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How the derivative becomes visible: the case of Daniel

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Abstract. This paper reports how an advanced 11^{th}-grade student (Daniel) perceived the derivative from a graph of a function at a task-based interview after a short introduction to the derivative. Daniel made very impressive observations using, for example, the steepness and the increase of a graph as well as the slope of a tangent as representations of the derivative. He followed the graphs sequentially and, for example, perceived where the derivative is increasing/decreasing. Gestures were an essential part of his thinking. Daniel's perceptions were reflected against those of a less successful student reported previously [Hähkiöniemi, NOMAD **11**, no. 1 (2006)]. Unlike the student of the previous study, Daniel seemed to use the representations transparently and could see the graph as a representation of the derivative.

Key words and phrases: case study, derivative, embodied world, gesture, graph, representation, transparency.

ZDM Subject Classification: C34, D54, I44.

1. Introduction

In recent years, the role of graphs in learning calculus has been studied extensively. Studies have pointed out that working with graphical representations may have positive influence on learning calculus [7, 22]. Several case studies have provided more detailed information on how students work with graphs [2, 3, 6, 10, 12, 16, 17, 20, 25, 28]. Cumulating on previous research, Tall [29, 30] has developed a theory of three worlds of mathematics. In the embodied world, students learn mathematics as they work, for example, with graphs. According to Tall's theory, the way in which students perceive graphs develops gradually to a more sophisticated way of seeing. This development is fundamentally different from learning in a symbolic environment and it cannot be described satisfactorily by, for example, the popular theories of Sfard [27] and Dubinsky [4].

This paper has two purposes. First, it presents a case study of one advanced student's ways to perceive the derivative from a graph of a function at an early stage of the learning of the derivative. Thus, the study contributes to the discussion on students' ways to work with graphs. Particularly, the aim is to find out effective tools that students may use for thinking about the derivative. Secondly, this case study is contrasted with a similar study of a less successful student, see Hähkiöniemi [10]. The aim of the comparison of the two students is to explore a viable way to describe differences between their perceptual activities.

There are different ways to build relations between a graph of a function and its derivative. Often, the relation is constructed procedurally: the algebraic expression of the function is differentiated, then the zero points of the derivative are determined, and, finally, it is concluded where the function is increasing/decreasing based on the sign of the derivative. Sometimes also the sign of the second derivative is used to determine the intervals where the graph of the function is concave up/down. This approach may cause difficulties because students may base their work only on memorized procedures. The problems of memorization were noticed in a study of Baker et al. [2] in which students drew the graph of a function on the basis of given information about the derivative. Also according to the study of Selden et al. [26], many calculus students who are able to solve routine calculus problems cannot solve corresponding non-routine problems. Indications of the problems of memorization are given also in Viholainen's [33] study. In his study, more than one fourth of 146 prospective mathematics teachers at the final stage of their studies in a written test answered that a certain discontinuous function is differentiable. Interviews of two students revealed that one reason for this was the inappropriate use of methods (e.g., differentiation rules) in reasoning.

There are also other than procedural ways to build relations between a function and its derivative. Baker et al. [2] studied how students sketched a graph of a function h when they were given several conditions on the values of the function, its derivative (e.g., $h'(x) > 0$ when $-2 < x < 3$) and its second derivative (e.g., $h''(x) < 0$ when $0 < x < 5$). According to them, to solve this problem, students need to coordinate different properties over a particular interval and coordinate different intervals related to a particular property. Baker et al. [2] developed a graphing schema in which success in both of these coordinations have three levels. Only eight students of the 41 interviewed university calculus students coordinated all the properties and the intervals and produced the completely correct graph [2]. The schema developed by Baker et al. is useful in describing how students determine properties of a graph and intervals for which the properties hold. However, Tall and Watson [32] reported another way to make observations about a graph. According to them, the derivative of a function may be perceived by following sequentially the graph of the function from left to right. This may be supported by moving a hand along the graph [32]. According to the study of Tall and Watson, students whose teacher emphasized this kind of activity sequentially observed how the gradient changes in a written test. These students also performed better than students of other teachers in graphing tasks [32]. This sequential nature can also be noticed in students' construction of distance-time graphs based on given velocity-time graphs in a case study reported by Berry and Nyman [3]. They found that eight university students moved from an instrumental understanding of calculus towards relational understanding when engaged in tasks where they sketched the distance-time graph, then created the corresponding movement and compared their graph to the graph produced by the motion detector.

