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Teaching
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On a special class of generalized conics with infinitely many focal points

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Abstract. Let a continuous, piecewise smooth curve in the Euclidean space be given. We are going to investigate the surfaces formed by the vertices of generalized cones with such a curve as the common directrix and the same area. The basic geometric idea in the background is when the curve runs through the sides of a non-void triangle ABC. Then the sum of the areas of some triangles is constant for any point of such a surface. By the help of a growth condition we prove that these are convex compact surfaces in the space provided that the points A, B and C are not collinear. The next step is to introduce the general concept of *awnings* spanned by a curve. As an important example awnings spanned by a circle will be considered. Estimations for the volume of the convex hull will be also given.

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1. Introduction

The concept of generalized conics as the set of points $\vec{x} \in \mathbb{R}^n$ such that

$$\int_G \alpha(\vec{g})\; d(\vec{g}, \vec{x})\, dg \leq \rho$$

can be found in [2], p. 742. Here $G \subset \mathbb{R}^n$ is the set of foci, $\alpha\colon G \to \mathbb{R}$ is a weight function and ρ is a real constant. As an open question the authors posed the

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problem whether *which results* (of the classical case) *can be extended to the cases of infinitely many focal points or to continuous set of foci*. We are going to deal with a similar problem based on special choices of G and a more general weight function admitting the points of the space in its argument. Our geometric idea can be formulated as follows: let three (or more, but finite many) points A, B and C be given in the three-dimensional euclidean space and consider the set of the points P such that

$$\mathcal{A}(ABP) + \mathcal{A}(APC) + \mathcal{A}(PBC) = \text{const.} \tag{1}$$

where $\mathcal{A}(ABP)$, $\mathcal{A}(APC)$ and $\mathcal{A}(PBC)$ denote the area of the triangles ABP, APC and PBC, respectively. We will show that (1) defines convex compact surfaces in the space provided that the points A, B and C are not collinear. In case of collinear points we have unbounded cylindrical surfaces. The global minimum of the left hand side as the function of P is attained at any point contained by the convex hull of the triangle ABC. The next step is the substitution of the sides of the triangle ABC with a continuous, piecewise smooth curve. We are going to investigate the niveau surfaces of the function $P \to \mathcal{A}(P)$ measuring the area of generalized cones with such a curve as the common directrix and the vertex P. They are called *awnings* spanned by the curve. As an important special case awnings spanned by a circle will be considered. We give estimations for the volume of the convex hull.

2. The basic geometric idea

Let the points A, B and C be given in the three-dimensional Euclidean space and consider the function

$$\Sigma(P) := \frac{1}{2}|(P-A)\times(P-B)| + \frac{1}{2}|(P-C)\times(P-A)| + \frac{1}{2}|(P-B)\times(P-C)|$$

measuring the sum of the areas of the triangles ABP, APC and PBC.

Lemma 1. *Σ is a convex function.*

Proof. It is enough to prove the convexity of the expression

$$|(P-A)\times(P-B)| = |P\times(A-B) + A\times B|$$

as the function of P. Here

$$f(P) := P \times (A - B) + A \times B$$

is an affine mapping preserving the affine, especially the convex combination of the points, i.e.

$$f(\lambda P + (1 - \lambda)Q) = \lambda f(P) + (1 - \lambda) f(Q).$$

The triangle inequality implies that Σ is convex. □

LEMMA 2. *If A, B and C are not collinear then the global minimum is attained at the points of the convex hull of the triangle ABC.*

PROOF. According to the triangle inequality it can be easily seen that Σ is great than or equal to the norm of the sum

$$\frac{1}{2}(P - A) \times (P - B) + \frac{1}{2}(P - C) \times (P - A) + \frac{1}{2}(P - B) \times (P - C)$$

which is just the area of the triangle ABC. If P is not in the plane of ABC then the orthogonal projection decreses the sum of the areas. If P is not in the convex hull of the triangle ABC then P and, for example, the point C are separated by the line AB. According to the *Plane separation postulate* the segment PC meets the line AB which implies that ABC is covered by the union of the triangles ABP, APC and PBC. □

LEMMA 3. *If a function satisfies the growth condition*

$$\lim_{|P| \to \infty} \frac{\Sigma(P)}{|P|} := \lim_{r \to \infty} \inf \left\{ \frac{\Sigma(P)}{|P|} \;\middle|\; |P| > r \right\} > 0$$

then it has bounded level sets.

