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Teaching
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Forming the concept of parameter with examples of problem solving

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Abstract. Pupils are encountering difficulties with learning algebra. In order for them to understand algebraic concepts, particularly the concept of parameter it was decided by the teacher of mathematics and Information Technology to integrate the teaching of these two subjects. The aim of this study is to investigate whether, and to what degree, software can be useful in process of forming the concept of parameter. This longitudinal study was conducted in a junior high school (13-16 year old children) using different computer programs.

Key words and phrases: investigating and problem solving, technological tools, educational research and planning (junior high school), educational technology and media research, mappings and functions (junior high school).

ZDM Subject Classification: D50, B13, I23.

1. Background

The most common area of interest while traditional mathematics teaching is connected with concepts like triangles, numbers, vectors etc. and their properties, and a teacher focuses pupils' attention mainly on these topics. Letters (variables) are treated as if they were already familiar to pupils and were represented by the alphabet which is known to them. The assumption that pupils will use mathematical symbols in a conscious manner is wrong provided nobody has familiarized them previously with these symbols [6]. The context determines their meaning in the algebraic text, phase of reasoning, subjective interpretation and also the conditions surrounding the traditional use of such letters. The proper interpretation of letters within a given algebraic structure is often a difficult task for pupils. Feeling and intuitive recognition turns out to be a more popular strategy than interpreting the letters using reasoning. Pupils know concepts like a variable, an unknown or a parameter. However, they do not always use them consciously, and therefore the concepts are not understood entirely [9].

As it is necessary to fully comprehend literal symbols in algebra, therefore this study focuses on the letter in the meaning of parameter. There is no definition for the concept of parameter analyzed in mathematical literature. The work of François Viète (1540–1603) is very important in the development of algebra, as he was the first to distinguish between the different roles of literal symbols. In his work, literal variables refer to given quantities. These are in fact parameters, "changing constants" that change at a higher level than the "ordinary" variables, namely at the global rather than the local level [3]. According to some descriptions *a parameter can be considered as a meta-variable: the* a *in* $y = a \cdot x + b$ *can play the same roles as an 'ordinary' variable, such as placeholder, unknown or changing quantity, but it acts on a higher level than is the case for a variable* [4]. Another more formal description states: *an equation or a function with a parameter is a (second order) function, the argument of which is the parameter, and the corresponding values of which are equations and functions (with the other letters as unknowns and variables)* [2].

It was necessary to be completely familiar with research connected with this area of mathematics for the study. Although there have been many studies abroad, research that discusses the problem of understanding parameters has not been considered in Poland until now. The research abroad was conducted using Information Technology. One of researchers was C. van de Giessen [5], who described the classroom experiences of 16–17 year old students while they were visualizing different meanings of parameters when solving problems using the Graphic Calculus Program. The meanings and descriptions have been simplified and listed in the following table:

Meaning of parameter	**Algebraic meaning**	**Graphic image**
placeholder	-one numerical value, -one at a time	-one graph, -one at a time -parameter in this role has static character
generalizer "family" parameter	a set of numerical values	a sheaf of graphs
changing quantity "sliding" parameter	a changing numerical value	-a graph is changing continuously with change of parameter -parameter has dynamic character

Another study in Holland, at the university of Utrecht, in 2003 P. Drijvers [3] also researched the problem of understanding the concept of parameter using Information Technology. In this experiment there were four meanings of parameter classified: placeholder, generalizer, changing quantity and unknown; the latter was described in the following way in the context of function:

Meaning of parameter	**Algebraic meaning**	**Graphic image**
unknown-to-be-found	a subset of numerical values	a subset of graphs in a sheaf

A majority of school-teaching concepts (especially arithmetic and algebra) are formed by their use (in a variety of tasks, accounts, etc.) rather than by definition. Students know the meaning of concepts in the course of mathematical activities [8]. An example of such a concept may be parameter. The analysis of the problems was carried out in the direction of establishing the type of activities undertaken by students in order to solve the problem with parameter:

– If the dominant action in student's solving the problem was accepting any number in place of parameter and solving the problem only for this parameter value, then it was accepting by the researcher that the student understands the concept of parameter in the meaning of placeholder.
– If the dominant action in student's solving the problem was accepting any number in place of parameter, changing parameter values and examining the solutions for each of them, then it was accepting by the researcher that the student understands the concept of parameter in the meaning of generalizer.

