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Teaching Mathematics and Computer Science

Probabilistic thinking, characteristic features

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Abstract. This paper is the first step in a series of a general research project on possible development in probability approach. Our goal is to check with quantitative methods how correct our presumptions formulated during our teaching experience were. In order to get an answer to this question, we conducted a survey among third-year students at our college about their general and scientific concepts as well as about the way they typically think.

Key words and phrases: analogy, probability approach, correlation thinking, scientific concept, general concept, intuition, event, certain event, impossible event, probability, multiplication event.

ZDM Subject Classification: C30, B50, E40, K50.

In the teaching of mathematics, probability is a field which deals with ideas related to chances. In terminology, we must be careful to make a distinction between 'development of probability approach' and 'calculation of probability'. The development of probability approach includes activities aimed at promoting the intuition of probability, e.g. predicting chance events of random phenomena, gambling, investigating irregular events etc. In teaching, these activities normally occur before the exact concept of probability is acquired. 'It is necessary to distinguish between the *concept* of probability as an explicit, correct computation of odds and the *intuition* of probability as a subjective, global estimation of odds' (Fischbein, 1975 p. 79). This point of view is considered to be fundamentally important by other researchers (for example Varga, Green, R. Kapadia, M. Borovcnik) as well. In junior classes, the concept of probability is developed in a rather lengthy process of instruction, via actual measurements, experiments and games. At this stage focus is laid on gaining experience without drawing any conclusions yet. Accordingly, in this initial stage a general, only vaguely defined meaning is assigned to the generally used terms of 'chance' and 'probability'. "We note life insurance and quality control as specific examples, or the substantial part played by statistical methods in the acquisition and conceptualization of knowledge in various siences. This fact now reflected by the increasing importance of probability within school curricula in most countries all over the world." (R. Kapadia- M. Borovcnik p. 2) In the beginning the meaning of probability is: what has actually occurred more often, will be deemed to be more probable to occur. At a later stage this definition will run, despite the fact that the statement may not be supported by actual experimental results, like this: what can occur more often, will be deemed to be more probable to occur. It is only at a much later stage that a number will be assigned to probability and the related Kolmogorov axiom will be applied.

'Plays an essential part in the domain of probability, perhaps more conspicuously and strikingly than it does in other domains of mathematics' (Fischbein, 1975, p. 5). Based on international research (Piaget, Fischbein, Kapadia- M. Borovcnik) it can be suspected that experimental 'dice throwing' activities are really important momentums in the teaching of mathematics. Based on the work of Tamás Varga and his colleagues, we have similar experience in Hungary. Despite the fact that their ideas were only partly realized in Hungary, Varga's approach is still highly acclaimed in Hungary.

In my teaching experience I have always found that there is a big gap between the development of probability approach and probability calculation. I have always been very careful to establish solid foundations for concepts which had not been discussed before. This is why, before discussing probability in an axiomatic way, I always spend quite a lot of time playing related games and making related experiments. And yet I have always had the feeling that probability calculation is not properly prepared by such activities. Detailed and plain foundations have not always necessarily helped prepare actual calculations. Also, whenever my students have come across some new type of problems, I have always observed that they hardly ever used their previously acquired knowledge in mathematics.

This is why I asked myself a few questions to which I wanted to find answers in a quantitative way as well. The most important questions were:

1. How can activities in the phase of development of a proper approach best help students acquire the skill of probability calculation?

2. How can knowledge acquired *by our students* during the learning of probability calculation be activated in everyday games or situations?

This paper discusses only the second question. I and my colleagues conducted a survey to learn what knowledge our college students had in this field and how they applied their knowledge in everyday practice. A great deal of information was obtained in the survey but in this paper I am only going to discuss the most important pieces of this information: information which clearly reveals the actual distance between developing probability approach and probability calculation as well as information which leads to further questions setting herewith the direction of future research.

I would like to thank my colleagues at ELTE-TÓFK Department of Mathematics, who helped me a lot get the students to answer the questionnaire. Julianna Szendrei, Eszter C. Neményi and myself jointly prepared a one-term-long course, we attended each other's classes and took detailed notes on our observations, which also helped us obtain lots of useful information. Special thanks to the student Anna Schönek, who was so kind as to contribute to achiving the data of the survey.

Survey background

Research objectives

Extensive research has confirmed the presumption: analogical thinking plays an important role in acquiring knowledge. (Vásárhelyi, 2006) Analogies are typically referred to as means of development of inductive thinking. Drawing conclusions analogically is a cognitive operation, which is used when based on the accordance in certain attribute, relation or structure of two or more phenomena or objects we assume them to accord in other attributes, relations or structures as well (Ambrus, 1995 p. 85.). They are of great importance, primarily, in connecting different elements in the students' knowledge but also in the uniform interpretation of their experience. This experience differs per each student since students acquire such experience from different sources. (Nagy Lászlóné, 2000.) The development of analogical thinking is one of the focus areas in the teaching of mathematics. However, our teaching experience suggests – and this is also substantiated by numerous articles/papers on the didactics of mathematics – that there are certain cases when these analogies turn out be wrong and, by jumping to general conclusions on the basis of actually existing analogies, can even lead to fals solutions. There are lots of traps in probability itself – quite a few students may be of the opinion that all you have to do is just think on the basis of analogies. (R. Kapadia- M. Borovcnik) Most of the paradoxes (Székely J, Gábor) in random mathematics are of this type. We say that in probability it is very-very difficult to tell whether a problem is related to a problem that has been solved before (perhaps in quite a different way) or not. Because of this, when trying to establish analogical links between different tasks, we frequently run into difficulties. In this paper we are making an attempt to identify some possible applications of analogies in probability.

