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Forming the concept of congruence I.

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Abstract. Teaching isometries of the plane plays a major role in the formation of the congruence-concept in the Hungarian curricula.

In the present paper I investigate the way the isometries of the plane are traditionally introduced in most of the textbooks, especially the influence of the representations on the congruence concept, created in the teaching process.

I am going to publish a second part on this topic about a non-traditional approach (Forming the concept of congruence II). The main idea is to introduce the isometries of the two dimensional plane with the help of concrete, enactive experiences in the three dimensional space, using transparent paper as a legitimate enactive tool for building the concept of geometric motion. I will show that this is both in strict analogy with the axioms of 3-dimensional motion and at the same time close to the children's intuitive concept of congruence.

Key words and phrases: teacher education, concept formation, acquisition of mathematical concepts, transformation geometry, manipulative materials, teaching methods and classroom techniques.

ZDM Subject Classification: B50, C30, E40, G50, Q60, U60.

1. Introduction

In the present article I use some important concepts connected to teaching of congruence, which have to be distinguished. My goal is to analyse different teaching methods of the *2-dimensional congruencies*. I use the term *3-dimensional motion* for the orientation preserving (direct) 3-dimensional isometry (which is also called rigid motion or rigid body move). When referring the concrete manipulative representation of the planar congruencies I will use the term *translocation.*

The present paper is a comparison of two different strategies for the teaching of the isometries of the plane, according to the following aspects:

- axiomatic background (mathematical aspect), based on the foundation of the concept of congruence in Hajós György's axiomatic system;
- psychological background (didactical aspect), based on Bruner's theory of representation, Tall and Gray's theory of procepts and the van Hiele model of spatial thinking;
- the role of manipulative techniques and their influence on the forming of the congruence concept (practical aspect).

In my consideration an isometry of the plane is restriction of a corresponding 3-dimensional motion (I will refer to that as *'motion given by flags'*). That can be represented by *'free translocation given by flags'.* In the traditional way teaching planar isometries is based on point-by point correspondences, represented with *'translocation through prescribed route'.*

I show in the present paper that the representation by translocation through prescribed route can cause problems in the formation of the concept of congruence.

In the II. paper on this topic I show, how these problems are tackled if we base the planar isometries on 3-dimensional motions and represent them with the help of translocations given by flags.

In the analysis of these methods I will show that congruence is a 'procept', a process based concept that unites the intuitive and the abstract properties of congruence.

2. Theoretical background

Congruence is a central concept of geometry. On one hand children have many everyday experiences about it, and a strong intuitive meaning of it. On the other hand in the mathematical theory it is defined by axioms. So we need to know both the pedagogical-psychological and the mathematical background of this concept.

2.1. Pedagogical-psychological background

In the teaching practice representations play a major role in the creation of a clear congruence concept. I mainly use Bruner's *theory of representation* and Tall & Gray's *theory of procept-formation*.

2.1.1. Representation - dangers and advantages

According to Bruner's (1971) classifification there are three main types of the representations: *enactive, iconic and symbolic representation*, and the effectiveness of learning depends strongly upon the representation we choose at a certain stage and also upon the sequence and the timing the child encounters.

Moreover the question is not only what type of representation, and in what sequence to choose, it is also important which concrete representation to use for a certain domain of knowledge. The representations, especially the enactive and the iconic ones, may lead to strong, although non-verbal, “concept images”[1] (Tall & Vinner, 1981) and help to get to the stage of a well working symbolic representation. But also because they are not verbal they may lead to misconceptions which result in a false meaning in some areas of the concerned domain. And just because they are not verbal, it is difficult to detect these kinds of thinking-faults. However, even if they are detected, it is very difficult, if not impossible, to correct them verbally.

Understanding is closely related to mental representation. “A principle, concept is understood if the inner representation of it becomes part of our net of representations. The level of understanding depends on the number, strength and stability of connections. So understanding is just the creation of connections between principles and concepts.” (Vásárhelyi 2006). The more possible connections are offered by the representations, the deeper is the understanding it results in. This way, a carefully chosen representation can help both the slower children to achieve a more stable understanding, and the more able children to get a ground on which they can deepen their knowledge.