In this study, a Finnish grade 11 student called Daniel was interviewed after a short teaching-learning sequence that introduced the derivative emphasizing perceptual activity. The analysis of the interview focuses on Daniel's use of representations in perceiving the derivative from a graph of a function. In this study a representation is considered as a tool for thinking about the derivative, see [10]. Also gestures are analyzed, as they play an important part in thinking, see, for example [13, 24].

2. Mathematical thinking and learning in the embodied world

This paper considers how the derivative may be perceived from a graph of a function. This kind of perceptual activity corresponds to working in the (conceptual-)embodied world in Tall's [29, 30] theory of three worlds of mathematics. The theory of three worlds combines several theories to describe coherently how humans learn mathematics. It tries to glue together seemingly very different theories, such as van Hiele levels, process-object development [4, 27] and theories of advanced mathematical thinking (e.g., [31]). According to Tall [29, 30], the embodied world consists of thinking about things that can be perceived and sensed in the physical and the mental world. The other two worlds are the (proceptual-)symbolic world in which symbols act dually as processes and concepts and the formal(-axiomatic) world which is based on the deduction of properties from axioms [29, 30]. These worlds have different warrants for the truth and a different mechanism for learning.

In the symbolic world, learning proceeds as students perform procedures and compress these procedures as thinkable concepts [29, 30]. Thus, in this world, a new concept is constructed from actions performed on an already existing object. This development corresponds to the reification theory [27] and the APOS theory [4]. However, Gray and Tall [5] have proposed that learning may begin also as students act with a concept and perceive its properties. According to them, perceptions may become more abstract constructs, which do not anymore refer to specific objects in the real world. For example, a conception of a line may develop from a line drawn by a ruler to a perfectly straight line that has no width and is arbitrarily extensible in either direction [5]. This description of the development is quite different from that of the symbolic world. However, Pegg and Tall [18] found a similarity underlying these two ways of learning. The similarity is that in the embodied world students may learn by shifting their focus from actions to the effects of those actions [18, 19, 30]. For example, in the case of the vector concept, students may shift their focus from translations of a hand to the effects of the translations [19, 30]. Similarly, the action of dividing a quantity into 6 equal parts and selecting 3 of them leads to the same effect as dividing the quantity into 4 equal parts and selecting 2 [18]. According to Pegg and Tall [18], this parallels the symbolic world. In the symbolic world, one may view the procedure of dividing 3 by 6 as different from dividing 2 by 4. If one focuses on the concept of fraction, he/she may notice that $3/6 = 2/4$.

The action-effect development has also some similarities with representations becoming more and more transparent. The transparency of a tool means that the tool is visible for acquiring detailed information about the tool but invisible in the sense that it gives access to a phenomenon that can be seen through the tool [11, 14, 23]. For example, when one is learning to ride a bicycle, the bicycle is first very visible, and all the attention is directed to handling it. Sometimes it happens that even when a child does not fall he/she hits something because of not paying attention to the environment. Gradually, the bicycle becomes invisible and the rider is able to concentrate on moving around and monitoring traffic. Sometimes the bicycle is so invisible that when a male is riding a women’s bicycle he still lifts his leg over the bicycle when getting off it. Like the bicycle metaphor illustrates, transparency is not a property of a tool (bicycle) but an emerging relation between the user and the tool [14, 23]. Also a graph is a tool, and it may become transparent to the user so that he/she sees the phenomenon behind the graph and does not only focus on the physical appearance of the graph [1, 23]. In the above mentioned examples from Tall [30], Poynter [19] and Pegg and Tall [18], the transparency of the embodiments would mean that one sees the vector through the hand movement and the fraction through selecting pieces of a quantity.

The hand movement embodiment discussed in [19, 30] is a good example of how gesturing plays an essential role in mathematical thinking. In this study, gestures are considered as external sides of representations. For example, when a student uses a hand movement for thinking about a vector, it has an external side (the hand movement that we see) but also some internal side that we cannot see (the hand movement is not always related to a vector). The use and the transparency of the representation consist of the interplay between the external and the internal side. Several authors have argued that gestures (as well as other external sides) play an important part in expressing, communicating and reorganizing one's thinking [10, 13, 15, 20, 21, 24]. In Roth and Welzel's [24] case studies gestures seemed to make abstract entities visible. In mathematics education, gestures are studied especially in graphical contexts. Radford et al. [20] reported how gestures with words allowed the student to make sense of a distance-time graph of a moving object. Also Mosckovich [15] highlights the importance of gestures when describing graphical objects.

In this study, it is examined how a student sees the derivative through a graph of a function. Instead of focusing on the graph as a single representation, more specific representations (such as the steepness, the increase, and the slope of a tangent accompanied with different gestures) inside the graph are examined. If a student sees the derivative transparently through these representations, it would mean that he/she perceives something else than only physical features of these representations. In the next section, results of the analysis of a student who did not seem to perceive the derivative especially transparently are summarized.

