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Examples of analogies and generalizations in synthetic geometry

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Abstract. Teaching tools and different methods of generalizations and analogies are often used at different levels of education. Starting with primary grades, the students can be guided through simple aspects of collateral development of their studies. In middle school, high school and especially in entry-level courses in higher education, the extension of logical tools are possible and indicated.

In this article, the authors present an example of generalization and then of building the analogy in 3-D space for a given synthetic geometric problem in 2-D.

The idea can be followed, extended and developed further by teachers and students as well.

Key words and phrases: analogy, generalization, synthetic geometry.

ZDM Subject Classification: D40, G10, G40.

1. Introduction

In the large process of studying and learning there are many methods, tools and achievement ways, some of them developed by themselves, others for interconnections. Two of them are the generalization and analogy.

In 1954 the mathematician George Polya ([15]) describes the process of doing mathematics. He identifies three iterative processes:

– Generalization is "passing from the consideration of a given set of objects to that of a larger set";

– Specialization is "passing from the consideration of a given set of objects to that of a smaller set, contained in the given one", and

– Reasoning from analogy involves identifying "similarity on a more definite and more conceptual level".

Today, in any instructional process of learning, in schools, colleges or universities, at any level of study, the notions of generalization and analogy are playing an important role.

In helping with these tools, a teacher needs to be prepared to offer situations in which the student has the possibility to expand their knowledge and be able to apply their knowledge to different analogical cases.

From our experience, we know that the instructors and the students struggle with topics in synthetic euclidean geometry, which are considered by many to be finished. The complexity and the beauty of this subject come from the fact that it's needing a strong rational and logical potential, as well as imagination and mathematical knowledge. The reasoning is essential and considered basic with this kind of geometry.

We have chosen in the article to present an example of extension of the regular routine study in a such class of synthetic geometry, opening the horizon of new discussions and topics for teachers and students. Starting from a given geometric problem with triangles, lines and points, we built a generalization of it, extending the properties or restraining the constraints. The new result is presented in synthetic geometry, too. Another part includes the analogies made in 3-D space.

In a regular high school or middle school talented-students Mathematics class the following problem is given (also it can serve for an introductory Geometry/Methodology course in higher education):

PROBLEM 1 (Starting Problem). *Let ABC be a triangle and d an exterior line of it. Consider G the centroid of the triangle and A_1, B_1, C_1 and G_1 be the projections of A, B, C and G, respectively on the line d. Then the following relation takes place:*

$$AA_1 + BB_1 + CC_1 = 3GG_1.$$

It can be solved using basic knowledge in geometry and reasoning.

First of all, let's recall an important concept often used along this article:

DEFINITION 1. The *centroid* (also called geometric center, or barycenter) of a plane figure is the intersection of all straight lines that divide the figure into two parts of equal moment about the line. Extended to any object in n-dimensional space, the centroid is the intersection of all hyperplanes that divide the object into two parts of equal moment.

In Physics, the centroid is called also center of mass or weight center.

For particular objects, like triangles (or tetrahedrons), the centroid is found at the intersection of the medians (or the intersection of all line segments that connect each vertex to the centroid of the opposite face).

Our study here is to extend the work with this kind of problems, using three steps:

(1) Build its analogous in 3-D space:

Using the analogy methods the *Starting Problem* can be "translated" in 3-D space as follows:

Problem 2 (The Analogous Problem). *Let $ABCD$ be an arbitrary tetrahedron and (d) an exterior plane of it. We denote by G the centroid of the tetrahedron. Through the points A, B, C, D and G we draw perpendicular lines to the plane (d) and denote the intersecting points with A_1, B_1, C_1, D_1 and G_1, respectively. Then*

$$GG_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

(2) Build the generalization of Analogous Problem:

Using methods of generalization, the *Starting Problem* can be extended to this:

Problem 3 (The Generalization Problem). *Let $ABCD$ be a tetrahedron, d an exterior plane of $ABCD$ and M an arbitrary interior point of the tetrahedron $ABCD$. We draw parallel lines through the points A, B, C, D and M, which intersect the plane d in A_1, B_1, C_1, and M_1, respectively. Then there are three constant real numbers m, n and k such that:*

$$MM_1 = \frac{kn(m-1)AA_1 + k(n-1)DD_1 + (k-1)CC_1 + BB_1}{mnk}.$$

(3) Find specializations for the generalization.

Will be given some examples of specializations, starting from the *Generalization Problem*. One of them will be the *Starting Problem*.

The later proofs for Problems 1-3 will endorse our work here.

2. Preliminaries

In our days, the first preoccupation for teachers is to enhance students' knowledge using different methods of teaching. All these methods give birth for a vast field of research called generic "education". Now the education is an expansive field of research, which encompass all academic departments (B. Sriraman, [21]).