2.1.2. Levels of spatial thinking

There are several models of the cognitive development of spatial thinking. The model worked out by the van Hiele couple is quite widely accepted as a basis

[1]A concept image is a personal internal representation. A meaning, which is associated with different objects and other representations, belonging to a certain mathematical concept.

for studies in this field (van Hiele, 1959). We will connect the van Hiele levels of thinking with the different stages in the formation of the 'procept' of congruence.

2.1.3. Processes and concepts

In this summary I follow Meissner's review of *procept* formation (Meissner, 2001).

There is a theory of "encapsulating a process", which says that the formation of a concept is a process that is often based on activities and procedures. These activities develop an individual "concept image" in the children's mind (Tall & Vinner, 1981).

The term procept means a concept, which is both process and product. Tall and Gray (1994) gives the following definition for the procepts: "An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object. A procept consists of a collection of elementary procepts which have the same object."

Tall et al. (2000a) give examples of procepts in the area of algebra, analysis, ... (like adding, derivation, ...), but not in geometry. They thought that there are no procepts there, because of "... the very different cognitive development in geometry".[2]

Meissner thinks about this differently, and gives a slightly modified interpretation of procept development.[3]

I argue that the concept of congruence is a procept in its original meaning, if it is based on the teaching of isometries. On one hand the concept of function is on the original list of Tall et al. (2000a), and in our approach congruence is a special function. On the other hand it is quite obvious that in the teaching practice the concept of congruence is strongly connected to processes and actions.

[2]They reckon that there are perceptions of real objects initially recognized as whole gestalts and classifications of prototypes. Reconstructions are necessary to give hierarchies of shapes and to see a shape not as a physical object, but as a mental object.

[3]He agrees with STRUVE (1987), who thinks that children in primary and lower secondary classes learn geometry like a natural science, they describe, explain, and generalize phenomena. Thus, for them, geometry becomes an empirical theory. In Meissner's view "an elementary procept in the meaning of Tall et al (2000a) just is the shift from the empirical stage to the theoretical stage. Consequently, following these ideas there should be no fundamental obstacle to find procepts in geometry, too."

2.2. Mathematical background

Let me quote Reimann (1986): "In the intuitive plane the statement that two shapes are congruent has a realistic meaning: it refers to the existence of a 'translocation' that makes one of the shapes coincide with the other. At the deductive, logic based foundation of geometry independent of intuition – the axiom-system of Hilbert takes the congruence of two line segments or angles as a basic concept, interpreted only by the axioms of congruence. But it is possible to accept the motion as a basic concept that results in an axiom-system which is different from the hilbertian."

2.2.1. Axioms of motion in Hajós's system

Hajós (1971) in his university textbook, "Introduction to geometry", formulates the axioms of 3-dimensional motions in order to give a foundation for the notion of congruence (instead of Hilbert's basic concept of congruence).

> "VII. A motion takes a line segment, connecting two points - A and B - into a line segment connecting the images of the end points - A' and B'. It takes a straight line into a straight line, a plane into a plane.
> VIII. There is one and only one 3-dimensional motion which takes a given half-plane with a given ray on its boundary into another (not necessarily different) given half-plane with a given ray on its boundary." (Hajós 1971)

The distance is also a basic concept in this system. The axioms of measurement speak about classes of pairs of points. Two pairs belong to the same class, if they can be made to coincide by a motion.

2.2.2. The significance of the axiomatic background in the teaching

In Hungary, we do not teach axiomatic geometry in primary and secondary schools, we introduce axioms only in the university maths courses. However, all these axiomatic foundations are indirectly present both in primary and secondary school teaching.

Before the sixties teaching geometric transformations was not an essential part of the Hungarian curricula. The congruence was based on the theorems of the congruence of triangles. In the background of this concept one can find Hilbert's axiom system.

From the middle of the sixties a new curriculum was gradually introduced in Hungary. In this curriculum the teaching of geometric transformations became a central idea as special cases of mapping concept. The intention of this curriculum was to build the concept of congruence on the congruence correspondences, i.e. the isometries. However the axiomatic background did not change much. When reasoning about congruency, schoolchildren still used arguments based on symmetry much less than the basic theorems on the congruence of triangles. This is so, even when the former would have been better suited to the problem.

