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Teaching Mathematics and Computer Science

The sum and difference of the areas of Napoleon triangles

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Abstract. The sum of the areas of the Napoleon triangles is the average of the areas of the three outward equilateral triangles on the sides of triangle ABC, and the differerence of these areas is the area of triangle ABC. In this paper we examine how to change these properties if we build on the sides of the triangle ABC, outwards and inwards, three similar triangles.

Key words and phrases: outer Napoleon triangle, inner Napoleon triangle.

ZDM Subject Classification: G05, G06.

1. Introduction

Let L, M, N be the circumcenters of the equilateral triangles BXC, CYA, AZB built outwards on the sides of an arbitrary triangle ABC and L', M', N' the circumcenters of the equilateral triangles $BX'C$, $CY'A$, $AZ'B$ built inwards on the sides of the triangle ABC. In [7] we find the following properties:

"*The sum of the areas of the Napoleon triangles LMN and $L'M'N'$ is the average of the areas of the three outward equilateral triangles on the sides of triangle ABC, and the differerence of these areas is the area of triangle ABC.*"

So

$$\sigma[LMN] + \sigma[L'M'N'] = \frac{\sigma[BXC] + \sigma[CYA] + \sigma[AZB]}{3}, \tag{1}$$

$$\sigma[LMN] - \sigma[L'M'N'] = \sigma[ABC] = \Delta, \tag{2}$$

where $\sigma[LMN]$ denote the area of the triangle LMN.

In this paper we examine the changing of the relations (1) and (2) if we build on the sides of the triangle ABC, outwards and inwards, three similar triangles. In the next part we use the law of sines: *if the sides of the triangle ABC are a, b and c and the angles opposite those sides are A, B and C, then*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the radius of the triangle's circumcircle.

2. The outward case

We build on the sides of the triangle ABC outwards the similar triangles with one another BCD, CAE and ABF in that way, that $BAF\sphericalangle = \alpha = CAE\sphericalangle$, $CBD\sphericalangle = \beta = ABF\sphericalangle$, $ACE\sphericalangle = \gamma = BCD\sphericalangle$ (Figure 1). Since $A+B+C = \pi = \alpha+\beta+\gamma$, therefore $BDC\sphericalangle = \alpha$, $CEA\sphericalangle = \beta$, $AFB\sphericalangle = \gamma$.

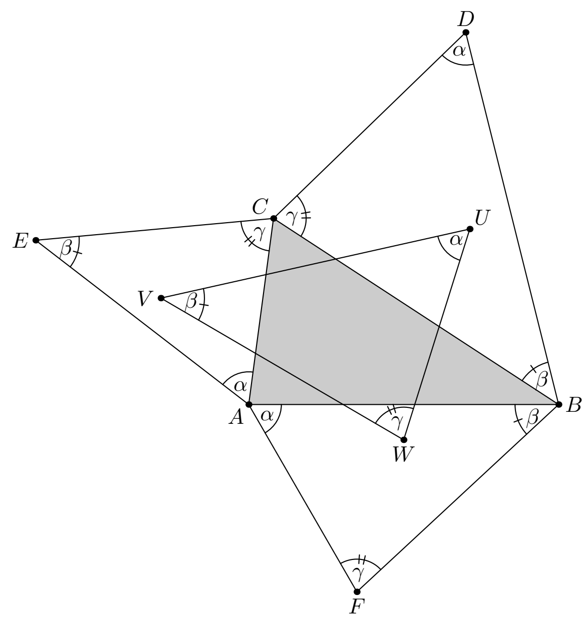

Figure 1

Applying the law of sinus to the triangles BCD, CAE and ABF, we obtain

$$\frac{a}{\sin A} = \frac{CD}{\sin B} = \frac{BD}{\sin C} = 2R_a, \tag{3}$$

$$\frac{CE}{\sin A} = \frac{b}{\sin B} = \frac{AE}{\sin C} = 2R_b, \tag{4}$$

$$\frac{BF}{\sin A} = \frac{AF}{\sin B} = \frac{c}{\sin C} = 2R_c, \tag{5}$$

where R_a, R_b resp. R_c denote the circumradius of the triangle BCD, CAE resp. ABF. The areas of the respective triangles are