– If the dominant action in student's solving the problem was changing parameter values with very small step value so that the image changed to the eye in a continuous manner which suggested that the parameter value is also changing in this way (practically to do only with the help of Information Technology) and then analyzing the solutions to these values, then it was accepting by the researcher that the student understands the concept of parameter in the meaning of changing quantity.

– If the dominant action in student's solving the problem was observation that the parameter can accept many values, but to meet the conditions of the problem the student selects only those that meet these conditions, then it was accepting by the researcher that the student understands the concept of parameter in the meaning of unknown-to-be-found.

2. Aims and methods

Within the confines of the Technology in the Teaching and Learning of Mathematics Seminar held at the Pedagogical University in Cracow further research on the formation and understanding of the concept of parameter among pupils during their three-year learning at junior high school (13–16 year-old pupils) was conducted and the results displayed here. The investigative problems are:
Can software be useful in the process of forming the concept of parameter? and *To what extent can computer algebraic software help students to understand the different meanings of the concept of parameter while maintaining continuous contact with this tool for mathematical learning?*

In other words, will using this software heighten understanding of the parameter from placeholder to changing quantity, as a generalizer and as an unknown.

Based on one of the final conclusions of previous research, it was decided that computer programs would also be used in this study: *a computer can help with the understanding of mathematical concepts taking a stand in a solved problem, illustrating, for example, images of these concepts* [5]. Such an observation was revealed in this research in connection with the concept of parameter. All decisions for using software for solving different mathematical problems in the lessons to help contribute to further knowledge and understanding of parameter were made by the teacher of mathematics and Information Technology. All lessons were conducted in the traditional manner as well as the use of the software. During lessons the teacher reacted to questions from pupils and discussed issues that were found relevant to stress. The students usually worked in pairs, although they could choose to work individually. Most lessons finished with feedback on the pupils' work through a whole-class discussion or explanation guided by the teacher. Pupils became accustomed to solving various mathematical problems with the use of software from these lessons.

The study had a continuous character and was conducted during a three-year period at junior high school (13–16 year old pupils). The subjects of the study were pupils from two classes of Jan Paweł II Junior High School in Kęty who formed randomly. The teacher in those classes is also the researcher of this study. The oldest class; D was studied for three years and the youngest class; B was also studied for three years. The preliminary research was conducted in the oldest class. The figure below shows this situation:

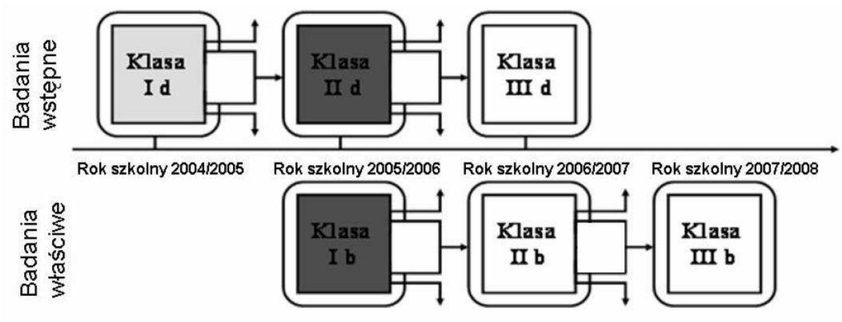

Figure 1. Schedule of preliminary and proper research

During the preliminary study, the researcher was able to create, develop and verify potentials lists of problems connected with parameters for the first, the second and third class of high school, all related to various sections of mathematics. Afterwards these lists were used during the normal research. After some time problems with parameters did not seem new to the pupils.

