA will yield `true` for all of the three inputs.

Afterwards the discussion centers on the right side of the inequality starting with elementary operations with fractions

$$\frac{x_1 + x_2 + x_3 + \sqrt[3]{x_1 \cdot x_2 \cdot x_3}}{4} = \frac{x_1 + x_2 + x_3}{4} + \frac{\sqrt[3]{x_1 \cdot x_2 \cdot x_3}}{4}.$$



These operations are followed by elementary transformations of inequalities

$$\left( \sqrt[3]{x_1 \cdot x_2 \cdot x_3} \leq \frac{x_1 + x_2 + x_3}{4} + \frac{\sqrt[3]{x_1 \cdot x_2 \cdot x_3}}{4} \right) - \frac{\sqrt[3]{x_1 \cdot x_2 \cdot x_3}}{4}$$

$$\left( \frac{3}{4} \cdot \sqrt[3]{x_1 \cdot x_2 \cdot x_3} \leq \frac{x_1 + x_2 + x_3}{4} \right) \cdot \frac{4}{3}$$

$$\sqrt[3]{x_1 \cdot x_2 \cdot x_3} \leq \frac{x_1 + x_2 + x_3}{3}$$

which will verify the desired relation finally.

*Pressing* ahead the strategy illustrated for  $n = 4$  and  $n = 3$  with  $n = 6, 8, 10$  and respectively  $n = 5, 7, 9$  will offer further exercise materials. Additionally it will underlay the generalized form of the inequality notably in the strategy of its proof by complete induction which is published at the website of the Austrian Mathematical Olympiad (AMO 2011) for example.

## 6. Numerically based assuming and analytically verifying

Teaching sequences and series means talking about unoffending (constant, convergent) behaviour on the one hand and unpleasant (cyclic, divergent) behaviour on the other hand. Again this teaching unit brings attention to the strategy of *constructing*.

The discussion starts with a sequence  $a$  of real numbers given by the formula  $a_n = 0.5^n$ ,  $n \in \mathbb{N}$  or alternatively by the iteration rule  $a_1 = 0.5 \wedge a_n = 0.5 \cdot a_{n-1}$ ,  $n \geq 2 \in \mathbb{N}$ . The boring job of generating ‘initial pieces’ of the sequence  $a$  is left to the CAS MAXIMA initially. With `a_c` we address the representation by the formula

```
a_c[n]:=0.5^n;
```

with `a_r` by the iteration rule

```
a_r[1]:0.5;
```

```
a_r[n]:=0.5*a_r[n-1];
```

Generating the ‘initial pieces’ by `makelist(a_c[n],n,1,10)`; or accordingly `makelist(a_r[n],n,1,10)` results in  $[0.5, 0.25, 0.125, 0.0625, 0.03125, 0.015625, 0.0078125, 0.00390625, 0.001953125, 9.765625 \cdot 10^{-4}]$ .

When *focusing* the list the students will assume that the decreasing numbers in the lists will approach 0 very fast. Verifying this conjecture analytically means

to pick up the definition of a limit  $\alpha$  of a sequence of numbers. It brings in a bundle of inequalities:  $\forall \varepsilon > 0 \exists n(\varepsilon) \forall n \geq n(\varepsilon) : |a(n) - \alpha| < \varepsilon$ .

In a first approach one may choose  $\frac{1}{1000}$  for  $\varepsilon$  which is a small positive real number in mathematical agreements.

Following the definition for the chosen  $\varepsilon$  stands for finding a corresponding  $n(\varepsilon)$  so that  $|a_c(n) - 0| < \frac{1}{1000}$  for almost all elements of the sequence. MAXIMA is asked for support.

Solving the corresponding equation

```
solve(abs(a_c[n]-0)=1/1000,n),numer;
```

the system yields  $[n=9.96578437701743]$ . According to this the natural number  $n(\varepsilon)$  is 10. The 10<sup>th</sup> element in the list generated before is  $9.765625 \cdot 10^{-4}$ . It might be purposeful for some of the students to translate the floating point notation  $9.765625 \cdot 10^{-4}$  into the decimal notation 0.0009765625. Manifestly it is the first element of the sequence which fulfills the condition  $|a_c(n) - 0| < \frac{1}{1000}$ .

In a second approach  $\varepsilon$  will be left as a symbol and MAXIMA will be consulted again:

```
solve(abs(a_c[n]-0)=epsilon,n);
```

The answer  $n=\log(1/\text{epsilon})/\log(2)$  calls for interpretation by the students. The knowledge of the prototypical attributes of the log - function must be activated.