In solving complex problems beside the known levels and classifications, there are the psychology of thinking involved. Funke (in [7]) defines the complex problem solving "by contrasting it with simple, noncomplex problem solving in terms of the non-orthogonal criteria". Based on his theory the psychological thinking increases the ability to understand the given data and the question of the problem; also it determines the goals of proofs and discovers the properties involved in solving the problems. To solve complex problems first the students are to learn to develop pattern-recognition and classification skills. Also they have to justify the answers using the empirical evidence and deductive reasoning. The progress of students consists in their capacity to formulate the generalizations and conjectures, evaluate conjectures and construct mathematical arguments (B. Sriraman in [21]). Many researchers like S. Epp ([5]) investigated the deductive reasoning in many domains such as deductive proof using empirical evidence and analytical approach.

Polya ([16]) wrote an ample paper about demonstrative reasoning and plausible reasoning. A particular case of plausible reasoning is the inductive reasoning (inductive logic), that invoke the inductive reasoning of generalization and analogy. The analogy and inductive reasoning are the most important contents in mathematical discovery. They are considered a particular case of plausible reasoning.

2.1. Analogies

"A conjecture becomes more credible if an analogous conjecture becomes more credible" (G. Polya)

Analogy is a powerful cognitive tool for discovery and learning new concepts based on existing knowledge. It allows students to learn intuitively using the existing knowledge as a bridge between similar situation and new situation. The analogy can represent an abstract idea in terms of concrete or physical structures. The success of intuitive learning is based on understanding and persuading in finding the similarities between two situations, and concrete structures from life.

The analogy has a significant role in teaching, learning and understanding geometrical properties. Its method is successful and important because it is the most familiar method of discovering the similarities and differences between figures. In Euclidean geometry and also in non-Euclidean geometry many geometrical properties are proved, using comparison and analogy with the existing properties. For this reason words as: "same", "similar", "just like" are used in teaching. The geometry itself offers a lot of analogies.

If we extend the dimension for the real space of geometry, we also can make analogies between, for instance, 3-D Euclidean space (our real space) and the 4-dimensional space-time.

Many of the ideas of Euclidean geometry have direct analogies in space-time geometry. This analogy makes some of the interesting ideas of space-time a little more intuitive.

2.2. Generalizations

"An idea is always a generalization, and generalization is a property of thinking. To generalize means to think."(Georg Hegel)

Generalization is a foundational element of logic and human reasoning. It is the essential basis of all valid deductive inference. The concept of generalization has broad applications in many disciplines, sometimes having a specialized context-specific meaning.

In general, for two related concepts, X and Y; we can say that the concept X is a generalization of the concept Y if and only if every exemplification of concept Y is also an exemplification of concept X. There are cases of concept X which are not instances of concept Y.

The generalization has an important significance in teaching and learning all mathematical subjects. The teachers can develop student's generalization tool, using inductive and deductive approach. The students are using inductive approach based on discovery of information, they are guided by the teachers to observe the differences and similarities between concepts, and to draw patterns and trends and finally to state the conclusion and a generalization. Based on discovery process students complete the inductive approach, which is related to learn by asking questions. The deductive approach is explanatory where students have to support or refute a given theory or hypothesis. In this approach teachers provide all the materials assisting the students to verify the generalization.

In teaching both inductive and deductive generalization used when the teachers want to teach specific topics in mathematics.

From the geometry point of view the generalization method has a great role of visualization and the perception of geometrical properties. Using generalization we can increase the student's ability in learning and understanding the geometrical concepts. Moreover we can investigate students capacity to generalize the geometrical figures and objects and the way of their developments.

In generalizing geometrical figures and objects that we can discuss, in function of their complexity, three important classifications are:

- generalization of definitions,
- generalization of geometrical properties of some objects and
- creative generalization.

Each type has a very important function in generalization process. For example in generalization of a definition the students have to state which figure is more general then the other from the same figure set. The generalization of properties usually refers to arbitrary points in plan and their properties. For the last type of classification, students have to change some features of geometrical objects such that to obtain a generalization of them.

A famous generalization is described by G. Polya in [14]: starting from the most known theorem in geometry – *The Pythagorean Theorem*, Euclid stated in one of his 13 books from *The Elements*: If three similar polygons are described on three sides of a right triangle, the one described on the hypothesis is equal in area to the sum of the two others. It's instructive to observe that this generalization is equivalent to the special case from which is started. Is instructive because we can learn from it something applicable to other cases, and the more instructive the wider the range of possible applications.