This research is based on results obtained by C. van de Giessen and P. Drijvers. However, the research of these men differs from this study as they did not have a continuous character and they were not researching for several years. Due to the fact that this research is continuous, instrumented techniques and interface problems had an influence on solving mathematical problems only in the initial period of the research. Besides, the Dutch team did research using teachers observing groups of young people. Contrary to their research, work of pupils of class D and B were monitoring by the teacher of mathematics and Information Technology; that is the researcher.

This research is consistent with the idea of natural experimentation. The *design research* method was also used while investigating, backing all observations with the analyzation of documents. The investigative method *design research* was used in the research "Design research on the understanding of the concept of parameter" carried out by P. Drijvers in 2003. This method has a cyclic character. We distinguish macrocycles and microcycles from this method. One macrocycle lasts a year and includes the phases: planning, conducting of experiments, analysis of results comparing final and initial states. In turn, one microcycle is more detailed within the confines of only one section of mathematics. The planning of research within the confines of one microcycle consists of preparing of a list of problems for pupils directed with understanding different meanings of the concept of parameter. Each list also includes problems with parameter that pupils solved during a final test of any given unit. The solutions of the problems throughout the whole year were analyzed using lesson work and final tests, both of which were aided using software.

In the Information Technology lessons, pupils were acquainted with the operating techniques of different software, for example TI InterActive!. Abilities with using this program played big role in solving mathematical problems only at the initial phase of experiment. Pupils did not need to use sheets of paper to solve problems with the aid of this program. This program could be used as a text editor, and it was possible to insert tools, for example SliderContol (for changing values), Graph Tool (for composing graph of function), and MathBox Tool (for symbolic accounts). When the students started working with this program, none of these important tools had been inserted. It was only at the moment when they needed it did students, on their own, decide to insert a tool. All of the files of TI InterActive! which included text and tools were saved onto Harddrive. Seeing this, it can be concluded that these files were necessary investigative tool documents of their importance for solving mathematical problems. However, the most important investigative tool was a screen recording program CamtasiaStudio which made it possible to record the work of each student. It is necessary to replay all the work and this program allows this. Thus, the files from computer programs and the recordings of the pupils' work were used for the analysis of the solutions of the problems. In order to achieve a deeper insight into the way of understanding, it was crucial to supplement the recording with a interview of some pupils about their problem solving capabilities.

The reason for choosing TI InterActive! program during this study was that instrumented techniques (a set of procedures) that were used for solving a specific type of problems were not difficult for students. Then students could focus on understanding of the concept of parameter. During this experiment students solved problems connected to different sections of mathematics: equations, linear systems of equations, functions, algebraic expressions, geometry, Pythagoras' theorem, powers and roots. Preliminary mathematical skills before solving the below problem was limited to a basic knowledge of linear functions. For example students solved such a problem:

For which values of the parameter a has the system of equations $\begin{cases} ax + y = 1 \\ 2x - y = 3 \end{cases}$

– no solution,

– exactly one solution?

3. Results within the system of equations problem

The application of the computer recording program enabled a thorough investigation and an exact reconstruction of the pupils' work while solving a problem using computer programs. At the beginning of the study in class I B it was conducted a pre test. It was necessary to know how students understand an 'extra letter' or parameter in a formula which occurred in different tasks. In this section is presented the analysis of a solution proposed to a problem found only by one girl in class II B. She was one of the two girls whose solutions of tasks on this pre test indicated that she understood the parameter as a finite generalizer at that moment. It was expected that after some time of solving problems with parameters she would perceive parameter as generalizer, changing quantity and unknown-to-be-found. Let me concentrate on the work of this girl. Here is the problem and her solution.

Consider the following system of equations:

$$\begin{cases} y = c \cdot x \\ y = 3 - c \cdot x \end{cases}$$

1) What is the effect of the value of parameter c on the positions of the intersection point of the graphs of functions defined by the above two formulas?
2) What values of parameter c result in the intersection point being to the right of the line $x = 4$?