There is a clear difference between everyday (spontaneous) and scientific interpretations of concepts about probability. The border between spontaneous and scientific interpretations seems to be rather flexible. This border is affected, on the one hand, by the person using this concept and, on the other hand, by the nature of the particular concept. The development of these two types of interpretation is an interrelated process characterized by a series of close and continuous interaction (Vigotszkij, 1966). In this paper we are going to discuss how a concept which has already been interpreted in a scientific way can affect the spontaneous development of the same concept.

We wondered to what degree the knowledge acquired in *probability calculation* becomes part of *probability approach*. In other words, how does a student who has finished his/her compulsory public education, has a GCSE but has perhaps no special interest in mathematics, apply his/her previously acquired knowledge?

Attention is focused in every institute, at regular intervals, on the programmes determining the training profile of the institute. Whenever we decide on our teaching line, it is very important that we take into consideration our students' preliminary knowledge and the way they approach different problems. So that we can find good solutions on how to best develop our students' skills in this field at the Teachers' Training Faculty we wanted to gather more information on this too.

We were interested to learn about our students' attitudes when they are estimating the probability of the occurrence of a random event, in particular when they

- try to estimate the chance of occurrence of a probable event,
- compare, based on probability, two or more events or
- make a distinction between deterministic and non-deterministic events.

In our investigation we started from the fact that probability was not ‘terra incognita’ for our students: As baccalaureate is a condition of being admitted to university, they have come to know the basic concepts during high school. They are familiar with the concepts of frequency and relative frequency. They have learnt how to solve common problems and how to calculate the probability of events based on the classical probability model. (Kerettanterv a gimnáziumok 9–12. évfolyamának számára) They had already learnt that a chance related to the occurrence of an event could be expressed by one number. Misunderstanding is less likely if the chance is expressed by a number rather than by words.

Questions, hypotheses

1. A student with a GCSE in mathematics will know that a number between 0 and 1 can be assigned to the probability of the occurrence of an event. It is not clear, however, whether such an estimation is expressed in numbers (and if yes, in what form) or in some other form. I am of the opinion that if someone communicates in numbers, it is more likely that his/her message will be properly understood. This is because, if a number is used, the actual estimation of the chance is expressed in a clear, unambiguous way. But if someone tries explaining the same chance in words, e.g. ‘It seems most unlikely that it will rain in Budapest tomorrow’, then he/she takes that his/her partner interprets the term ‘most unlikely’ exactly in the same way as he/she does. Our assumption is that our students express the probability of events by figures based on their high school knowledge. We have also presumed that if a chance is expressed by numbers, the value will typically be given in percentage despite the fact that everyone has learnt about common fractions at school.

2. In the teaching of mathematics a large number of terms have differing meanings in everyday usage and in special technical terminology (Szendrei, 2005, p. 400.). This is even truer in the case of probability. E.g, the term ‘probability’ has an unambiguous, exact meaning in mathematics whereas the term ‘probable’ is typically used spontaneously in everyday usage. I have presumed that it is difficult to make a distinction between such terms and also that they are always interpreted spontaneously, depending on the actual situation and on the person involved.

3. We have presumed that decisions on probability are also largely influenced by the level of completeness of the related mathematical model.

4. If a mathematical problem is set by words (by text explanation), the first step is to normally interpret/understand the actual problem, which is then followed by working out a model, i.e. enciphering it into the language of mathematics (Pólya, 1945.). The so arising plan to solve the problem is already independent of the original context. A scientific thesis will only be authentic if it can be demonstrated in an exact form exempt from all disturbing elements, from emotional involvement. The question arises thus: can probability approach really be exempt from emotions? We do not want to exclude students' motivation (an element of emotion) in the stage of concept formation at all. It is good to be motivated regardless of the type of the particular problem which must be solved. It is rather the students' presumption about the outcome of a problem we should talk about. Can our students exclude their presumptions about the chance of the occurrence of an event if they have sufficient knowledge (and can also mobilize this knowledge) to solve a particular problem or not? Are they able to think independently of the particular context? We presume that the presence of emotional factors in probability prediction is so intense that in some cases it can obscure the existing mathematical knowledge.

Place of the survey, students involved in the survey

The survey was conducted at the Faculty of Elementary and Nursery School Teachers' Training, Budapest University ELTE in 2008. The participating students had already completed five terms in mathematics and had been instructed in the theory of numbers, clusters, logics, functions, geometry and teaching methodology. Our students also have some knowledge in probability calculation: the basic concepts and methods of calculation were taught in primary and secondary schools. In addition, since this area has been given a bigger emphasis in the new GCSE, probability calculation has been a focus area in mathematics teaching for quite some time now.

Development of skills and practical application of knowledge – both are important elements in the National Curriculum and in the curriculum of our institute alike. In training, bearing the above in mind, we have our students re-interpret the basic concepts of mathematics. Our students again go through the same process as young children in the foundation stage. We analyze the development of the different concepts jointly with our students, who receive information about typical features of the way children think, then analyze this information and compare it with the way they are thinking now. Based on this we have come to the following conclusion: students who have studied probability calculation at school but have perhaps no special interest in mathematics are well represented by our students.