2.3. Basic facts in synthetic geometry

Because in solving the *Starting Problem* we need some basis facts in synthetic geometry and moreover, for building the analogy and its generalization we need basic facts, too, we will recall fundamental results and reasoning in 2-D and 3-D geometry. Most of them can be found in any elementary or advanced geometric textbook, like [4], [11], [10], or [17].

Let's recall some properties of the centroid of a triangle. First, we have the following:

Definition 2. A *median* of a triangle is a line segment joining a vertex to the midpoint of the opposing side. Every triangle has exactly three medians.

Then a property of medians is given by:

PROPOSITION 1 ([11]). *If ABC is a triangle and A_1, B_1 and C_1 are the midpoints of the sides (BC), (AC) and (AB) respectively, then the segments (AA_1), (BB_1) and (CC_1) intersect in a point G, which is the centroid of the triangle.*

Now, there is following a metric basic property related to the centroid of a triangle:

LEMMA 1 ([17]). *If ABC is an arbitrary triangle and G is its centroid G, then*

$$\frac{A_1G}{AG} = \frac{B_1G}{BG} = \frac{C_1G}{CG} = \frac{1}{2} \tag{2.1}$$

where A_1, B_1, and C_1 are the midpoints of the triangle sides (BC), AC and (AB), respectively.

The next fundamental and famous result is called from early time to be one of the Thales Theorems, but also is known as Similar Theorem in triangle.

THEOREM 1 ([17]). *(Thales' Theorem) For any triangle ABC, if $M \in AB$ and $N \in AC$ such that $MN \| BC$, then*

$$\frac{AM}{AB} = \frac{AN}{AC} = \frac{MN}{BC}. \tag{2.2}$$

The Thales's Theorem is used for an arbitrary triangle, but also can be extended for a trapezoid. The analogy takes place because the trapezoid is the first particular king of quadrilaterals in 2-D, having a pair of parallel sides.

Let's consider the trapezoid $ABCD$ (see Figure 1). If $M \in AD$ and $N \in BC$ such that: $MN \| CD \| AB$, and

$$\frac{AM}{AD} = \frac{BN}{BC} = \frac{1}{q}.$$

If we connect the points B and D, the line segment (BD) intersects the line segment MN in a point P.

Applying (2.2) for triangle ADB and line segment MP, we obtain:

$$\frac{AM}{AD} = \frac{1}{q} \rightarrow \frac{AD}{MD} = \frac{q}{q-1},$$

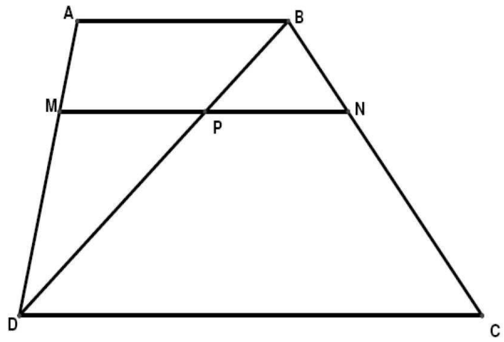

Figure 1. Extension of Thales Theorem

such that

$$\frac{AB}{MP} = \frac{q}{q-1},$$

and it follows

$$MP = \frac{q-1}{q} AB. \tag{2.3}$$

Also, applying (2.2) in the triangle BCD for the line segment PN, we find that

$$PN = \frac{1}{q} CD. \tag{2.4}$$

Adding the relations (2.3) and (2.4) we obtain:

$$\begin{aligned} MN &= MP + PN \\ &= \frac{q-1}{q} AB + \frac{1}{q} CD \\ &= \frac{CD + (q-1)AB}{q}, \end{aligned}$$

which is exactly (2.6).

There are several different methods in proving this result, and we described above just one of them.

Based on the above reasoning, we can now state:

PROPOSITION 2. *(Extension of Thales' Theorem in trapezoid) Let $ABCD$ be a trapezoid. If $M \in AD$ and $N \in BC$ such that: $MN \| CD \| AB$, and*

$$\frac{AM}{AD} = \frac{BN}{BC} = \frac{1}{q} \tag{2.5}$$

where $q > 1$, then

$$MN = \frac{CD + (q-1)AB}{q}. \tag{2.6}$$

In preparing the extension to 3-D space case, we recall the following results:

PROPOSITION 3 ([10]). *If $ABCD$ is a tetrahedron and G_1, G_2, G_3, and G_4 are the centroids of the faces BCD, ADC, ABD and ABC respectively, then the segments AG_1, BG_2, CG_3 and DG_4 will intersect in an interior point G, which is the centroid of the tetrahedron.*

PROPOSITION 4 ([10]). *If $ABCD$ is a arbitrary tetrahedron and G is its centroid, then*

$$\frac{AG}{AG_1} = \frac{BG}{BG_2} = \frac{CG}{CG_3} = \frac{DG}{DG_4} = \frac{3}{4} \tag{2.7}$$

where G_1, G_2, G_3, and G_4 are the centroids of the faces BCD, ADC, ABD and ABC respectively.