I will show that teaching transformations in the school can be based straight on the notion of 'translocation', in such a way, that in the background the axioms of motion stand.

3. Introduction of the congruence correspondences on the comprehensive level in Hungarian textbooks

The concept of congruence in the lower grades is based on the natural and sensible idea that two plane figures are congruent if they can be made to coincide. The congruence is a correspondence that tells which points of one figure coincide with which points of another figure. By making a trace of one shape and making it to coincide with another one is a meaningful method of determining the congruency of the two shapes.

The next step in building the concept of congruence is teaching special isometries, as a new topic. In all the textbooks the isometries are introduced parallel in two different ways.

- An isometry is *a rule* which tells, for each point of the plane, one by one, how to obtain its corresponding image. I will call this a *point-by-point definition.* In this case the transformation is specified as a correspondence among points of the plane.
- An isometry is viewed as *a translocation* of a piece of paper, representing a plane (or part of a plane), which takes each point into its corresponding image. In this case the transformation is represented by an action performed on a given figure or on the whole plane.

3.1. Transformations given with the help of *a rule*

In this case the correspondence is given by a rule which maps an arbitrary point of a plane to another (not necessarily different) point of the same plane. Let us see some examples:

Example 1 (Szeredi, Kovács, 2003, p. 53):

"Play a 'rule-game' with the points of the plane!

We gave a few points with their corresponding image in Figure 1. For each of the six cases, find a possible rule which explains how to find the image X' for any point X."

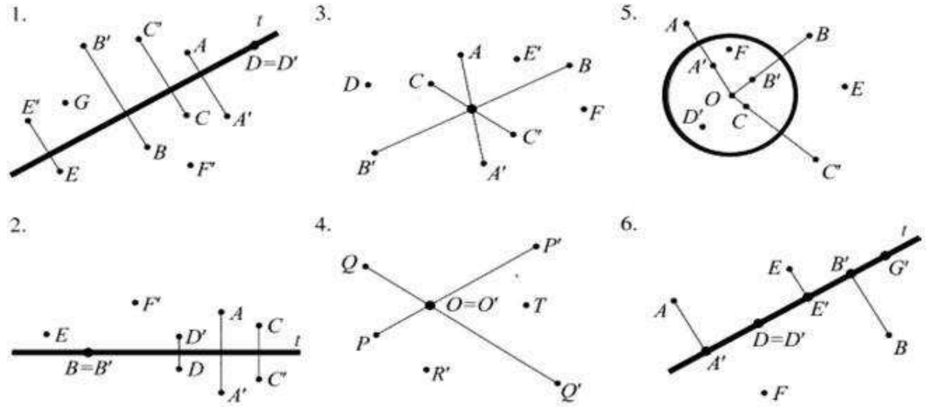

Figure 1. point-by-point definitions

"Ad case 2: A possible rule is axial reflection.
There is a given line t – the axis. To get the corresponding image of any point P, draw a line through it which is perpendicular to the axis. That intersects the axis in a point T. The image P' of P is on the perpendicular line, at equal distance from T on the other side of line t, i.e. $PT = P'T$. If P is on the axis, $P' = P$."
"Ad case 5: A possible rule can be called reflection on a circle.
There is a given circle, with a centre O. For any point P, different from O, take the ray OP. This ray intersects the circle in exactly one point, M. P' (the image of P) is on the line OP, at equal distance from M, on the other side of the circle: $PM = P'M$. If P is on the given circle, $P' = P$. It is an interesting issue how to define the image of O, a simple solution is to make it O itself."

Axial reflection is a congruence transformation of the plane. It keeps the distances, the angles, the straight lines, etc. *Reflecting on a circle* is not a congruence transformation. It distorts the straight lines, changes the distances and the angles (although there can be exceptions).

These examples show that these kind of point-by-point definitions do not guarantee the isometry of a correspondence. With the help of these kind of examples we can point out the necessity of proof.

3.2. Transformations given as 3-dimensional motions

Introducing planar isometries as restrictions of 3-dimensional motions represented by spatial translocations is a straight continuation of the childrens early, natural experiences of congruence, but it is not easy to provide exact definition for a specific mapping as a motion.