$$\sigma[BCD] = \frac{BD \cdot CD \sin\alpha}{2} = \frac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}, \tag{6}$$

$$\sigma[CAE] = \frac{AE \cdot CE \sin\beta}{2} = \frac{b^2 \sin\gamma \sin\alpha}{2\sin\beta}, \tag{7}$$

$$\sigma[ABF] = \frac{AF \cdot BF \sin\gamma}{2} = \frac{c^2 \sin\alpha \sin\beta}{2\sin\gamma}. \tag{8}$$

Let U, V resp. W denote the circumcenter of the triangle BCD, CAE resp. ABF (Figure 1). The triangle UVW is called the **outer generalized Napoleon triangle**. For computing the sides of the outer Napoleon triangle, we use the law of cosines.

Since $\frac{AC}{AV} = \frac{AF}{AW} = \frac{CF}{VW} = 2\sin\beta$ and $\frac{AE}{AV} = \frac{AB}{AW} = \frac{BE}{VW} = 2\sin\gamma$, then $\triangle ACF \sim \triangle AVW \sim \triangle AEB$, consequently $VAW\sphericalangle = A+\alpha$. Similarly is justifiable then $\triangle BFC \sim \triangle BWU \sim \triangle BAD$ and $\triangle CDA \sim \triangle CUV \sim \triangle CBE$, therefore $WBU\sphericalangle = B + \beta$ and $UCV\sphericalangle = C + \gamma$.

We determine the side VW of the triangle UVW:

$$\begin{aligned} VW^2 &= R_b^2 + R_c^2 - 2R_bR_c\cos(A+\alpha) \\ &= \frac{b^2}{4\sin^2\beta} + \frac{c^2}{4\sin^2\gamma} - 2\frac{bc}{4\sin\beta\sin\gamma}\cos(A+\alpha) \\ &= \frac{4R^2\sin^2 B}{4\sin^2\beta} + \frac{4R^2\sin^2 C}{4\sin^2\gamma} - 2\frac{4R^2\sin B\sin C}{4\sin\beta\sin\gamma}\cos(A+\alpha) \\ &= \frac{R^2}{\sin^2\beta\sin^2\gamma}\left[\sin^2 B\sin^2\gamma + \sin^2 C\sin^2\beta - 2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)\right]. \end{aligned}$$

Let's introduce the following notation:

$$\begin{aligned} \lambda &= \lambda(\alpha,\beta,\gamma,A,B,C) \\ &= \sin\alpha\sin\beta\sin\gamma(\cot\alpha\sin^2 A + \cot\beta\sin^2 B + \cot\gamma\sin^2 C + 2\sin A\sin B\sin C) \\ &= \frac{\sin\alpha\sin\beta\sin\gamma}{4R^2}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma + 4\Delta\right). \end{aligned}$$

Are available the following conditional identities:

$$\sin^2 B\sin^2\gamma+\sin^2 C\sin^2\beta-2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)=\lambda, \tag{9}$$

$$\sin^2 C\sin^2\alpha+\sin^2 A\sin^2\gamma-2\sin C\sin A\sin\gamma\sin\alpha\cos(B+\beta)=\lambda, \tag{10}$$

$$\sin^2 A\sin^2\beta+\sin^2 B\sin^2\alpha-2\sin A\sin B\sin\alpha\sin\beta\cos(C+\gamma)=\lambda. \tag{11}$$