PROOF. Suppose, in contrary, that the level set

$$M := \{P \mid \Sigma(P) \le c\}$$

is unbounded. Then we can choose a sequence P_n of the elements of M such that the norm $|P_n|$ tends to the infinity. Therefore

$$\lim_{|P| \to \infty} \frac{\Sigma(P)}{|P|} \le \lim_{n \to \infty} \frac{\Sigma(P_n)}{|P_n|} \le \lim_{n \to \infty} \frac{c}{|P_n|} = 0,$$

which is a contradiction. □

Remark 1. Requiring a function to have bounded level sets is also a kind of growth conditions. For convex functions these conditions are actually equivalent to each other as it was stated in [1] but the proof is left as an exercise.

Theorem 1. *If A, B and C are not collinear then the levels of the function Σ are convex and compact.*

Proof. Since Σ is continuous the levels are closed subset of the space. In order to see the compactness we prove that it satisfies the growth condition

$$\lim_{|P|\to\infty} \frac{\Sigma(P)}{|P|} := \lim_{r\to\infty} \inf \left\{ \frac{\Sigma(P)}{|P|} \;\Big|\; |P| > r \right\} > 0$$

which implies that the levels are bounded. First of all note that

$$\frac{1}{4}|(P-A)\times(P-B)|^2 + \frac{1}{4}|(P-C)\times(P-A)|^2 + \frac{1}{4}|(P-B)\times(P-C)|^2 \le \Sigma^2(P).$$

The left hand side can be written into the form

$$L_2(P) + L_1(P) + L_0$$

with a quadratic form , a linear functional and a real constant denoted by L_2, L_1 and L_0, respectively. The steps of the proof can be summarized as follows:

- We prove that L_2 is positive definite.
- Taking $L_1(P) = \langle L, P\rangle$ we have that

$$-|L| \le \frac{L_1(P)}{|P|} \le |L|.$$

- If

$$k := \min \left\{ \frac{L_2(P)}{|P|^2} \;\Big|\; |P| \ne 0 \right\} > 0$$

then

$$\frac{\Sigma^2(P)}{r^2} \ge k - \frac{|L|}{r} + \frac{L_0}{r^2}, \quad \text{where } r = |P|.$$

Taking the limit $r \to \infty$ it follows that

$$\lim_{|P|\to\infty} \frac{\Sigma(P)}{|P|} \ge \sqrt{k} > 0$$

as was to be stated.

Since

$$(P-A)\times(P-B)=P\times(A-B)+A\times B=P\times A-P\times B+A\times B$$

we have that

$$|(P-A)\times(P-B)|^2=\langle P\times A,P\times A\rangle+\langle P\times B,P\times B\rangle+\langle A\times B,A\times B\rangle$$
$$-2\langle P\times A,P\times B\rangle+2\langle P\times A,A\times B\rangle-2\langle P\times B,A\times B\rangle.$$

Using the identity

$$\langle X\times Y,Z\times W\rangle=\langle X,Z\rangle\langle Y,W\rangle-\langle Y,Z\rangle\langle X,W\rangle$$

to separate the quadratic term of the sum, it follows that

$$|(P-A)\times(P-B)|^2=|A|^2|P|^2-\langle A,P\rangle^2+2\langle A,P\rangle\langle B,P\rangle$$
$$-2|P|^2\langle A,B\rangle+|P|^2|B|^2-\langle P,B\rangle^2+\text{ linear}+\text{const.}$$

and, consequently,

$$|(P-A)\times(P-B)|^2=|P|^2\big(|A|^2-2\langle A,B\rangle+|B|^2\big)$$
$$+\big(\langle A,P\rangle-\langle B,P\rangle\big)^2+\text{linear}+\text{const.}=$$

Therefore

$$|(P-A)\times(P-B)|^2=|P|^2|A-B|^2-\langle P,A-B\rangle^2+\text{linear}+\text{const.}$$

and the quadratic form L_2 can be expressed as

$$L_2(P)=\frac{1}{4}\big(|P|^2|A-B|^2-\langle P,A-B\rangle^2\big)+\frac{1}{4}\big(|P|^2|C-A|^2-\langle P,C-A\rangle^2\big)$$
$$+\frac{1}{4}\big(|P|^2|B-C|^2-\langle P,B-C\rangle^2\big).$$

The Cauchy–Schwarz inequality shows that $L_2(P)$ vanishes if and only if P is linear dependent with each of the vectors $A-B$, $C-A$ and $B-C$. Therefore P is equal to zero or the points A, B and C are collinear. □