Hajós's explanation of 3-dimensional motion is based on the physical experiences about moving. "When a figure moves, then its points get to a new position, but it is possible that some of the points stay in the same place. The shape of the figure does not change while moving. We can speak about the moving of the whole space, namely that all the points of the space move together. The moving of a figure can be extended into the moving of the whole space, so we can think that the moving space takes the figures into their new position. We are not interested in the route of the points of the moving figure, only in the original position and the end position of the points." (Hajós 1970)

In teaching the planar isometries are usually represented with the help of a piece of paper. This piece of paper represents a duplication of the plane, which can be translocated into a given position so, that the points of the duplication end up as points of the same plane. In the teaching practice there are two types of translocations, which are essentially different from the didactical point of view:

- *translocation through prescribed route* (when they are defined by prescribing the route of the points) and
- *free translocation given by flags* (when they are defined by the starting position and the end position with the help of two flags without any prescription of the route).

4. The representation by translocation through prescribed route

In this part I show, how isometries are represented by translocation through prescribed route and analyse the influence of this kind of representation on the evolution of the congruence concept.

In this case the points are dragged into their image along a prescribed route, the translocations are used mainly as illustrations for the 'point-by-point definitions'. For example, folding a paper along the axis illustrates axial reflection, etc. The aim of such an illustration is to show that certain special correspondences – given point-by-point – are congruencies.

4.1. The role of representation by translocation through prescribed route

In the following analysis I use axial reflection as the main example. However, the observations apply to the teaching of other isometries as well.

Mathematics textbooks introduce axial reflection in various ways, but there is a common structure of building the concept of axial reflection and its properties. Most textbooks give two different techniques for getting the reflected image:

(a) reflecting the shape by folding a piece of paper along the axis, and rotating a half plane with 180° around the axis or rotating of the plane around the axis;

(b) constructing the reflected image of certain points using ruler and compass.

The technique (a) which translocates a shape, is used during the introduction of axial reflection. Its main purpose is twofold. First, it helps to establish some basic statements about reflection or axial symmetry, because it is close to the intuitive concept of congruence. Second, the technique is used to support the development of a 'correct definition', the formulation of a point-by-point rule, which specifies how to get the corresponding image of an arbitrary point. Having obtained the definition and the statements about the main properties of axial reflection all these books put aside the technique (a), such as paper folding and rotating around the axis. Subsequently, they switch to technique (b): constructing the corresponding image with ruler and compass.

4.2. The representation by translocation through prescribed route and the procept of congruence

This method contains all the elements of procept formation: the process, the symbol and the object.

- The processes are:
  (a) the translocation of a piece of paper, and
  (b) the construction of shapes by ruler and compass.
- The symbol is the given axis t, or the twin picture of shape and image.[4] (The symbol can be later a mathematical symbol R_t, as a formal notation for reflection on line t.)

[4]A symbol in sense of procept formation may mean visual and symbolic representation as well in Bruner's classification.

- The object is the pair of shape and its congruent image.
  The concept is the congruence correspondence between these shapes.

The technique (a), the translocation of a piece of paper, is a procedure closely connected to the intuitive concept of congruence. However, it is used very briefly. The new process, technique (b) is taught to the children soon after the "point-by-point" rule is defined. They learn how to construct the images of points, lines and circles. The new process, the construction by ruler and compass is practised a lot.

Davis (1984) says about the procedures (processes): "When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity - it becomes a *thing*."

The process of construction by ruler and compass is quite difficult in some cases (rotation is especially complicated), and because too much effort is needed to produce the image, it may not become a *thing*.

Although the translocations may illustrate very clearly the congruence properties of a certain transformation, they cannot be conveniently and correctly used for constructing the images of shapes, that means that they cannot be used as substitutes for constructions with the ruler and compass. Therefore they are used only for a short time at the very beginning of the introduction of the different kind of isometries. The too early change of techniques causes, that the amount of experiences about the corresponding congruent (reflected, rotated, ...) images are very limited. So the concept of congruence too soon breaks away from the intuitive congruence concept of the young child, which was based on everyday experiences about translocating and coinciding shapes, and in the upper classes is based on abstract statements about the congruence of triangles. More, clearer and in time less concentrated experiences would support the stability and compatibility of the inner representations of isometry.