3. The Starting Problem

Our presentation starts from a synthetic geometric problem in plane (2-D):

Let's consider an arbitrary triangle ABC, and d an exterior line of it. Also, let's consider G be the centroid of the triangle ABC and A_1, B_1, C_1 and G_1 be the projections of A, B, C and G, respectively, on the line d (see Figure 2).

We denote by M the midpoint of the side (BC), so that $\frac{CM}{CB} = \frac{1}{2}$, and by M_1 its projection on d. In this situation, based on Lemma 1, we have also that $\frac{GM}{AM} = \frac{1}{3}$.

Now, in the trapezoid AMM_1A_1, because $\frac{GM}{MA} = \frac{1}{3}$, we may use (2.6) for $q = 3$, and obtain

$$GG_1 = \frac{AA_1 + 2MM_1}{3}.$$

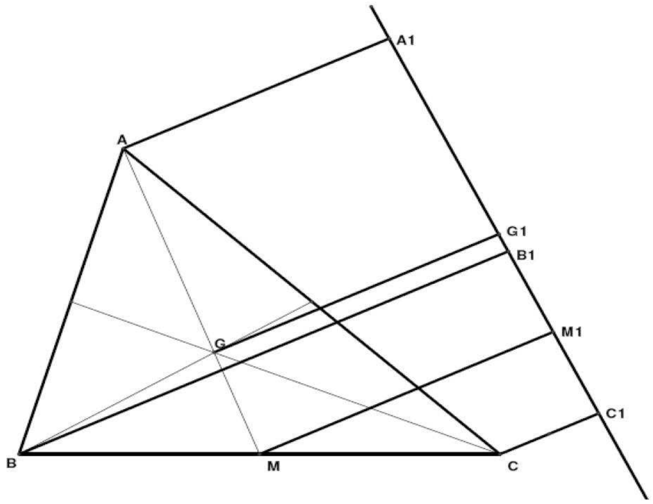

Figure 2. Starting Problem

Repeating the procedure in the trapezoid BCC_1B_1, because $\frac{CM}{CB} = \frac{1}{2}$, we may use (2.6) for $q = 2$, and obtain

$$MM_1 = \frac{BB_1 + CC_1}{2}.$$

Combining the above two relations, is easy to see that

$$GG_1 = \frac{AA_1 + BB_1 + CC_1}{3}.$$

We may conclude all of the above as a proof of the following:

Proposition 5 ([4]). *(The Starting Problem) Let ABC be a triangle and d an exterior line of it. Consider G the centroid of the triangle and A_1, B_1, C_1 and G_1 be the projections of A, B, C and G, respectively, on the line d. Then the following relation takes place:*

$$\frac{AA_1 + BB_1 + CC_1}{3} = GG_1.$$

The previous result is proven using synthetic geometry at an elementary level, just using basic facts.

Other analogous proofs can be used using coordinate or vector geometry.

4. The analogy in 3-D

First step in exploring, extending and developing the study regarding this simple geometric 2-D problem is to see what happens if we move from plane to a 3-D space. For sure here we'll need not only the basic concepts as points, lines and figures, but also other basic geometric concepts as plane and solid figures.

4.1. Describing the analogy

Our extension starts from the following analogy:

Starting Problem	*to*	*Analogous Problem*
(synthetic geometry 2-D)		*(synthetic geometry 3-D)*

In the *Starting Problem* we face with the 2-D geometric facts. Translated into 3-D it's possible to obtain something different, but still analogous.

The analogous figure in 3-D for a triangle is a tetrahedron.

In 2-D each triangle has a centroid. Analogous, in 3-D each tetrahedron has a centroid. The metric relation related to the position of the centroid in triangle has analogous in tetrahedron (Proposition 4).

In 2-D we had perpendicular lines on the exterior line d, through the vertices and G; analogous in 3-D we can draw perpendicular lines on the exterior plane d, through the vertices of tetrahedron and its centroid.

Summarizing the construction, we represent the analogies in the following table:

Starting Problem	to	*Analogy in 3-D*
Triangle ABC	-	Tetrahedron $ABCD$
Exterior line (d)	-	Exterior plane (d)
Intersection of 2 lines	-	Intersection of 2 planes or plane-line
Centroid for the triangle	-	Centroid for the tetrahedron
Parallel lines in $2-D$	-	Parallel lines in $3-D$
Perpendicularity in $2-D$	-	Perpendicularity in $3-D$
Colinearity	-	Coplanarity

The 3-D analogy is another important creative moment for our students. An analogy of two analogue geometries is important because the students have to find for each 2-D geometrical figure its 3-D analogue. This is one of the conjectures, which have to be found based on existing properties.