3. The case of a student who did not perceive the derivative especially transparently

Hähkiöniemi [10] presented an analysis of a less successful student's ways to perceive the derivative in a similar setting to that in the present study. In this section this case is reviewed. The student named Susanna perceived from a graph of a function the sign of the derivative, the zero point and the maximum point of the derivative, and the interval where the derivative is constant. For making these observations she used representations of the increase, the steepness and the horizontalness of the graph. For Susanna the increase was a property of an interval that concerned the quality of the derivative, whereas the steepness was a point wise property and gave the magnitude of the derivative. She used gestures as an essential part of her thinking. She imitated a graph as an external side of the increase-representation, and the horizontalness was accompanied by drawing a horizontal line in the air. Placing a pencil as a tangent to a graph was an external side of the steepness. She also made some mistakes: she used a differentiation rule inappropriately, determined the minimum point of the derivative incorrectly from a graph of a function, perceived a function and its derivative to change in the same way and drew an incorrect distance-time graph given a velocity-time graph.

Susanna seemed to focus on the graph as a physical object while she recognized some aspects of the derivative. She noticed such things as the graph going upward/downward and the steepness of the graph, but she did not spontaneously perceive the rate of change of the derivative. Considering the rate of change of the derivative would have required perceiving aspects that need a more disciplined way of seeing. Such methods as concavity, for seeing the second derivative, were not discussed before the interview. Susanna’s use of physical objects to see these aspects and her mistakes suggest that she still focused on very concrete aspects. She did not seem to use her representations very transparently because she focused more on the representations than on the derivative which can be seen through them.

However, Susanna demonstrated some conceptual knowledge in the embodied world by connecting features of a function and its derivative. When compared to her work in the symbolic world (in which she, for example, used differentiation rules arbitrarily), she demonstrated more mathematical reasoning. She also seemed to consider the derivative as an object which has some properties, such as sign and magnitude.

4. Methodology

Daniel was invited into a task-based interview after the derivative was introduced in a five-hour specially designed teaching-learning sequence and five normal lessons. The teaching-learning sequence began by examining motion graphs and by perceiving the rate of change of a function from its graph. Moving a hand along the curve, placing a pencil as a tangent, looking how steep the graph was and the local straightness of the graph were used as representations. In the class it was discussed how the above-mentioned representations can be used to see the sign and the magnitude of the rate of change. The students were engaged in determining the value of the instant rate of change. The derivative was defined through the solution of this problem. After the specially designed teaching-learning sequence, the course continued with, for example, differentiation rules and investigations of polynomial functions using the derivative function.

Daniel was selected to the interview because he had been a very successful student in mathematics in previous courses and in a pre-test. In the about 45-minute task-based interview, the most important tasks discussed in this paper were:

Task 2. The graph of a function f is given in the figure (Figure 1). What observations can you make about the derivative of the function f at different points?

Task 3. Estimate as accurately as possible the value of the derivative of the function $f(x) = 2^x$ at the point $x = 1$.

Task 5. A car starts at the time $t = 0$ from the starting point. The figure (Figure 2) represents the velocity v (m/s) of the car as a function of time t (s).

b) When does the distance traveled by the car increase and when does it decrease?

c) Sketch the graph of the distance traveled s (m) by the car as a function of time t (s) in the given (t, s)-coordinates.

f) Sketch the graph of the acceleration a (m/s^2) of the car as a function of time t (s) in the given (t, a)-coordinates.

Task 2 was designed to give information on how students can see the derivate from the graph of the function. The equation of the function was not given, so that all the conclusions would be based on the graph. Task 3 was chosen to get information about how students estimate the derivative of a function for which they do not know the differentiation rule, and the use of the limit of the difference quotient is too difficult. Task 5 was intended to be similar to task 2 but in a different context. The difference is that this task corresponds to the situation where the graph of a function (velocity) is given, and students are asked about the integral function (the distance) and the derivative function (the acceleration). In this task they were also asked to draw the graphs.

The interview was video-recorded and Daniel was directed to think aloud. The video data were transcribed and analyzed inductively to find out how Daniel perceived the derivative and what kind of representations he used in thinking about the derivative. The situations where Daniel used some representation were located. From each situation it was analyzed how he used these representations and what they allowed him to do. Then all the situations were compared to each other. This way, an analysis of one representation was reflected against the analysis of other representations. Daniel's uses of representations were also compared to the uses of the other four interviewed students, including Susanna. These comparisons allowed noticing common and distinct features in students' use of representations, see Hähkiöniemi [8]. Furthermore, it made possible to deepen the analysis of Daniel as he was contrasted against the other students. The excerpts from the interview given in the following section are translated from Finnish by the author, and "[...]" in the transcripts means that the text is snipped. Gestures are described in brackets [ ].