We will proove the identity (9):

$$\begin{aligned}
&\sin^2 B\sin^2\gamma+\sin^2 C\sin^2\beta-2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)\\
&\quad=\sin^2 B\sin(\alpha+\beta)\sin\gamma+\sin^2 C\sin(\alpha+\gamma)\sin\beta\\
&\qquad-2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)\\
&\quad=(\sin\alpha\cos\beta\sin\gamma+\cos\alpha\sin\beta\sin\gamma)\sin^2 B\\
&\qquad+(\sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma)\sin^2 C\\
&\qquad-2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)\\
&\quad=\cos\alpha\sin\beta\sin\gamma(\sin^2 B+\sin^2 C)+\sin\alpha\cos\beta\sin\gamma\sin^2 B+\sin\alpha\sin\beta\cos\gamma\sin^2 C\\
&\qquad-2\sin B\sin C\sin\beta\sin\gamma\cos(A+\alpha)\\
&\quad=\cos\alpha\sin\beta\sin\gamma(\sin^2 A+2\cos A\sin B\sin C)\\
&\qquad+\sin\alpha\cos\beta\sin\gamma\sin^2 B+\sin\alpha\sin\beta\cos\gamma\sin^2 C\\
&\qquad-2\sin B\sin C\sin\beta\sin\gamma(\cos A\cos\alpha-\sin A\sin\alpha)\\
&\quad=\sin\alpha\sin\beta\sin\gamma(\cot\alpha\sin^2 A+\cot\beta\sin^2 B+\cot\gamma\sin^2 C+2\sin A\sin B\sin C)\\
&\quad=\lambda.
\end{aligned}$$

Consequently $VW=\frac{R\sqrt{\lambda}}{\sin\beta\sin\gamma}$. Similarly we obtain the sides WU and UV:

$$VW=\frac{1}{2}\sqrt{\frac{\sin\alpha}{\sin\beta\sin\gamma}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)}, \tag{12}$$

$$WU=\frac{1}{2}\sqrt{\frac{\sin\beta}{\sin\gamma\sin\alpha}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)}, \tag{13}$$

$$UV=\frac{1}{2}\sqrt{\frac{\sin\gamma}{\sin\alpha\sin\beta}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)}. \tag{14}$$

Now we determine the angles of the triangle UVW:

$$\begin{aligned}
\cos VUW\sphericalangle&=\frac{UV^2+UW^2-VW^2}{2UV\cdot UW}\\
&=\left(\frac{R^2\lambda}{\sin^2\alpha\sin^2\beta}+\frac{R^2\lambda}{\sin^2\gamma\sin^2\alpha}-\frac{R^2\lambda}{\sin^2\beta\sin^2\gamma}\right)\frac{\sin^2\alpha\sin\beta\sin\gamma}{2R^2\lambda}\\
&=\frac{\sin^2\beta+\sin^2\gamma-\sin^2\alpha}{\sin^2\alpha\sin^2\beta\sin^2\gamma}\cdot\frac{\sin^2\alpha\sin\beta\sin\gamma}{2}
\end{aligned}$$

$$= \frac{\sin^2\beta + \sin^2\gamma - \sin^2\alpha}{2\sin\beta\sin\gamma} = \frac{2\cos\alpha\sin\beta\sin\gamma}{2\sin\beta\sin\gamma}$$
$$= \cos\alpha \quad \Rightarrow \quad VUW\sphericalangle = \alpha.$$

Similarly we obtain that $WVU\sphericalangle = \beta$ and $UWV\sphericalangle = \gamma$. So the triangle UVW is similar to the external triangles BCD, CAE and ABF.