Survey methods

The survey was based, on the one hand, on a questionnaire and, on the other, on our observations made during a one-term-long period.

Questionnaire

Before the one-term-long course in combination theory and in probability calculation was launched, our third-year students were asked to fill a questionnaire of 23 questions on their concepts about probability. With a view to a comparative analysis in the future, 12 questions were identical to the Hungarian questions of the Simulo project on the levels of probability completed by 5–8th grade schoolchildren in 1990. Three countries shared in this poject: Brasil, Hungary and Canada (Quebec). The goal of the project was to compare the probability thinking of the students age of 10–14 in these countries. John Izard summarized the conclusions of the research in 1991. The project was connected to the measurement of David Green in the early 1980s in England, wherein he examined the probability thinking of 2390 students age of 11–16.

In our examination borrowing a part of those questions seemed practical, because pre-qualification isn't required to answer these questions, also we would like to use that in a comparative analysis later.

Our questionnaire, which had to be completed in 60 minutes, was returned by 178 students.

Regarding the evaluation system, it was decided beforehand that the data would be entered in Excel because here you can not only compare the data but make clear, easy-to-interpret graphs as well.

We don't attach the complete questionnaire because of its extent. But in the present article we announce the questions literally and without formal change.

Aspects of compilation of the questionnaire

1. The questionnaire should consist of questions which can be answered even by students who have little knowledge in probability. With this in view, several questions were formulated so that the answers could be formulated with

common everyday terms, not requiring any previous professional knowledge. Regarding the questions about apparent mathematical problems, they were so easy that even schoolchildren at primary schools could solve them.

2. In general, we did our best to create a situation where the student filling in the questionnaire does not feel that some kind of special previous knowledge is required. At the same time, research normally has to face the following dilemma: depending on the actual context, can knowledge be classified as special knowledge for school and common knowledge for everyday use? Since we were aware of this possible distinction, we formulated some of our questions in two ways: in everyday context (hiding thus their mathematical contents) and also in a way where it was evident for the student that mathematical knowledge or some kind of analogy with a previously solved exercise was necessary. We were eager to learn whether the related answers would differ from each other or not.

Observation

At the course which followed this questionnaire we observed 64 out of the a.m. 178 students focusing on their concepts about probability during some kind of activity. Since some day these students would work as school teachers at the foundation stage, priority was given not to the actual calculation of probability but rather to the development of the probability approach and the teaching methodology. We observed (the method actually being observation) the development of the probability approach in three seminar groups. In practice this meant that such lessons were attended by a teacher, by the students and also by a second teacher who was taking notes during class.

We intend to make use of the this material (the Questionnaire and the notes) in future research projects too.

Our findings

1. Estimation of probability, different ways of expressing the estimation

From the answers given to Question 4 we were hoping to be able to conclude whether students typically express probability with numbers or words and also, if numbers, in percentage or in a common fraction.

4. What do you think the chance is that the following events will occur?

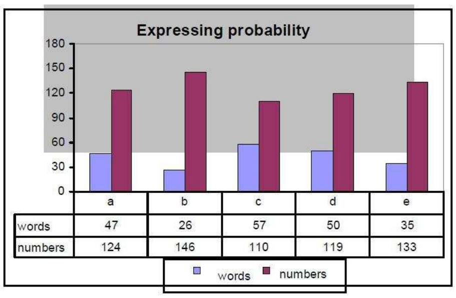
a) It's going to rain in Budapest tomorrow. ______________

b) If you toss a coin, you'll get heads. ______________

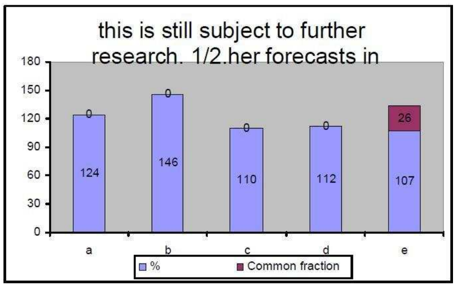
c) A randomly chosen person in Budapest will speak Japanese. ______________

d) A randomly chosen 10-year-old child will like chocolate. ______________

e) If you throw a dice, you'll get a 3.



(a)



(b)

Diagram 1

Diagram 1(a) shows whether the students expressed their estimates for the different sub-questions in numbers or in percentage. Sub-question *b* was answered by the vast majority of students (146) with a number. This is so because it is easy to express in a number (1/2) what is the probability to get heads in tossing of a coin. In the other sub-questions, in particular in *c* (little chance) and *d* (big chance), the students a bit more preferred words to numbers. Typical answers were 'Most likely', 'Almost certain', 'Most unlikely', 'Not likely'.) Even in sub-question *e*, which is not at all difficult to calculate, 35 students opted to answer in words ('In one-third of the throws', 'Maybe yes, maybe no') rather than in one number.