5. Daniel's perceptions of the derivative

In task 2, Daniel perceived the maximum point, the minimum point and the sign of the derivative, the interval where the derivative is constant and the point where the derivative does not exist. Furthermore, he also perceived how the derivative changes from right to left:

> Daniel: When we start to go forward down here [traces the graph with a finger from 2 to 1], then over there it actually increases all the time, because it is steepest there [points to the graph at 1.9]. No, it's decreasing, because it's positive there [traces the graph with a finger from 2 to 0.8]. It goes here, it is zero down here [points to the graph at 0.8]. Then it becomes negative here, we go upward. It starts to increase again here somewhere in the middle [points to the graph at −0.6] and there it is zero again [points to the graph at −1.5]. And then we go downward, it decreases.

Daniel made his perceptions of the derivative otherwise correctly but stated that it decreases as x decreases from -1.5. The correctness of the increasing/decreasing of the derivative depends on the direction: Daniel proceeded from right to left. He seemed to use the steepness of the graph to see the magnitude of the derivative and how it changes. When asked about the minimum point of the derivative, he compared the steepness of the graph at two points:

Interviewer: When would the derivative be at its greatest and when at its smallest?

Daniel: It is at its smallest at this interval about in the middle after the origin [points to the graph at −0.4]. And then at greatest... No, it's by the way smallest here [points to the graph at 3.8], this is actually more diagonal. No, how is it? No [makes a gesture with a hand, Figure 1], it is smaller here [points to the graph at −0.4], of course, it is steeper [makes a gesture with a hand, Figure 1], and negative.

When finding the steepest part of the graph, he made a gesture shown in Figure 1. This gesture was an external side of the steepness representation.

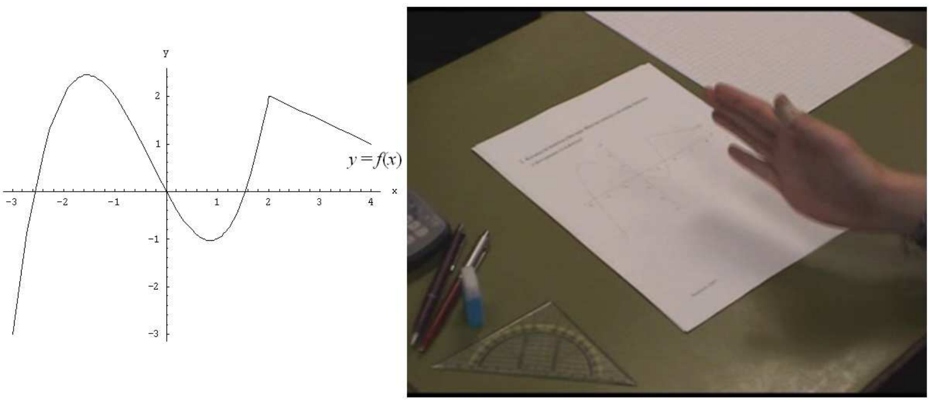

Figure 1. The graph given in task 2 and Daniel's gesture when considering the steepness of the graph.

He also seemed to relate the graph going upward/downward to the sign of the derivative. However, because of proceeding from right to left, the positive derivative corresponded to going downward. The movement as he followed the graph was essential to his perceptions: "we start to go", "it goes", "it becomes" and "it starts to". Corresponding to the movement, he traced the graph with a finger. He also used the tangent-representation when arguing that there is no derivative at the point 2: "you can draw the tangent in any direction [swings hand in the air]".

Daniel used the increase of the graph also in task 3. He calculated mentally some points of the graph and noticed that "it rises upward". He estimated the value of the derivative to be "something between 1 and 2" without making any inscriptions. When asked, he explained how he did it:

Interviewer: How did you figure that out?

Daniel: Because it is a kind of half a parabola, if I imagine it correctly. [. . .] The value at the point 1 was 2 and at the point 2 it would be 4, so at that interval the derivative would be something, it would be 2 at that interval. [. . .] Yes, it would be like a half a parabola. If we then assume that it would continue to decrease in the same way also after the 2 to the power 1, then it would be, it would be about slightly above 1.

Apparently, he calculated mentally the derivative over an interval (the average rate of change), imagined the shape of the graph and considered the tendency of the decrease. This allowed him to construct an estimate for the value of the derivative. Later, in another task, he called the derivative over an interval "the average derivative" and used it in relation to the difference quotient, see Hähkiöniemi [9].