4.2. The Analogous Problem

Approaching to an analogy in 3-D for the *Starting Problem*, let's consider the tetrahedron $ABCD$ and an exterior plane (d), as in the Figure 3. We denote by G the centroid of the tetrahedron. Through the points A, B, C, D and G we draw perpendicular lines to the plane (d) and denote the intersecting points with A', B', C', D' and G', respectively.

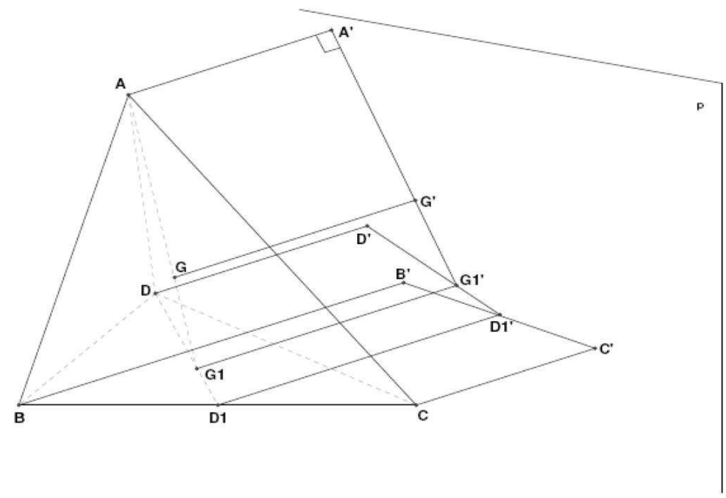

Figure 3. Analogy in 3-D

Also we denote by D_1 the midpoint of the segment (BC), by G_1 the centroid of the triangle BCD and by G the centroid of the tetrahedron $ABCD$. Through all these three points we draw also the lines D_1D_1', G_1G_1' and GG', perpendicular on the plane d, such that the points D_1', G_1' and G' belong to the plane.

In this situation all the segments AA', BB', CC', DD', D_1D_1', G_1G_1' and GG' are mutual parallel. Because of this, the following triplets of segments belong to the same plane:

$$(AA', GG', G_1G_1') \quad \text{to the plane} \quad (AA'G_1'G_1);$$
$$(DD', G_1G_1', D_1D_1') \quad \text{to the plane} \quad (DD'D_1'D_1);$$
$$(BB', D_1D_1', CC') \quad \text{to the plane} \quad (BB'C'C).$$

Let's work separately and successively in this three planes.

In the plane $(AA'G_1'G_1)$ the figure $AA'G_1'G_1$ is a right trapezoid with the basis AA' and G_1G_1'. The segment GG' is also parallel to the bases, such that $\frac{AG}{AG_1} = \frac{3}{4}$. Using Proposition 2 with relation (2.6) for $q = \frac{4}{3}$, we have

$$GG' = \frac{3G_1G_1' + AA'}{4}. \tag{4.1}$$

In the plane $(DD'D_1'D_1)$ the figure $DD'D_1'D_1$ is a right trapezoid with the basis DD' and D_1D_1'. The segment G_1G_1' is also parallel to the bases, such that $\frac{DG_1}{DD_1} = \frac{2}{3}$. Using Proposition 2 with relation (2.6) for $q = \frac{3}{2}$, we have

$$G_1G_1' = \frac{2D_1D_1' + DD'}{3}. \tag{4.2}$$

In the plane $(BB'C'C)$ the figure $BB'C'C$ is a right trapezoid with the basis DD' and D_1D_1'. The segment D_1D_1' is also parallel to the bases, such that $\frac{CD_1}{CB} = \frac{1}{2}$. Using Proposition 2 with relation (2.6) for $q = 2$, we have

$$D_1D_1' = \frac{CC' + BB'}{2}. \tag{4.3}$$

Finally, if we plug in the expression 4.3 into 4.2, and then 4.2 into 4.1, we obtain easy the relation

$$GG' = \frac{AA' + BB' + CC' + DD'}{4}.$$

Under the above considerations and proofs, we are able now to give a form to the analogy in 3-D to the *Starting Problem*, as follows:

Theorem 2. *(The Analogous Problem) Let $ABCD$ be an arbitrary tetrahedron and (d) and exterior plane of it. We denote by G the centroid of the tetrahedron. Through the points A, B, C, D and G we draw perpendicular lines to the plane (d) and denote the intersecting points with A', B', C', D' and G', respectively. Then*

$$GG' = \frac{AA' + BB' + CC' + DD'}{4}. \tag{4.4}$$

5. The generalization of analogy

The second step in our presentation here is obtained from the *Analogous Problem* in 3-D. We try to generalize it, substituting the particular properties of the problem by more general. The space of work is considered to be also the three dimensional one.