Diagram 1(b) shows what is the proportion, in the case of those answering with a number, between the answers given in percentage or in common fractions. It can be seen that, apart from sub-question e, the students decided to give the chance of the occurrence of the events in percentage. This percentage-type expression of chance has been widely populated by the weather forecasts in the Hungarian media. On the other hand, exercise books on probability in Hungarian secondary schools prefer common fractions. It turns out thus that even if students have some previous mathematical knowledge, they still stick to the common, everyday practice (percentage). This applies even to cases where the actual chance can more precisely be expressed by a common fraction than by percentage: for e, only 26 students wrote '1/6' but as many as 107 students used (more or less correct) percentage. It is well-known to everyone that percentage corresponds to a particular number. Still, there are different concepts associated to 50% and 1/2 This became evident during classes where notes were taken but this issue is still subject to further research.

What kind of estimates were given by the students? In order to make an estimate, you certainly need some kind of a model. Probability models can also be made about deterministic events of which you have little information. There are indeed different opinions as to what events we consider to be deteministic. "Various interpretations of quantum physics contradict each other if there exists real randomness in the quantum world (Lovász, 2000). For example, 'Is it going to rain in Budapest tomorrow?' is a deterministic event to which we do not have sufficient information to give an exact answer. In cases like this some kind of model is used to express the chance of the occurrence of the event. Such models are:

- Subjective estimation of probability. Here you express your own belief, the rate how much *you* think it's going to rain in Budapest tomorrow. One thinks based on its own experience, knowledge and impressions.
- Statistical model. This expresses the chance of the occurrence of an event based on previous experience. (Like: in this season of the year it frequently rains in Budapest.)
- Classical probability model. This is the quotient of the favorable elementary events and of all the elementary events. (In this example there are two options: it is going to rain or it is not going to rain.) This model cannot be applied in our case.

From these estimates we could make some conclusions about the way our students think. It can be concluded, for example, what kind of model has been used for making an estimate.

According to the estimates (expressed in numbers or words) of appr. 10% of the students (17 students), there is an equal chance of the occurrence of an event. This fact reveals the way these students think. Regardless of the actual problem, they used the classical probability model to make an estimate about the chance of the occurrence of an event. Some of them were interviewed in class too and they reverted to their estimates by saying: 'I guess it is 50% certain that someone speaks/does not speak Japanese. He either does or does not.' It is evident that these students have some kind of knowledge, they are aware of a model and they use this model to make their estimates. The complexity of probability calculation; however, is that you must always be able to choose a proper model which suits the actual problem. If the outputs or possibilities of which one can occur, than the classical model cannot be applied. On the whole, answers to Question 4 confirm the validity of hypothesis Nr 1.

2. Estimating probability in everyday usage

By chance? Likely? Not likely? Less likely? No chance at all? Such questions are often asked in everyday life and, as a rule, no 'good' or 'bad' answers exist. It is always dependent on the actual subjective circumstances how much we think an event will occur or not. In problems like this it is always the subjective model that is used. 'Probable' is not a special technical term in mathematics. 'Probability', on the other hand, is a special technical term: it refers to a number not less than 0 and not bigger than 1. And you cannot say whether a number in itself is big enough or small enough. You can only say that a number is bigger than another one. It follows that if we are interested in assessing our students' knowledge, we should formulate our question like this: 'Is the chance of the occurrence of an event bigger than that of another event?' In mathematics, the terms 'certain' and 'unlikely' are precisely defined. The actual related chances exist irrespective of any subjective factors. 'Unlikely' is 0, 'certain' is 1. What to do then with the term 'not likely'? Is the chance 'not likely' if its probability is somewhere near 0? Or even 0?

5. Write a sentence with this ending: „ ... – there is a random chance of the occurrence of this event."
____ – there is a random chance of the occurrence of this event.

6. Write a sentence with this ending: „ ... – there is little chance of the occurrence of this event."
____ ... – there is little chance of the occurrence of this event.

7. Complete the following sentence as you like.
It is not likely that Budapest ______________________________________

8. Complete the following sentence as you like.
It is most likely that Sydney ______________________________________

When evaluating the answers, we tried to make a guess at the occurrence of the event and then we put them into different categories. During the evaluation we did not take into consideration those events proposed by students which we could not guess properly.

For question 5 ('there is a random chance of the occurrence of this event') 60% of the students (106 students) chose to describe a situation whose probability of occurrence was much less than 1/2: 'I win on the lottery', 'A brick falls on my head in the street' etc. We can conclude thus that in everyday usage there is not much difference between 'random' chance and 'little chance'. For some of our students, 'random' equals 'little chance'. True, in the Hungarian language the term 'random' can be interpreted like this too, e.g. 'Fancy (in Hungarian = 'Random') meeting you here!' However, in mathematics we speak of a random phenomenon when the outcome is not unambiguously defined by the existing conditions. It follows that the probability of a random event does not ab ovo mean little probability.