3) Explain what happens to the intersection point when the value of parameter c is negative.
4) It can be noticed that the value of parameter c does not affect the height of the intersection point. Explain why this is the case using the formulas.
5) What values of c lead to an intersection point that is on the left of the line $x = \frac{1}{100}$? Provide an exact answer.
6) Find two linear functions which both contain a parameter c in their formulas and have the following property: If the value of c gets smaller, the intersection point of the two graphs goes straight upwards [1].

Below are presented the student's solutions proposed to the first, third, fourth and sixth points of the forementioned problems and their analysis. The student used the TI InterActive! program and while solving *the first problem* she inserted the tool called SliderControl. She established a letter meaning parameter for such a tool. Initially, in this program, the current value of this parameter is equal to 0 and the value of the step of assigning consecutive values to the parameter is equal to $0,1$. This student did not change these settings at that moment. The Graph Tool was also placed into the program and it is here where the equations of both linear functions were typed. Graphs of the functions were plotted in the coordinate plane. The first parameter value tested by this pupil was an initial value. The situation given in this problem was examined and illustrated in the coordinate plane for this parameter value. In this case the lines were not intersecting. The value of parameter c was gradually changed one by one using the SliderControl tool.

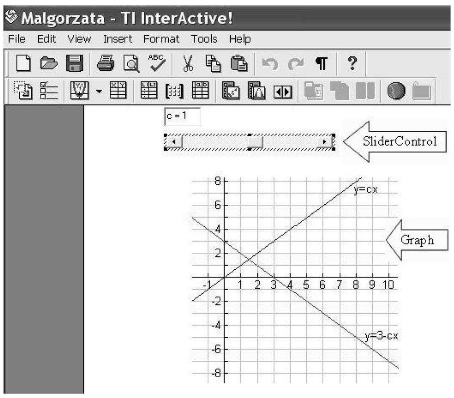

Figure 2. The first problem illustrated by means of TI InterActive!

Changing the current value of the parameter caused the graphs to be updated automatically in the Graph tool to reflect the new parameter value. When parameter c accepted different values from 0, then the straight lines were intersecting and the intersection point had been moving through the first and the second quadrant of the coordinate plane. Next the tool Graph was edited and then the option Calculate and next Intersection was chosen. The linear functions from the equations were indicated and then the y-coordinate of the intersection point was noticed. It was equal to $1,5$. Then, it was ascertained, that while changing values of parameter c the intersection point of these lines was only moving along in horizontal direction, therefore, its y-coordinate always remained constant. One more regularity was noted: if the value of parameter c was growing from the initial value equal to 0, then the intersection point of the graphs approached the y-axis. Incidently, if the value of the parameter was growing for a value equal to 0, then the intersection point also approached the y-axis.

At first, the situation contained in the problem was observed through one established parameter value. The basic form was a static graph and parameter took precedence in placeholder. By means of the SliderControl tool, one by one parameter values were accepted and it was observed how the values changed in jumping steps. This situation is characteristic of parameter as a generalizer. Besides this, it could be observed how each value of parameter was connected with one position of graphs in the coordinate plane. In turn, by choosing values of parameter in a continuous manner it was possible to view a moving image showing the positions of the graphs depending on the values of parameter. This girl investigated the dynamic changes in the coordinate plane caused by varying values on slider. This situation led her to another meaning of parameter; that of changing quantity.

In working with *the third problem*, the value of parameter c was changed by means of the SliderControl according to step value equal to $0,1$. She changed these values in a continuous manner, which caused dynamic translations of the intersection point of graphs. She observed in this way, that when parameter accepted negative values, the intersection point was placed in the second quadrant of the coordinate plane.

The use of SliderControl tool in a dynamic manner allowed the observation of the situation that represented many different values accepted by parameter c. Following this was a generalization of the situation shown in the coordinate plane.

While working on *the fourth problem* the participant noticed the following parameter values did not decide about the vertical slip of the intersection point of the graphs. This fact was checked algebraically using TI InterActive! as this program also allows for the use of symbolic accounts. The equations were solved using the substitution method. To help with this, she used the MathBox tool and entered $solve(c \cdot x = 3 - c \cdot x, x)$ and she received $x = \frac{3}{2c}$.