An interpretation might be the following: One *remembers* that  $\varepsilon$  - as chosen  $0 < \varepsilon < 1$  - is a small real number. Hence  $\frac{1}{\varepsilon} > 1$  the numerator  $\log\left(\frac{1}{\varepsilon}\right)$  in the output is greater than 0. As the nominator  $\log(2)$  is positive the fraction  $\log(1/\text{epsilon})/\log(2)$  is positive. Stringently one can bring forward the argument for the sequence  $a_c$ : For any small real number  $\varepsilon$  ( $0 < \varepsilon < 1$ ) exists an  $n(\varepsilon) = \left\lceil \frac{\log \frac{1}{\varepsilon}}{\log 2} \right\rceil + 1$  at least with  $|a_c(n) - 0| < \varepsilon$  for all  $n \geq n(\varepsilon)$ .

A look at number series will be taken finally.

```
Sum(a_c[n],n,1,k),simpsum;
```

yields  $-2.0(2^{-k-1} - \frac{1}{2})$ . One question may be the verification of the output which addresses the formula for finite series. As focusing on inequalities in this paper consequently the symbolic proof of the formula is left untreated. Rather the variable  $k$  in the expressions  $-2.0(2^{-k-1} - \frac{1}{2})$  will be prolonged to infinity. The ideas of *monotonicity* and *boundedness* come to the fore.

A list of values is generated by the CAS initially.

```
Makelist(-2.0*(2^(-k-1)-1/2),k,1,10);
```

produces [0.5, 0.75, 0.875, 0.9375, 0.96875, 0.984375, 0.9921875, 0.99609375, 0.998046875, 0.9990234375].

The values in the list are increasing and they seem to approach 1.

The first part of the assumption addresses *monotonicity*. The sequence of sums seems to be strictly monotonic increasing:

$$\forall k \in \mathbb{N} : -2.0 \left( 2^{-k-1} - \frac{1}{2} \right) < -2.0 \left( 2^{-(k+1)-1} - \frac{1}{2} \right) = -2.0 \left( 2^{-k-2} - \frac{1}{2} \right).$$

MAXIMA is confronted with this result: The CAS

```
-2.0*(2^(-k-1)-1/2)<-2.0*(2^(-k-2)-1/2),pred;
```

answers **true**.

The second part of the assumption addresses *boundedness*. The sequence of sums seems to be bounded by 1:  $\forall k \in \mathbb{N} : -2.0(2^{-k-1} - \frac{1}{2}) < 1$ .

As expected the input

```
-2.0*(2^(-k-1)-1/2)<1,pred;
```

is evaluated **true** by the CAS.

It is the right moment to introduce the statement that any strictly monotonic increasing sequence having an upper bound is convergent which means it has a limit. The CAS should find out the limit in Black – Box – Mode.

```
Limit(-2.0*(2^(-k-1)-1/2),k,\inf);
```

yields 1.

Other problems quantified by inequalities for example the application – oriented approach to continuity (see Knoche & Wippermann 1986, page 118) or the discussion of discrete random variables  $P(X \leq x_k) = \sum_{i=1}^k P(x = x_i)$  serve the students with meaningful mathematics. These discussions should be taken on by further research studies.

## 7. Perceptions and Perspectives

Actually the course presented in the paper has been an essential element in the lecture on ‘Computer Algebra Systems in Teaching Mathematics’ for teacher students at the University of Salzburg. The intention was that the students who were mainly trained in the CAS MATHEMATICA<sup>5</sup> during this course and other courses on the use of mathematical software should be provoked in their

<sup>5</sup><http://www.wolfram.com/>

knowledge and skills by confronting them with the topic of inequalities in the context of another CAS system, in particular MAXIMA. In reality the targeted plan worked out even. Vivid discussions broke out in the proseminar associated with the lecture.

In general the students demonstrated a rich repertoire of strategies which was enriched by the idea of Estimating through this topic. Concerning the context the students agreed to the prototypical character in particular. The discussion of inequalities opens up numerous educational perspectives of various complexities for them (compare with the first and second step of justification in chapters 2, 3).

Hence modern goal – oriented teaching and learning processes in mathematics shouldn’t abstain from the use of CAS software even though this use of New Media will boost the calls for applicable topics together with some well argued methods enormously.

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