5.1. Describing the generalization

Here our study continues with the following construction:

Analogous Problem	*to*	*Generalization*
(synthetic geometry, 3-D)		*(synthetic geometry, 3-D)*

In the *Analogous Problem* we have a tetrahedron $ABCD$ and an exterior plane d. We keep these.

The considered point in the *Analogous Problem* is G, the centroid of the tetrahedron $ABCD$. For sure G has a lot of particular properties related to the tetrahedron, especially the metric relations (as given in (2.7)). The most general property that G has is to lie in the interior of $ABCD$. So, instead of the centroid G, we'll consider a point M, for which the only condition is to lie inside of $ABCD$.

In this new, more general situation, we need to built some metric relations regarding the point M.

For the point G we had that $\frac{AG}{AG_1} = \frac{3}{4}$, which is an explicit fraction. Also, we had that G lies on the segment AG_1, where $G_1 is$ the centroid for the triangle BCD. For sure $G_1 \in \operatorname{int}(BCD)$.

For the point M we have only that lies inside of the tetrahedron $ABCD$. For this reason, if we connect the vertex A with M and extend the line, it intersects the interior of the triangle BCD in a point N. So, $N \in \operatorname{int}(BCD)$ as G_1 did, and we may consider $\frac{AM}{AN} = \frac{1}{m}$, where m is an arbitrary real number. It's easy to see that $m > 1$.

Continuing the construction, we draw a line through the points D and N. Because $N \in \operatorname{int}(BCD)$, this line intersects the segment (BC) in a point P, such that that $P \in \operatorname{int}(BC)$. Moreover, we can assume that $\frac{DN}{DP} = \frac{1}{n}$, where n is an arbitrary real number. It's easy to see that $n > 1$.

Finally, because $P \in \operatorname{int}(BC)$, we may consider that $\frac{CP}{CB} = \frac{1}{k}$, where k is an arbitrary real number, such that $k > 1$.

In the construction above we generalize the centroid G by an arbitrary interior point M.

In the *Analogous Problem* we have that the lines AA_1, BB_1, etc. are perpendicular to the plane d. The perpendicular property that they have is very strong (is given a precisely measure of the angle). A more general property that these lines have is that they are mutual parallel. If we take into consideration only this little constraint and asking not to be parallel to the given plane d, we obtain a more general situation for the *Analogous Problem*.

We can summarize our steps in the following table:

Analogous Problem	to	*Generalization*
tetrahedron $ABCD$	-	tetrahedron $ABCD$
exterior plane d	-	exterior plane d
centroid of $ABCD$	-	arbitrary interior point M inside $ABCD$
centroid G_1 of ΔBCD	-	interior point N of triangle BCD
midpoint D_1 of (BC)	-	interior point P of segment (BC)
perpendicular lines on d	-	mutual parallel lines intersecting d.

Here we use the following:

– Generalization of definition(s): the centroid has the property to be an interior point of a trapezoid. It will be replaced by another variable interior point of the trapezoid;

– Generalization of geometrical properties: the centroid G describes a given and well-known proportion on the corresponding line segments. Replaced by M, there will be other proportions on the corresponding line segments, using arbitrary variable. Also, the perpendicular lines on (d) are parallel to each other. We keep the property that they are parallel, but take off the perpendicularity condition, such that we get only mutual parallel lines, intersecting (d).

– Generalization by creativity: to get the corresponding proportions for the line segments, we need to connect one vertex of the tetrahedron an interior point; then extend the line and intersect it by the opposite plane/side to the vertex. In this way we may define new and arbitrary proportions on the line segments described by intersections.

In addition, this learning task requires basic knowledge of triangle, tetrahedron, the properties of line in tetrahedron, and the properties of parallel and perpendicular lines. The level of computational off-loading for this generalization is low because the students have a lot of inference to do. They have to find relation between proportions of lines, relations between geometrical figures and also they have to draw auxiliary lines in order to complete the proof.

5.2. The Generalization Problem

In this section we'll approach to the generalization.

Let's consider an arbitrary tetrahedron $ABCD$, d an exterior plane of it and M be an arbitrary interior point of the tetrahedron. We draw parallel lines through the points A, B, C, D and M, which intersect the plane d in A_1, B_1, C_1, and M_1, respectively (Figure 4).