Remark 2. The linear term can be expressed as

$$L_1(P)=\frac{1}{2}\langle P,A\rangle\big(\langle A,B+C\rangle-|B|^2-|C|^2\big)+\frac{1}{2}\langle P,B\rangle\big(\langle B,A+C\rangle-|A|^2-|C|\big)$$
$$+\frac{1}{2}\langle P,C\rangle\big(\langle C,A+B\rangle-|A|^2-|B|^2\big)$$

and we have that

$$L_0=\frac{1}{4}|A\times B|^2+\frac{1}{4}|B\times C|^2+\frac{1}{4}|C\times A|^2.$$

Corollary 1. *If A, B and C are not collinear then the niveau surfaces of the function Σ are convex and compact.*

3. A special class of generalized conics with infinitely many focal points

Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a continuous, piecewise smooth curve under the partition $a = t_0 < t_1 < \ldots < t_{n-1} = t_n = b$ and consider the function

$$P \to \mathcal{A}(P)$$

measuring the area of generalized cones with such a curve as the common directrix and the vertex P.

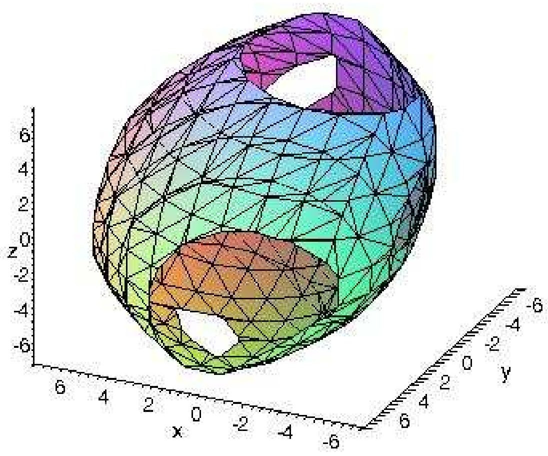

Figure 1. A niveau surface with $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

Definition 1. Niveau surfaces of the form $\mathcal{A}(P) =$ const. are called *awnings* spanned by γ.

Using the parameterization

$$\vec{r}\colon [0, 1] \times [a, b] \to \mathbb{R}^3, \quad \vec{r}(u, v) := uP + (1 - u)\gamma(v)$$

an easy calculation shows that

$$D_1\vec{r}(u,v) = P - \gamma(u), \quad D_2\vec{r}(u,v) = (1-u)\gamma'(v)$$

and, consequently,

$$D_1\vec{r}(u,v) \times D_2\vec{r}(u,v) = (1-u)(P-\gamma(v)) \times \gamma'(v)$$

provided that v is not one of the parameters $t_i's$. Then the area function can be calculated by the formula

$$\mathcal{A}(P) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_0^1 |D_1\vec{r} \times D_2\vec{r}|\, du\, dv.$$

Particularly,

$$\mathcal{A}(P) = \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |(P-\gamma) \times \gamma'|\, dv.$$

Theorem 2. *Any awning is the boundary of a generalized conic with the set of foci $G := \{\vec{g} \mid \vec{g} = \gamma(v),$ where $v \in [a,b]\}$ and the weight function*

$$\alpha(\vec{g}, P) := \frac{1}{2} \sin\big(\text{the angle of } P - \vec{g} \text{ and the tangent line of } G \text{ at } \vec{g}\big).$$

Proof. Let $\vec{g} = \gamma(v)$ be a point in G. Then

$$\frac{1}{2}|(P-\gamma(v)) \times \gamma'(v)| = |P-\vec{g}|\,|\gamma'(v)|\alpha(\vec{g},P) = d(\vec{g},P)\alpha(\vec{g},P)|\gamma'(v)|.$$

Integrating by v we have that

$$\mathcal{A}(P) = \int_G d(\vec{g},P)\ \alpha(\vec{g},P)\ dg$$

and the niveau surfaces of $\mathcal{A}$ are the boundary of generalized conics of the form

$$\int_G d(\vec{g},P)\ \alpha(\vec{g},P)\ dg \leq \text{const.}$$

as was to be stated. □

Lemma 4. *The function $P \to \mathcal{A}(P)$ is convex.*

Proof. The proof is similar to that of Lemma 1 taking the affine term

$$f(P) = P \times \gamma'(v) - \gamma(v) \times \gamma'(v)$$

for some fixed parameter v. □

Remark 3. If the curve γ runs through the sides of the triangle ABC then we get back to the basic geometric idea.