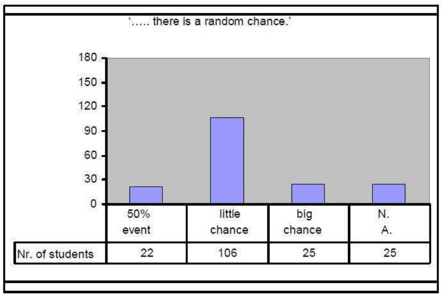
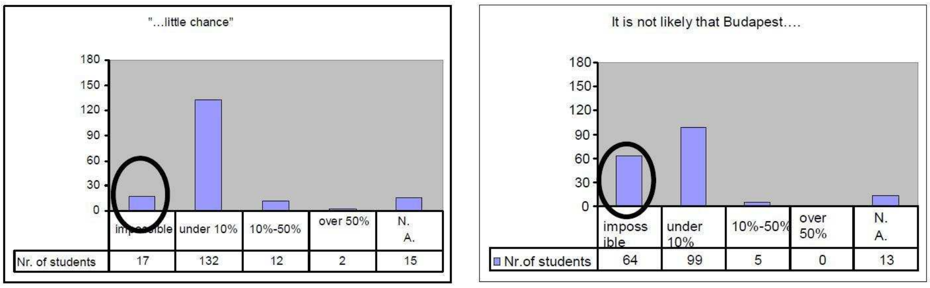


Diagram 2

Regarding the statements 'Little chance' and 'Not likely' (diagrams 3(a) and 3(b)), we should expect that students perhaps described events with a similar chance of probability, for these two phrases can also be interpreted as equal. Interestingly enough, there was a significant difference, in particular, in forecasting an event as 'impossible'. An 'impossible' event is far more associated with 'not likely' than with 'little chance' (64 students and 17 students, resp.).



(a) (b)

Diagram 3

The events described by the students were taken from everyday life. In the next question we specifically asked about the same terminology but this time without any connection to examples taken from everyday life. The question was formulated so that answers with a more scientific approach could be given.

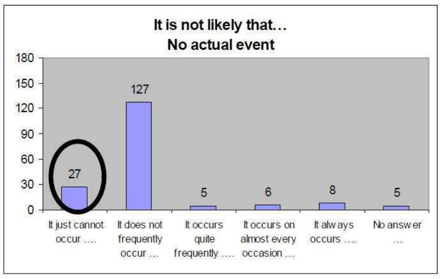
9. The following five phrases have been numbered:
   1. It just cannot occur ….
   2. It does not frequently occur …
   3. It occurs quite frequently ….
   4. It occurs on almost every occasion …
   5. It always occurs ….

Below you'll find five sentence beginnings. Assign the numbers of the above phrases to the sentence beginnings below that you think are the most appropriate. (You can assign the same number to different sentence beginnings.)

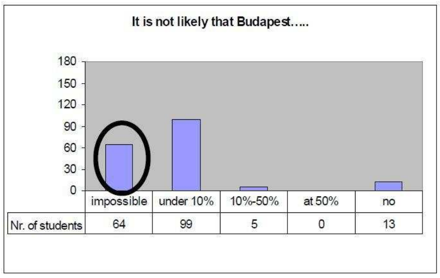
   a) 'It is almost certain that ...'
   b) 'It is not likely that ...'
   c) 'It is most likely that ...'
   d) 'It seems perhaps likely ...'
   e) 'There is little chance that ...'
   f) 'There's no chance that ...'

Such types of abstract questions, which resemble typical maths exercise, can often be found in course-books.

In this question there were only 27 students who assigned, without giving any actual events, the phrase 'It just cannot occur' to the sentence beginning 'It is not likely that' (i.e. an impossible event).



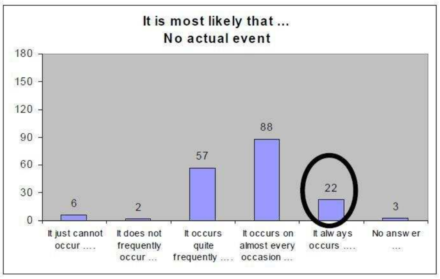
(a)



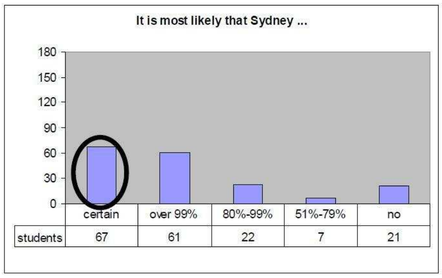
(b)

Diagram 4

We received similar results regarding a certain event. When an actual everyday event was chosen, 67 students completed the sentence 'It is most likely that Sydney . . . ' with a certain outcome whereas only 22 students assigned the phrase 'It always occurs' (i.e. a certain event) to the sentence beginning 'It is most likely that . . . '.



(a)



(b)

Diagram 5

We found that in the field of probability, both at its foundation stage as well at a later stage, there are some terms to which no definitions can be assigned. These concepts develop in a spontaneous way and each and every student links his/her own images/concepts to every particular problem.

According to the diversity and irregularity of the answers given to the Questions 5-9 our students have different thoughts on the concepts of probable, unlikely, no chance etc. On the whole, answers to Questions 5–9 confirm the validity of hypothesis Nr 2.

3. Conjunctive fallacy

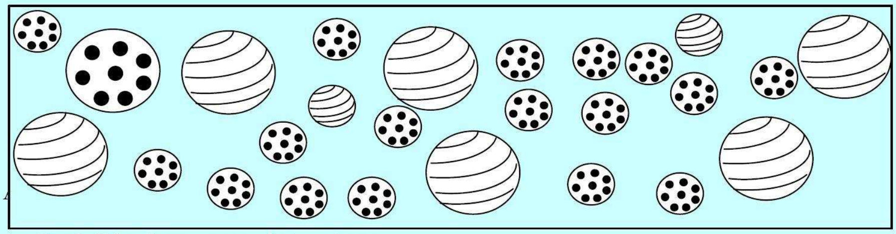
Previously the students had already learnt that the chance of two events occurring at the same time (their products) cannot be bigger than the chance of either event. $P(AB) \leq P(A)$ and $P(AB) \leq P(B)$

Despite this, however, when they must give a quick and intuitive answer, they very often tend not to use this knowledge.