The area of the triangle UVW is

$$\sigma[UVW] = \frac{UV \cdot UW \sin\alpha}{2} = \frac{R^2\lambda}{2\sin\alpha\sin\beta\sin\gamma}$$
$$= \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) + \frac{\Delta}{2}. \tag{15}$$

Let R_O be the circumradius of the triangle UVW:

$$R_O = \frac{UV \cdot VW \cdot WU}{4\sigma[UVW]} = \frac{R^3\lambda\sqrt{\lambda}}{\sin^2\alpha\sin^2\beta\sin^2\gamma} \cdot \frac{\sin\alpha\sin\beta\sin\gamma}{2R^2\lambda} = \frac{R\sqrt{\lambda}}{2\sin\alpha\sin\beta\sin\gamma}.$$

Therefore

$$R_O = \frac{1}{4}\sqrt{\frac{a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma + 4\Delta}{\sin\alpha\sin\beta\sin\gamma}}. \tag{16}$$

3. The inward case

We build on the sides of the triangle ABC inwards the similar triangles with one another BCD', CAE' and ABF' in that way, that $BAF'\sphericalangle = \alpha = CAE'\sphericalangle$, $CBD'\sphericalangle = \beta = ABF'\sphericalangle$, $ACE'\sphericalangle = \gamma = BCD'\sphericalangle$ (Figure 2). Since $A + B + C = \pi = \alpha + \beta + \gamma$, therefore $BD'C\sphericalangle = \alpha$, $CE'A\sphericalangle = \beta$, $AF'B\sphericalangle = \gamma$.

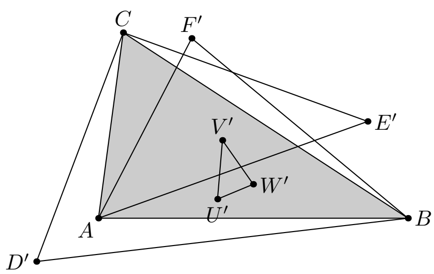

Figure 2

Let U', V' resp. W' denote the circumcenter of the triangle BCD', CAE' resp. ABF' (Figure 2). The triangle $U'V'W'$ is called the **inner generalized Napoleon triangle**. Here we introduce the following notation:

$$\begin{aligned}\lambda' &= \lambda'(\alpha,\beta,\gamma,A,B,C)\\ &= \sin\alpha\sin\beta\sin\gamma(\cot\alpha\sin^2 A+\cot\beta\sin^2 B+\cot\gamma\sin^2 C-2\sin A\sin B\sin C)\\ &= \frac{\sin\alpha\sin\beta\sin\gamma}{4R^2}\left(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma-4\Delta\right).\end{aligned}$$

For determine the sides of the inner Napoleon triangle, here we use the following conditional identities:

$$\sin^2 B\sin^2\gamma+\sin^2 C\sin^2\beta-2\sin B\sin C\sin\beta\sin\gamma\cos(A-\alpha)=\lambda', \tag{17}$$

$$\sin^2 C\sin^2\alpha+\sin^2 A\sin^2\gamma-2\sin C\sin A\sin\gamma\sin\alpha\cos(B-\beta)=\lambda', \tag{18}$$

$$\sin^2 A\sin^2\beta+\sin^2 B\sin^2\alpha-2\sin A\sin B\sin\alpha\sin\beta\cos(C-\gamma)=\lambda'. \tag{19}$$

Since $V'AW'\sphericalangle=|A-\alpha|$, $V'BU'\sphericalangle=|B-\beta|$ and $U'CV'\sphericalangle=|C-\gamma|$ therefore

$$\begin{aligned}V'W'^2 &= R_b^2+R_c^2-2R_bR_c\cos(A-\alpha)\\ &= \frac{b^2}{4\sin^2\beta}+\frac{c^2}{4\sin^2\gamma}-2\frac{bc}{4\sin\beta\sin\gamma}\cos(A-\alpha)\\ &= \frac{4R^2\sin^2 B}{4\sin^2\beta}+\frac{4R^2\sin^2 C}{4\sin^2\gamma}-2\frac{4R^2\sin B\sin C}{4\sin\beta\sin\gamma}\cos(A-\alpha)\\ &= \frac{R^2}{\sin^2\beta\sin^2\gamma}\left[\sin^2 B\sin^2\gamma+\sin^2 C\sin^2\beta-2\sin B\sin C\sin\beta\sin\gamma\cos(A-\alpha)\right]\\ &= \frac{R^2\lambda'}{\sin^2\beta\sin^2\gamma}\quad\Rightarrow\quad V'W'=\frac{R\sqrt{\lambda'}}{\sin\beta\sin\gamma}.\end{aligned}$$