Remark 4. Using the parameterization

$$\vec{r}(u, v_1, \ldots, v_{n-2}) = uP + (1-u)\gamma(v_1, \ldots, v_{n-2}),$$

where $\gamma\colon H \to \mathbb{R}^n$ is a parameterized surface of dimension $n-2$, the awnings can be given by

$$\frac{1}{n-1}\int_H |(P-\gamma) \times D_1\gamma \times \ldots \times D_{n-2}\gamma|\, dv_1 \ldots dv_{n-2} = \text{const.}$$

in case of higher dimensional spaces.

4. Awnings spanned by a circle

Let γ be a parameterized circle, i.e.

$$\gamma(v) := (r\cos v, r\sin v, 0),$$

where $0 \le v \le 2\pi$. Awnings spanned by γ can be given by the integral equation

$$\frac{r}{2}\int_0^{2\pi} \sqrt{z^2 + (x\cos v + y\sin v - r)^2}\, dv = \text{const.}$$

They are obviously invariant under the rotation around the z-axis. In what follows we investigate them in details.

Lemma 5. *Awnings spanned by a circle are convex compact surfaces in the space. The global minimum of the function*

$$\mathcal{A}(P) := \frac{r}{2}\int_0^{2\pi} \sqrt{z^2 + (x\cos v + y\sin v - r)^2}\, dv, \quad P = P(x, y, z)$$

is attained at any point contained by the convex hull of the circle.

Proof. Let $n \ge 3$ and consider the vertices $A_0, \ldots, A_n = A_0$ of the inscribed regular n-gon with a fixed initial point A_0 as the set of foci. It can be easily seen that

$$\lim_{n\to\infty} \Sigma_n(P) = \mathcal{A}(P)$$

and the set of the extremal points is just the limit

$$\lim_{n\to\infty} \operatorname{conv}\{A_0, \dots, A_n\}$$

of the convex hulls. □

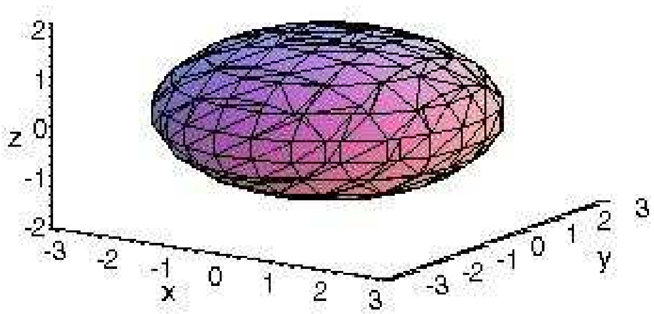

Figure 2. An awning spanned by the unit circle.

5. Approximation of awnings by quadrics

Theorem 3. *The volume enclosed by the awning*

$$\frac{r}{2}\int_0^{2\pi} \sqrt{z^2 + (x\cos v + y\sin v - r)^2}\, dv = c$$

is greater than or equal to

$$V_1 := \frac{8}{3}\pi\left(\frac{c^2}{r^2\pi^2} - r^2\right)^{\frac{3}{2}}.$$

Proof. Since

$$\left(\int_0^{2\pi} f\, dv\right)^2 \le 2\pi\int_0^{2\pi} f^2\, dv$$

we have that

$$c^2 \le \frac{r^2}{2}\pi \int_0^{2\pi} z^2 + (x\cos v + y\sin v - r)^2\, dv.$$

Therefore the surface defined by the equation

$$c^2 = \frac{r^2}{2}\pi \int_0^{2\pi} z^2 + (x\cos v + y\sin v - r)^2\, dv$$

is contained by the convex hull of the awning. Here

$$c^2 = \frac{r^2}{2}\pi^2\big(x^2 + y^2 + 2r^2 + 2z^2\big),$$

which gives an ellipsoid

$$1 = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2}$$

with

$$\alpha^2 = \beta^2 = 2\left(\frac{c^2}{r^2\pi^2} - r^2\right) \quad \text{and} \quad \gamma^2 = \frac{c^2}{r^2\pi^2} - r^2.$$

Its volume is stated as a lower bound. □

Remark 5. The surfaces has a contact point of first order with

$$x_0 = y_0 = 0 \quad \text{and} \quad z_0 = \pm\left(\frac{c^2}{r^2\pi^2} - r^2\right)^{\frac{1}{2}}.$$

To find an upper bound for the volume enclosed by the awning we can use the vertices $A_0, \dots, A_n = A_0$ of the inscribed regular n-gon with a fixed initial point A_0. According to the convexity, we have that

$$\Sigma_n(P) \le \mathcal{A}(P)$$

and, consequently,

$$\frac{1}{4}\sum_i |(P - A_i) \times (P - A_{i+1})|^2 \le \big(\Sigma_n(P)\big)^2 \le c^2.$$