Numerous authors have already worked up this subject (Tversky, A. and Kahneman, 1983). Our goal is now more complex. We were not merely interested to know if our students fall for the trap but to observe if it depends on the way of questioning.

The Questionnaire contained the following two exercises:

22. Take a look at these balls. There are small/big and striped/spotted balls.



One ball has been removed at random.

Circle the statement that you think is the most probable.

a) A small spotted ball has been removed.
b) A big ball has been removed.
c) A small ball has been removed.
d) A striped ball has been removed.

23. Katie is keen on sport. She goes to the gym five times a week. She also likes cakes and goes to the nearby cake shop 4-5 times a week.

Circle the statement that you think is the most probable.

a) Katie is going to the gym tomorrow.
b) Katie is buying a cake tomorrow.
c) Katie is going to the gym tomorrow and is buying a cake tomorrow.
d) Katie is not going to the gym tomorrow and is not buying a cake tomorrow.

We can feel that some analogy exists between the two exercises but this analogy often remains hidden. A drawing is attached to the first exercise. In the second exercise this drawing must be created by the student himself/herself. It is interesting to know in which exercise the students prefer to use the rules of multiplication. If you give a quick and shallow answer to these exercises, you may find yourself trapped. It is true that there are many small and spotted balls and more small balls. The same is true for the other exercise: Katie does go both to the gym and to the cake shop quite often but still, the probability that these two events should occur at the same time is much less than the probability that either of them occurs on its own, independently of the other.

Most of the students chose the product-type solution in both exercises, i.e. they said that it was most probable that a small spotted ball had been removed (exercise 22) and Katie would go to the gym and buy a cake alike (exercise 23). Since in probability calculation there are numerous exercises which are based on this approach, our presumption was thus confirmed by these answers. It is worth noting here that many more students were trapped in exercise 23 (72% or 130 students) than in exercise 22 (42% or 74 students).

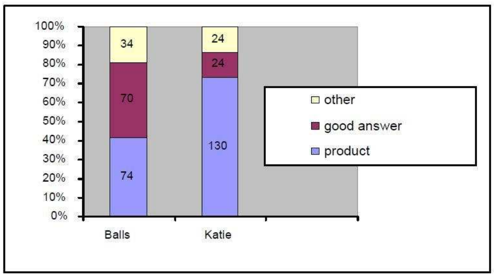


Diagram 6

Can this great difference perhaps be explained by the way these two exercises were formulated? In the exercise where a drawing was attached, it was easy to count how many small/spotted/striped/big balls there were. Also, it was clear to see that there were less small spotted balls than small balls. This kind of help was not provided in the second exercise, where the students, consequently, may have drawn their own, very differing 'inside drawings'. It is actually not an easy task to make an inside image of this situation. For example, if a student makes an inside image of a calendar with the days of the week, then he/she will only concentrate on the fact that there are quite a lot of days when Katie may go to the gym and buy a cake as well.

Finally, there is one more important difference between the two exercises. Namely, exercise 22 is about a problem which does not typically occur in everyday situations but is often practised at maths lessons at school. No doubt, this suggests to the students that a mathematical model should be applied here. The second exercise, on the contrary, is a very common everyday situation and here students are prompted to listen to their intuitions rather than try and apply some kind of mathematical considerations.

In the light of the above it can be concluded that when you are working on a mathematical problem, it is not at all easy to abstract from the particular context.

Tasks that can be associated with well defined mathematical models were solved more easily. Therefore we believe that in our third hypothesis our assumption was confirmed.

4. Interrelation between two events

When we solve mathematical problems, we normally follow the rules of formal logic. For example, the statement 'Every prime number is an odd number' is not true. The truth value of this statement is modified by the only one exception to this rule. In the world of social sciences and natural sciences, unless it is of some importance in the particular problem, such cases are dealt with much more liberally – if there are only one or two exceptions, the truth value of a statement is not affected at all. For example, if you say 'A calf has one head', this statement will remain valid despite the fact that everyone has heard of cases when calves were born with two heads. In other words, truth here is treated as a kind of tendency.
A number of problems can be interpreted in two different ways:

- If our experience shows that whenever 'A' appears, 'B' appears too and vice versa, then we will tend to think that these two events are synchronous.
- In a 'cause and effect' type phenomenon this pattern will be formulated as follows: 'Whenever 'A' appears, 'B' will appear too but not vice versa.'

Piaget did research only in the latter type whilst others, incl. József Nagy, did research in the interpretation of accompaniment. (See: Bán Sándor, 1998, pp. 221–227.)