Moreover representing isometries by translocation through prescribed route induces additional association between the correspondence and route. If the main emphasis is on the route, or on other concrete rule of producing the corresponding image instead of on the corresponding pairs of points, it is possible to generate a hidden conflict factor in the "concept image". The associated route can hinder the evolution of a correct abstract transformation concept, namely the equivalence of correspondences.

The dominant use of constructing the iconic symbol, the twin picture of shape and its image by ruler and compass causes other problems:

- the construction lines and the approximate image does not provide a convincing concept of congruence;
- the correspondence of the inner points of segments becomes easily forgotten because of too big emphasis on the vertices.

5. Summary

According to the threefold aspects stated in the introduction, I gave an analysis of a method based on representing the isometries by translocation through prescribed route. The main thesis of the present paper is that manipulation has only a small role in this method; that the representations serve mainly as illustrations; and they may lead to false concept images (the concept of transformation is perturbed by the route of the points). This is in accordance with the view of András Ambrus (1995) on teaching geometry in Hungary. Ambrus says: "There is a gap between the primary and the lower secondary level. In the grades 5–6, the formal approach dominates, the teaching of geometry is system oriented, the actual activities are not continued."

In the II. paper on this topic I will show an alternative way of representing the isometries of the plane which I think to be appropriate to create a more coherent congruence concept.

References

[1] A. Ambrus, Some trends in Hungarian geometry teaching, *In: Perspectives on the teaching of geometry for the 21st century, pre-proceedings for Catania Conference*, Department of Mathematics University of Catania, 1995.

[2] T. Andrássy, T. Czeglédy, I. Czeglédy, S. Hajdu and L. Novák, *Matematika 7. osztály*, Mathematics textbook, grade 7, Calibra Kiadó, Budapest, 1994.

[3] J. S. Bruner, *Toward a Theory of Instruction*, The Belknap Press of Harvard University Press, Cambridge - Massachusetts, 1971.

[4] J. Cotrill, E. Dubinsky, D. Nichols, K. Schwingendorf, K. Thomas and D. Vidakovich, Understanding the limit concept, Beginning with a coordinated process schema, *Journal of Math. Behaviour* **15**, 1996.

[5] R. B. Davis, *Learning mathematics: the cognitive science approach to mathematics education*, Ablex, Norwood, NJ, 1984.

[6] T. Gallai and R. Péter, *Matematika a középiskolák I. osztálya számára*, Mathematics for the grade I of secondary schools, Tankönyvkiadó, Budapest, 1952.

[7] E. M. Gray and D. O. Tall, *Duality, Ambiguity and Flexibility in Successful Mathematical Thinking*, In: Proceedings of PME-XV. vol. 2, Assisi, Italy, 1994.

[8] Gy. Hajós, *Bevezetés a geometriába*, Introduction to geometry, Tankönyvkiadó, Budapest, 1971.

[9] H. Meissner, *Procepts in geometry*, In: European Research in Mathematics Education II, 2001.

[10] I. Reimann, *A geometria határterletei*, Topics from the frontiers of geometry, Gondolat Kiadó, Budapest, 1986.

[11] A. Sfard, *Two conceptions of mathematical notions: operational and structural*, In: Proceedings of PME-XIII. vol. 3, Montreal, Canada, 1987.

[12] Sulinova Educatio Kht, 2008, Packages of materials for teaching mathematics, `http://www.sulinovadatbank.hu/index.php`.

[13] É. Szeredi and Cs. Kovács, *Geometric transformations*, In: E. Csahóczi,K. Csatár, Cs. Kovács, É. Morvai, G. Széplaki, É. Szeredi: Matematika 7. osztály (Mathematics textbook for 6. grade), Apáczai Kiadó, Celldömölk, Hungary, 2003.

[14] D. O. Tall and S. Vinner, Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics* **12**, 1981.

[15] D. O. Tall, *Advanced Mathematical Thinking*, Kluwer Academic Publishers, Dordrecht, NL, 1991.

[16] P. van Hiele, *The Childs Thought and Geometry*, Brooklyn, NY: City University of New York 1, 1959.

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