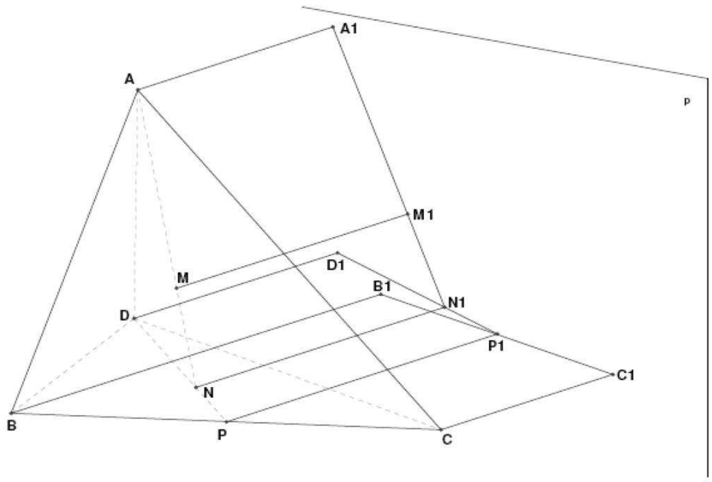

Figure 4. Generalization

Because M is an interior point of the tetrahedron $ABCD$, we may consider the points:

$$\{N\} = AM \cap (BCD)$$
$$\{P\} = DN \cap (BC).$$

Each of the points M, N and P divides the line on which belongs as follows: there are the real numbers m, n and k such that:

$$\frac{AM}{AN} = \frac{1}{m}, \quad \frac{DN}{DP} = \frac{1}{n}, \quad \text{and} \quad \frac{CP}{CB} = \frac{1}{k},$$

where m, n and $k \in (1, \infty)$.

From each vertex of the tetrahedron A, B, C and D, and from the points M, N and P we draw mutual parallel lines which intersect the plane d in $A_1, B_1, C_1,$ D_1, M_1, N_1, and P_1, respectively.

In this situation all the segments (AA_1), (BB_1), (CC_1), (DD_1), (MM_1), (NN_1) and (PP_1) are mutual parallel. Because of this, the following triplets of segments belong to the same plane:

$$(AA_1, MM_1, NN_1) \quad \text{to the plane} \quad (AA_1N_1N);$$

$$(DD_1, NN_1, PP_1) \quad \text{to the plane} \quad (DD_1P_1P);$$

$$(BB_1, PP_1, CC_1) \quad \text{to the plane} \quad (BB_1C_1C).$$

Let's work separately and successively in this three planes.

In the plane (AA_1N_1N) the figure AA_1N_1N is a trapezoid with the basis (AA_1) and (NN_1). The segment (MM_1) is also parallel to the bases, such that $\frac{AM}{AN} = \frac{1}{m}$. Using Proposition 2 with relation (2.6) for $q = m$, we have

$$MM_1 = \frac{NN_1 + (m-1)AA_1}{m}. \tag{5.1}$$

Similar, in the plane (DD_1P_1P) the figure DD_1P_1P is a trapezoid with the basis (DD_1) and (PP_1). The segment (NN_1) is also parallel to the bases, such that $\frac{DN}{DP} = \frac{1}{n}$. Using Proposition 2 with relation (2.6) for $q = n$, we have

$$NN_1 = \frac{PP_1 + (n-1)DD_1}{n}. \tag{5.2}$$

The same way in the plane (BB_1C_1C) : the figure BB_1C_1C is a trapezoid with the basis (BB_1) and (CC_1). The segment (PP_1) is also parallel to the bases, such that $\frac{CP}{CB} = \frac{1}{k}$. Using Proposition 2 with relation (2.6) for $q = k$, we have

$$PP_1 = \frac{BB_1 + (k-1)CC_1}{k}. \tag{5.3}$$

Finally, if we plug in the expression 5.3 into 5.2, and then 5.2 into 5.1, we obtain easy the relation

$$MM_1 = \frac{kn(m-1)AA_1 + k(n-1)DD_1 + (k-1)CC_1 + BB_1}{mnk}.$$

Our conclusion follows:

Theorem 3. *(The Generalization Problem) Let $ABCD$ be a tetrahedron, d an exterior plane of $ABCD$ and M an arbitrary interior point of the tetrahedron $ABCD$. We draw parallel lines through the points A, B, C, D and M, which intersect the plane d in A_1, B_1, C_1, and M_1, respectively. Then there are three constant real numbers m, n and k such that:*

$$MM_1 = \frac{kn(m-1)AA_1 + k(n-1)DD_1 + (k-1)CC_1 + BB_1}{mnk}. \tag{5.4}$$

If we are looking in the relation (5.4) for the coefficients of the side lengths, we see that they are:

$$\text{the coefficient of } AA_1 : \frac{kn(m-1)}{mnk},$$

$$\text{the coefficient of } BB_1 : \frac{1}{mnk},$$
$$\text{the coefficient of } CC_1 : \frac{k-1}{mnk},$$
$$\text{the coefficient of } DD_1 : \frac{k(n-1)}{mnk}.$$