Therefore the surface defined by the equation

$$\frac{1}{4}\sum_i |(P - A_i) \times (P - A_{i+1})|^2 = c^2$$

contains the awning in its convex hull. To simplify the calculation as far as possible consider the case $n = 3$ with $r = \sqrt{\frac{2}{3}}$. Rotating in the euclidean sense if necessary we can suppose that

$$A_0 = (1, 0, 0), \quad A_1 = (0, 1, 0) \quad \text{and} \quad A_2 = (0, 0, 1);$$

see Figure 1. Using the notations as above

$$\frac{1}{4}\sum_i |(P - A_i) \times (P - A_{i+1})|^2 = L_2(x, y, z) + L_1(x, y, z) + L_0.$$

By the help of a straightforward calculation

$$L_2(x, y, z) = x^2 + y^2 + z^2 + \frac{1}{2}(xy + yz + zx)$$
$$L_1(x, y, z) = -(x + y + z) \quad \text{and} \quad L_0 = \frac{3}{4}.$$

Taking the matrix

$$\begin{pmatrix} 1 & 1/4 & 1/4 \\ 1/4 & 1 & 1/4 \\ 1/4 & 1/4 & 1 \end{pmatrix}$$

of the quadratic form, the eigenvalues are $3/2$ and $3/4$ with a system of eigenvectors

$$b_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad b_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Using the coordinate transformation

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

it follows that

$$L_2(\tilde{x}, \tilde{y}, \tilde{z}) + L_1(\tilde{x}, \tilde{y}, \tilde{z}) + L_0 = \frac{3}{2}\tilde{x}^2 + \frac{3}{4}\tilde{y}^2 + \frac{3}{4}\tilde{z}^2 - \sqrt{3}\tilde{x} + \frac{3}{4}$$

and the left hand side of the equation

$$L_2(x, y, z) + L_1(x, y, z) + L_0 = c^2$$

can be written into the form

$$\frac{(\tilde{x}-\tilde{x}_0)^2}{\alpha^2}+\frac{\tilde{y}^2}{\beta^2}+\frac{\tilde{z}^2}{\gamma^2}=1$$

with

$$\alpha^2=\frac{2c^2}{3}-\frac{1}{6}\quad\text{and}\quad\beta^2=\gamma^2=2\left(\frac{2c^2}{3}-\frac{1}{6}\right).$$

Theorem 4. *The volume enclosed by the awning*

$$\frac{r}{2}\int_0^{2\pi}\sqrt{z^2+(x\cos v+y\sin v-r)^2}\,dv=c,\quad r=\sqrt{\frac{2}{3}}$$

is less than or equal to

$$V_2:=\frac{8}{3}\pi\left(\frac{2c^2}{3}-\frac{1}{6}\right)^{\frac{3}{2}}.$$

Theorem 5. *The volume enclosed by the awning*

$$\Sigma_3(P)=c,\quad A_0=(1,0,0),\quad A_1=(0,1,0)\quad\textit{and}\quad A_2=(0,0,1)$$

is between

$$V_1:=\frac{8}{3}\pi\left(\frac{2c^2}{9}-\frac{1}{6}\right)^{\frac{3}{2}}\quad\textit{and}\quad V_2:=\frac{8}{3}\pi\left(\frac{2c^2}{3}-\frac{1}{6}\right)^{\frac{3}{2}}.$$

Proof. According to Theorem 4 it is enough to investigate the lower bound. The Cauchy–Schwarz inequality shows that

$$c^2=\big(\Sigma_3(P)\big)^2\le 3\big(L_2(x,y,z)+L_1(x,y,z)+L_0\big).$$

Therefore the surface defined by the equation

$$c^2=3\big(L_2(x,y,z)+L_1(x,y,z)+L_0\big)$$

is contained by the convex hull of the awning. Similar calculations as above give rise to the ellipsoid

$$\frac{(\tilde{x}-\tilde{x}_0)^2}{\alpha^2}+\frac{\tilde{y}^2}{\beta^2}+\frac{\tilde{z}^2}{\gamma^2}=1$$

with

$$\alpha^2=\frac{2c^2}{9}-\frac{1}{6}\quad\text{and}\quad\beta^2=\gamma^2=2\left(\frac{2c^2}{9}-\frac{1}{6}\right).$$

Its volume is stated as a lower bound. □

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