Similarly we obtain the sides $W'U'$ and $U'V'$:

$$V'W'=\frac{1}{2}\sqrt{\frac{\sin\alpha}{\sin\beta\sin\gamma}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma-4\Delta)}, \tag{20}$$

$$W'U'=\frac{1}{2}\sqrt{\frac{\sin\beta}{\sin\gamma\sin\alpha}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma-4\Delta)}, \tag{21}$$

$$U'V'=\frac{1}{2}\sqrt{\frac{\sin\gamma}{\sin\alpha\sin\beta}\,(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma-4\Delta)}. \tag{22}$$

The angles of the triangle $U'V'W'$ are $V'U'W'\sphericalangle=\alpha$, $W'V'U'\sphericalangle=\beta$ and $U'W'V'\sphericalangle=\gamma$. So the triangle $U'V'W'$ is similar to the internal triangles BCD', CAE' and ABF', too.

The area of the triangle $U'V'W'$ is

$$\sigma[U'V'W'] = \frac{U'V' \cdot U'W' \sin\alpha}{2} = \frac{R^2\lambda'}{2\sin\alpha\sin\beta\sin\gamma}$$
$$= \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) - \frac{\Delta}{2}. \tag{23}$$

Let R_I be the circumradius of the triangle $U'V'W'$:

$$R_I = \frac{U'V' \cdot V'W' \cdot W'U'}{4\sigma[U'V'W']} = \frac{R^3\lambda'\sqrt{\lambda'}}{\sin^2\alpha\sin^2\beta\sin^2\gamma} \cdot \frac{\sin\alpha\sin\beta\sin\gamma}{2R^2\lambda'} = \frac{R\sqrt{\lambda'}}{2\sin\alpha\sin\beta\sin\gamma}.$$

Therefore

$$R_I = \frac{1}{4}\sqrt{\frac{a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma - 4\Delta}{\sin\alpha\sin\beta\sin\gamma}}. \tag{24}$$

4. The sum and difference of the areas of the generalized Napoleon triangles

Between the areas of the two generalized Napoleon triangles of the same triangle ABC exist the following relations:

$$\sigma[UVW] + \sigma[U'V'W'] =$$
$$= \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) + \frac{\Delta}{2} + \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) - \frac{\Delta}{2}$$
$$= \frac{1}{4}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right),$$
$$\sigma[UVW] - \sigma[U'V'W'] =$$
$$= \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) + \frac{\Delta}{2} - \frac{1}{8}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right) + \frac{\Delta}{2}$$
$$= \Delta.$$

Summed up:

$$\sigma[UVW] + \sigma[U'V'W'] = \frac{1}{4}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right), \tag{25}$$
$$\sigma[UVW] - \sigma[U'V'W'] = \Delta = \sigma[ABC]. \tag{26}$$

We can express the sum of the areas of the generalized Napoleon triangles with the areas of the external triangles BCD, CAE and ABF:

$$\sigma[UVW] + \sigma[U'V'W'] = \frac{1}{4}\left(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma\right)$$

$$= \frac{1}{2}\left(\frac{\cos\alpha}{\sin\beta\sin\gamma}\cdot\frac{a^2\sin\beta\sin\gamma}{2\sin\alpha}\right.$$
$$\left.+\frac{\cos\beta}{\sin\gamma\sin\alpha}\cdot\frac{b^2\sin\gamma\sin\alpha}{2\sin\beta}+\frac{\cos\gamma}{\sin\alpha\sin\beta}\cdot\frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}\right)$$
$$= \frac{1}{2}\left(\frac{\cos\alpha}{\sin\beta\sin\gamma}\cdot\sigma[BCD]+\frac{\cos\beta}{\sin\gamma\sin\alpha}\cdot\sigma[CAE]+\frac{\cos\gamma}{\sin\alpha\sin\beta}\cdot\sigma[ABF]\right).$$