If we add all these coefficients we obtain:

$$\frac{kn(m-1)}{mnk} + \frac{1}{mnk} + \frac{k-1}{mnk} + \frac{k(n-1)}{mnk} = \frac{knm - kn + 1 + k - 1 + kn - k}{mnk}$$
$$= \frac{knm}{mnk} = 1.$$

We may state the following:

REMARK 1. The sum of the coefficients of the side lengths in (5.4) is equal to 1.

6. Specializations

6.1. Obtaining the Analogous Problem

From any general statement, making particular considerations, we are able to obtain specializations.

Instead of having an arbitrary interior point M for a given tetrahedron $ABCD$, we consider a more particular one, the centroid G. In this case the point N is the centroid of the triangle BCD, and P is the midpoint of the side (BC), such that we have the exactly values: $m = \frac{4}{3}$, $n = \frac{3}{2}$, and $k = 2$.

We may represent the specialization in the following table:

Generalization	to	*Specialization*
tetrahedron $ABCD$	-	tetrahedron $ABCD$
exterior plane d	-	exterior plane d
arbitrary interior point M inside $ABCD$	-	centroid of $ABCD$
interior point N of triangle BCD	-	centroid G_1 of ΔBCD
interior point P of segment (BC)	-	midpoint D_1 of (BC)
mutual parallel lines intersecting d	-	perpendicular lines on d.

In relation (5.4) we plug in the particular values for m, n and p. The resulting specialization gives us exactly the *Analogous Problem*:

THEOREM 4. *Let $ABCD$ be an arbitrary tetrahedron and (d) and exterior plane of it. We denote by G the centroid of the tetrahedron. Through the points A, B, C, D and G we draw perpendicular lines to the plane (d) and denote the intersecting points with A_1, B_1, C_1, D_1 and G_1, respectively. Then*

$$GG_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

6.2. Obtaining other specializations

We can continue now to built a lot of specializations, directly from the *Generalization Problem*. One more example presented here is the following:

In the tetrahedron $ABCD$ we denote by P the midpoint of the segment (BC) and by N the midpoint of the segment (DP). *It* follows immediately that $\frac{DN}{DP} = \frac{1}{2}$ and $\frac{CP}{CB} = \frac{1}{2}$, such that in (5.4) we consider $n = k = 2$.

Finally, we denote by M the midpoint of the segment (AN), such that $\frac{AM}{AN} = \frac{1}{2}$ and $m = 2$ in (5.4).

Thus, our specialization here can be represented in the following table:

Generalization	to	*Specialization (2)*
tetrahedron $ABCD$	-	tetrahedron $ABCD$
exterior plane d	-	exterior plane d
interior point P of segment (BC)	-	midpoint P of (BC)
interior point N of segment (DP)	-	midpoint N of (DP)
arbitrary interior point M of $ABCD$	-	midpoint M of (AN)
mutual parallel lines intersecting d	-	perpendicular lines on d.

Plugging in the values for m, n and p into (5.4) we state the following:

THEOREM 5. *Let $ABCD$ be an arbitrary tetrahedron and (d) and exterior plane of it. We denote by P, N and M the midpoints of the line segments (BC), (DP) and (AN) respectively. Through the points A, B, C, D and M we draw perpendicular lines to the plane (d) and denote the intersecting points with A_1, B_1, C_1, D_1 and M_1, respectively. Then*

$$MM_1 = \frac{4AA_1 + BB_1 + CC_1 + 2DD_1}{8}.$$

7. Conclusion

The generalization in learning process has the important role to eliminate some particularities of a conjecture make it to represent a larger class of conjectures where the first is included. By generalization in fact the students' judgment is exploring the abstract areas, which require high knowledge related to the problem or theory.

The analogy is one of the most used reasoning in teaching and learning. In teaching with analogies the teacher usually are based on the following steps: introduce the target concept, access the source of analogy, identify relevant features of the source of target, map similarities of source and target, investigate where the analogy is brake down, draw conclusions. The analogy is designed to help students in develop judgment about a good choice for the source from a number of sources, which are the best analogy with the given task. In geometry in special, the analogies are tools to aid visualization rather than deep analogues.

The specialization gives the possibility to make particular consideration, starting from the general one. The role of it is to show how nice is the entire process completed and how the "chain" is closed.

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