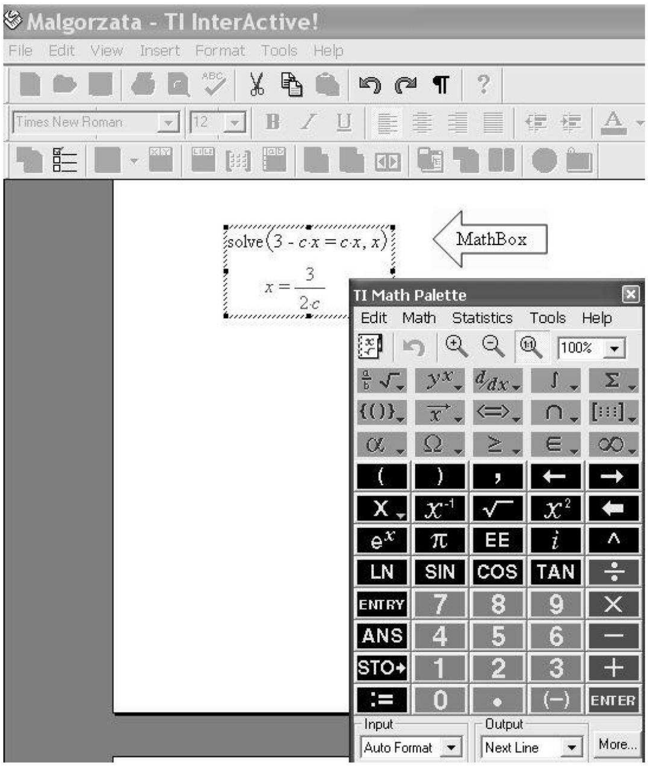

Figure 3. The file of program TI InterActive! with the fourth problem

Next she typed $solve\left(y = c \cdot \frac{3}{2 \cdot c}, y\right)$ and received $y = \frac{3}{2}$. From this it was ascertained, that the y-coordinate of the intersection point of the graphs had not been subordinated from the value of parameter and thus it always amounted to $1,5$. However, the x-coordinate was subordinated from the value of parameter c. Knowing this, it can be said that only the x-coordinate was changeable and only it was able to execute horizontal translations.

Evidence showed a certain regularity while changing the parameter values in the coordinate plane which was verified by calculations. Both the coordinates of the intersection point was calculated generally. Attention was given to the fact if the coordinates were dependent in formulas from parameter c. All the consequences of these dependencies with graphic regularities observed at the beginning of problem solving were confronted.

The sixth problem enabled the student to find formulas of functions with parameter using the *trial and error process* provided the condition was accurate. The assumption they have to be linear was omitted and it was suggested to try the following two functions: $y = \frac{c}{x}$ and $y = 3 - \frac{c}{x}$. Two hyperbolas were obtained which surprised the student. Until the moment, the girl had not faced such graphs.

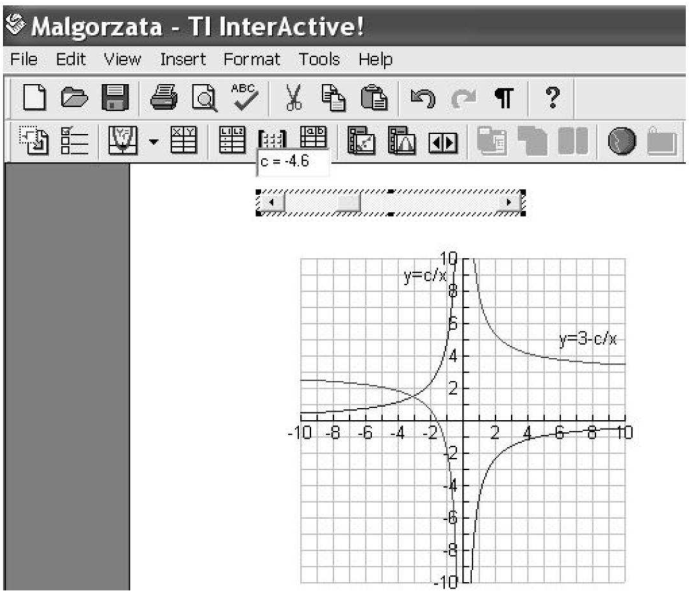

Figure 4. The file of program TI InterActive! with the sixth problem

She analyzed the translations of the intersection point of these graphs with change of parameter c and found out that conditions of problem had not been met. She considered two other functions: $y = \frac{x}{c}$ and $y = \frac{3}{c} - \frac{x}{c}$. The student checked that with the change of parameter c, the intersection point executed only vertical translations using SliderControl tool.