Consequently the relation (1) changes in the following way:

$$\sigma[UVW]+\sigma[U'V'W'] = \tag{27}$$
$$= \frac{1}{2}\left(\frac{\cos\alpha}{\sin\beta\sin\gamma}\cdot\sigma[BCD]+\frac{\cos\beta}{\sin\gamma\sin\alpha}\cdot\sigma[CAE]+\frac{\cos\gamma}{\sin\alpha\sin\beta}\cdot\sigma[ABF]\right).$$

The relation (2) remaines invariable.

5. Special case

If $\alpha=\beta=\gamma=\frac{\pi}{3}$ then the outer and the inner Napoleon triangles are equilateral and its areas are

$$\sigma[LMN] = \frac{\sqrt{3}}{24}\left(a^2+b^2+c^2\right)+\frac{\Delta}{2}, \tag{28}$$
$$\sigma[L'M'N'] = \frac{\sqrt{3}}{24}\left(a^2+b^2+c^2\right)-\frac{\Delta}{2}. \tag{29}$$

The circumradius of the outer resp. inner Napoleon triangle is

$$R_O = \frac{\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}}{3\sqrt{2}}, \tag{30}$$
$$R_I = \frac{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}{3\sqrt{2}}. \tag{31}$$

Other arrangements of the triangles BCD, CAE and ABF are possible. We will present two of them (Figure 3 and Figure 4). To investigate the valabilities of the relations (27) and (2) in this cases is a possible subject for further researches.

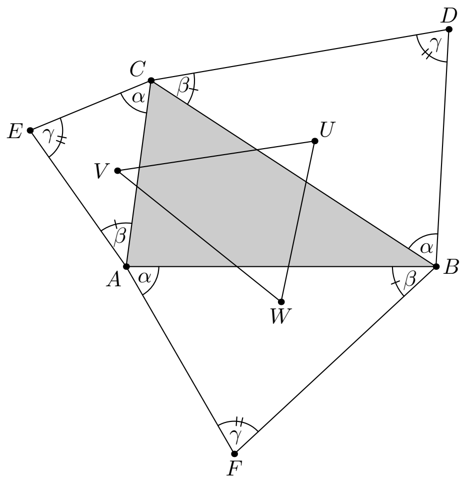

Figure 3

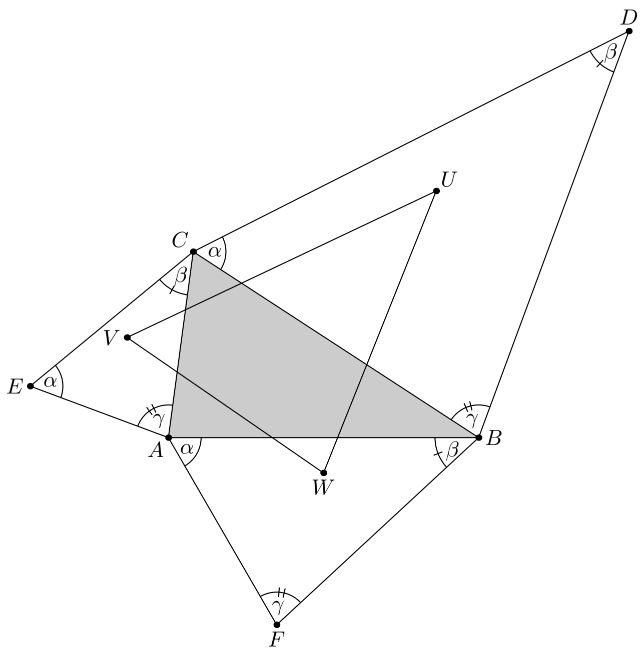

Figure 4

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