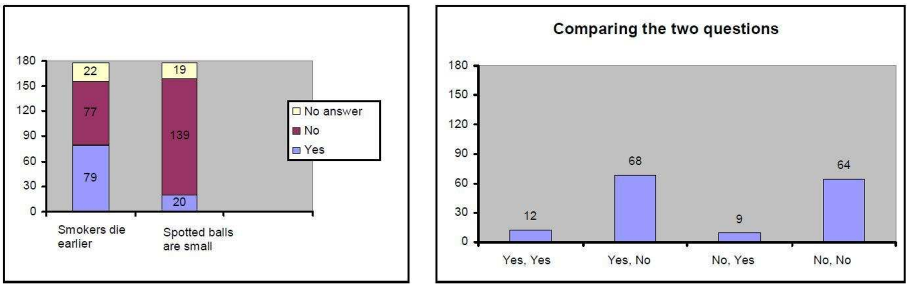
If it is not clear for a student what the actual context of the teacher's question is or what type of answer he/she is expected to give, he/she gets often confused. In our research we were very much interested to learn how our questions were interpreted by our students. Is their interpretation of a cause and effect or of an accompaniment type? Our related questions were as follows.

20. Do you think the following statement is true? 'Smokers die earlier than non-smokers.'
YES / NO
Give your reasons. ______________________________

21. Do you think the following statement is true? 'Spotted balls are small.'
YES / NO
Give your reasons. ______________________________

If you think in the 'cause and effect' scheme, NO is the correct answer in both cases. (Not every spotted ball is small. Some smokers die at a very old age.) If you think in the 'accompaniment' scheme, YES can be the correct answer in both cases.

A YES answer was given to the first question by 44% (79 students), but only by 11% (20 students) to the second question. There were only as few as 12 students who answered YES to both questions. These students applied the same scheme for both answers although the two questions were formulated in different ways. A NO answer was given to the first question by 77 students, out of whom 64 applied the same scheme for the second question too.



(a) (b)

Diagram 7

Only 43% (76 students) applied the same scheme to both questions. They interpreted the two questions in the same way, either in this scheme or in the other one.

What is the reason for this randomness in the answers? Is it perhaps the way the questions were formulated? One of the questions was familiar to the students – they had learnt an equivalent situation in their previous studies, at the course

Logics. If they remembered to apply the same scheme here, then of course they said that the statement 'Spotted balls are small' is NOT true (because not every spotted ball is small, counter-examples can also be given). Regarding the other question, here they could feel more independent because the way the question was formulated (quasi in the world of natural sciences) allowed them to ignore eventual exceptions to a greater degree. Appr. 38% (68 students) answered YES to the first question and NO to the second question. This leads us to believe that these students applied different schemes to the two questions.

In these questions students felt a bit confused because we had not made it clear to them what our expectations were. In their mathematics studies, students normally adjust our suspected expectations to some general experience. The second part of these questions ('Give your reasons') was a good means to ease this confusion because by explaining their reasons they were able to reveal the schemes they had chosen to apply and, in the second question, also to provide an explanation from the word of natural sciences.

There are valid arguments for both schemes.

'YES – as far as I know, statistics show that smokers die at 77 whilst non-smokers die at 84'.

'YES – that's what I have often heard and I guess it must be true.'

'NO - non-smokers can also die from lung cancer or from other diseases.'

'YES – most spotted balls are small.'

'NO – I can see one big spotted ball as well.'

It follows from the above that the same problem needs to be solved by different schemes, depending whether we speak about everyday life or the world of mathematics (where, apart from logical thinking, no other approaches are allowed). In the teaching of probability at school, this fact is often not paid due attention to and expectations, as a rule, are not adjusted to the particular problems.

We believe that the third hypothesis is verified by the above mentioned and the point 4.

5. Estimation of probability, emotional factors

Question Nr 3 was:

3. You throw a standard coin and you get a 'tail' in five consecutive throws.

   Circle the statement that you think is the truest.

   a) There's a bigger chance that you'll get a 'head' in the next throw.
   b) There's a bigger chance that you'll get a 'tail' in the next throw.
   c) There's an equal chance that you'll get either a 'head' or a 'tail' in the next throw.

Give your reason. ______________________________

Mind you, we did not ask about the chance of getting a 'tail' in six consecutive throws. What we asked was what is the chance of the sixth throw. There is really little chance that you'll get a 'tail' in six consecutive throws (this is the product of the probabilities of independent events, in this case: $(1/2)^6$). But, if you take the chance of getting a 'tail' in the sixth throw (after you have had a 'tail' in the five previous throws as well), it is again $1/2$. This means you must make a distinction between the chance of the occurrence of a product event and that of a probable occurrence. We wondered whether our students were aware of this rule or not and also, if they were, whether they would remember it in games.

The answers were in line with our expectations. Since probability calculation was not a new field for our students (and they must have come across this problem in secondary school), 90% (160 students) gave a correct answer to this question.