Due in a large part to TI InterActive! computer program this schoolgirl was able to interpret the situation and analyze unknown functions to this point.

Without such a program or tools, perhaps she might not have been able to solve this problem at all. Students' solutions of other problems with parameter can be found in the publication [10].

4. Conclusions

Analysis of the registration contained in the CamtasiaStudio program and file of program TI InterActive! enabled the opportunity to remark on this participant's mathematical activities and allowed the researcher to make conclusions about the level of understanding of parameter at the moment of solving the above-mentioned problem.

a) *Mathematical activities revealed while solving the above-mentioned problem*

- *Detecting analogies*
  By substituting different values of parameter, the student repeated problem solving procedures for these particular values. Then she detected analogies between these procedures.
- *Generalization*
  Solving the problem with parameter in computer programs allowed varying parameter values and calculated generic algebraic solutions (Drijvers, 2003). As recalled in the above analysis, solving the problem with parameter also fostered generalizing by assigning different parameter values and observing the situations connected with those values.
- *Stating and verifying hypotheses*
  The results were not regarded as a final solution of the problem. They were treated as a hypothesis only which could then be verified by calculations.
- *Use of symbolic language*
  She calculated the distances between points in general. Her solving strategy did not seem to be confused by the presence of the parameter.

b) *Conclusions concerning the understanding of the concept of parameter by the student*

On base of preliminary test I have concluded that the student probably understood the idea of parameter as a placeholder and finite generalizer at the beginning of the first class of junior high school. An analysis of all collected material gave me the chance to formulate partial answers to previously asked questions.

Computer programs can help in understanding different meanings of the concept of parameter:

- Due to the use of the SliderControl tool for solving problems with help of TI InterActive! program she could substitute different parameter values. Any number could be substituted for the parameter, not only integer (not only positive as well), as usually it is accepted during solving problems without use of computer programs. This girl observed the situation given in the problem at different parameter values. The pupil deepened her understanding of parameter as a generalizer, because she claimed that the parameter can accept an infinite number of values according to an established step value. While working with software she could perceive how values of the parameter were varying in jumps according to the established value of the step of assigning consecutive values of parameter equal to $0,1$. Next she formulated hypotheses and verified them.
- Changes of parameter value were made step by step and the consequences of these changes could not be as clearly noticed when solving this problem using "paper and pencil". Such observations could be imagined only if the changes had to concern a finite set of values. However it could also involve an infinite set of values, e.g. with the value of the step of changing parameter values equal to $0,1$. This girl noted, along with change of parameter values, dynamic changes of the problem in the coordinate plane. Her solution to the problem indicated how her knowledge of parameter developed from the finite generalizer to changing quantity.
- In TI InterActive! program it is possible to solve problems with parameter without an assignment of any particular value. For example, while solving an equation step by step, it is possible to treat parameter as a given established value. The pupil used *the solve* command. She solved the equation with respect to parameter c. In this situation her comprehension of parameter developed from the finite generalizer to unknown-to-be-found.

This participant was very good in mathematics, had a good attitude and was willing to talk about her work. The above example shows that computer algebra software not only helps in understanding different meanings of parameter through continuous contact in mathematics learning, but it also makes it possible to achieve extensions in knowledge of the parameter from placeholder to changing quantity, as a generalizer and as an unknown. However, computer algebra software is a difficult tool for students of the age and level of those in this research. I conjure that integrating CAS into algebraic education would be better suited for higher grades, when pupils have more experience.

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