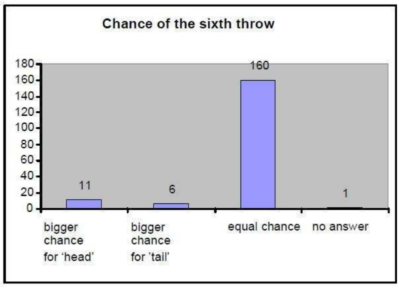


Diagram 8

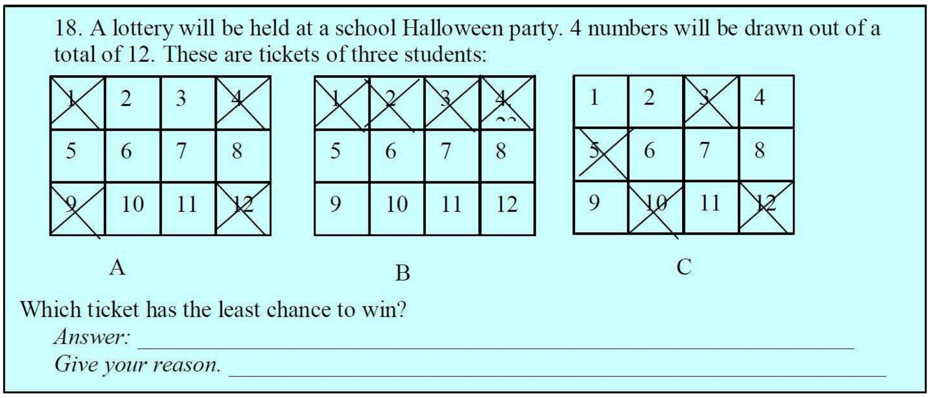
We went on to investigate how our students would respond in 'real' situations. Four weeks after the questionnaire was filled in, we played the same game in two groups. First I did not make any reference to the related question in their questionnaires. With the help of a colleague, who was taking notes, I started throwing a two-coloured disc (red on one side, blue on the other). We arranged the scene so that the throws could not be seen by the students. The students had to make guesses whether the disc would fall on the table with its red or blue colour side upwards.

Previously we had agreed with my colleague that she would say ‘It’s red’ for every throw, regardless of whether it was red or blue. The two of us even did some acting and tried to pretend via our facial expressions, gestures and comments that we too were very astonished to get such a series of outcomes.

Before the first throw, as we had expected, half of the students (appr. 15) said it would be red, the other half (appr. 15) said blue. However, after every new throw, there were more and more students who said that the next throw would be blue. After the fifth throw, there were only three students in the first group and two in the second group who said ‘It will be red’. Before the sixth throw I asked them: ‘What do you think the chance is now that I will be throwing blue at last?’ Typical answers were: ‘Red has been thrown five times in a row, it should be blue now’, ‘The more reds you throw, the more likely it is to throw a blue at last’, ‘Miss, try throwing now with your left hand’, ‘It is very-very unlikely to throw six reds in a row’, ‘I know of course that red and blue again have equal chances but, after all, we have had so many reds now’. This latest remark led to a heated debate in one of the groups, which ended by everyone agreeing that yes, the chance to throw a red or blue is equal (1/2) for each and every throw. Later, when this game was jointly evaluated, I drew a comparison between their approaches during the game and their answers in the Questionnaire.

And here we were all surprised to come to the conclusion: the students have some knowledge on this problem and still they do not use this knowledge in a game. This statement is substantiated by the fact that as the game proceeded and more and more throws were made, less and less students said that the chance to get a red or blue was still equal. In the game, emotional expectations (hoping to balance between red and blue in the short run) became more and more typical than the activation of already existing knowledge.

Question 18, a lottery type game, was about a similar problem. We knew that in Hungarian secondary schools most students had already made their calculation regarding their chances to win. This problem is often discussed in Hungarian coursebooks. In this question we asked the students to find the solution which has the least chance to win.


18. A lottery will be held at a school Halloween party. 4 numbers will be drawn out of a total of 12. These are tickets of three students:

Which ticket has the least chance to win?

Answer: ____

Give your reason. ____

Only 37% (66 students) said that each of the three tickets had equal chances to win. The majority of the students (92) thought that the ticket with 1,2,3,4 had the least chance to win.

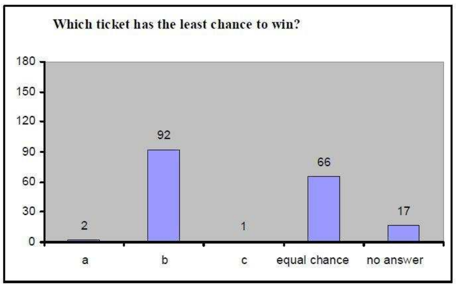


Diagram 9

Different reasons were given for and against. Some students even made mention of a Hungarian blockbuster movie (A Kind of America), in which the jackpot is hit with numbers 1,2,3,4,5 in a lottery of 90 numbers.

In the light of the above as well as on the basis of our observations in class we suspect that emotional factors may be involved in the field of probability. This confirms our 4. Hypothesis. In other words, beliefs occasionally may affect or even inhibit the use of existing knowledge of a theoretical model in estimating chances. Such beliefs, their types and effects are subject to new research.

Conclusion

Based on the evaluation of the answers to the Questionnaire and on our findings in class we have come to some conclusions which are worth further pondering upon. Estimation of probability is largely influenced by the way a particular problem is formulated and by the type of the activity involved. Our findings suggest that this has an effect

- on expressing probability,
- on making use of analogies,
- on differences between scientific terms and everyday usage
- on the role of beliefs when estimates are being made.

In school education the concept of probability is prepared in a lengthy process including practical activities, measurements and experiments. However, it is difficult to foresee how a particular game or experiment helps develop the concept of probability or how actual calculations can be based on games/experiments at a later stage. Vice versa: it is difficult to see how knowledge acquired in the process of learning about probability at school is used when the actual chance of the occurrence of non-deterministic events is calculated. This latter statement is supported by our findings too. The interrelation between the development of probability concepts and probability calculation is subject to further detailed research in didactics.

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