Also in task 5 Daniel used the representations of the steepness, the increase and the tangent. In task 5b he considered where the velocity was increasing and where decreasing and made inferences of the movement of the car:

Daniel: The distance increases, of course, because it is going forward. If it goes forward, the speed increases all the time [traces the graph from 0 to 11]. There it starts to slow down, the speed [points to the graph at 11]. It still has to go forward [traces the graph from 11 to 20] because it cannot have turned at any point. Here the speed is zero [points to the graph at 20]. Then it must have stopped here.

However, when Daniel drew the distance-time graph (Figure 2) in task 5c, he assumed, against his observation in task 5b that the velocity is decreasing at the beginning and related this to the steepness of the distance-time graph:

Daniel: At the beginning it [distance] is zero, of course. It starts to increase as time goes forward. It gets less steep because also the speed decreases [starts to draw a parabola, Figure 2]. And the distance starts to. The distance would be zero somewhere there. I mean not zero but it would stay the same, it would be constant [draws the top of the parabola]. Then the distance starts to decrease [continues the parabola downward].

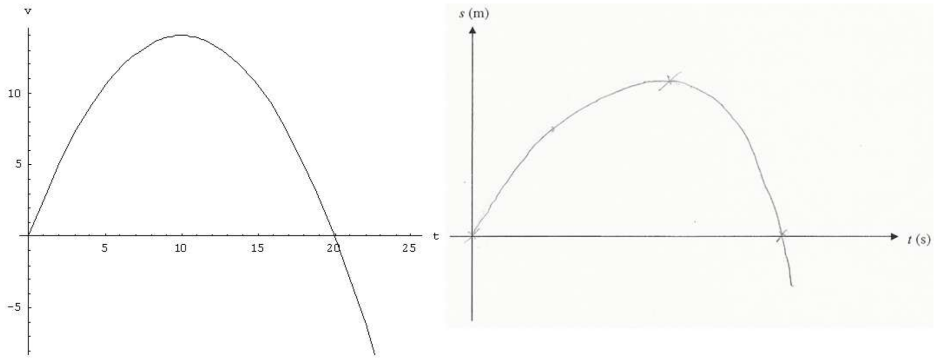

Figure 2. The velocity-time graph given in task 5 and Daniel’s distance-time graph.

Although Daniel’s distance-time graph resembles the velocity-time graph, it seems that he did not assume resemblance between these two graphs, see Nemirovsky and Rubin [16]. This is evidenced as he related the x-intersection of the velocity-time graph to the maximum point of the distance-time graph:

Daniel: At this point it turns back, that is, the velocity becomes negative [points to the x-intersection of the velocity-time graph]. Here it would be this point [points to the maximum point of the distance-time graph] when it turns back, that is, 20 seconds.

When the interviewer asked about the maximum point of the velocity according to the two graphs, Daniel noticed that they gave different answers:

Interviewer: When would the velocity of the car be at its greatest?

Daniel: The velocity of the car would be at its greatest when the slope of a tangent drawn here [sketches a tangent to the distance-time graph in the air near to the origin] would be greatest. Well, it would be here, at the beginning because the velocity decreases all the time. [...]

Interviewer: If you look at this graph [points to the velocity-time graph], then what would be the value of the velocity at the beginning?

Daniel: Well, yeah. Oh yeah, of course. It would then be... The velocity starts from zero, of course. The velocity would be greatest here at the top [points to the velocity-time graph at $t = 10$], about 13 kilometers per hour. [Pause.] In this figure [points to the velocity-time graph] the velocity would be greatest at the top. [...] In this figure [the distance-time graph] it would be. [Pause.] Well, it would be at the beginning [sketches a tangent to the distance-time graph in the air near to the origin] if the distance is divided by the time. [. . .] Basically, it should be at the same, that is, at 10 seconds.

The point is that, although Daniel made a mistake, he used the representations of the increase, the steepness and the slope of a tangent to perceive essential aspects from the graphs. Also in this task he made these observations sequentially following the graph from left to right. He also used a gesture of sketching a tangent in the air as an external side of the slope of a tangent. In task 5f Daniel used a new kind of representation: drawing tangents in the air along the graph of velocity (Figure 3):

Daniel: The acceleration is greatest here, gets smaller, there the acceleration would be zero and then the acceleration would become negative [draws small tangents in the air along the graph of velocity]. So here at the point 10, the acceleration of the car would be zero. [. . .] There the acceleration of the car would be zero. There the tangent would be straight.

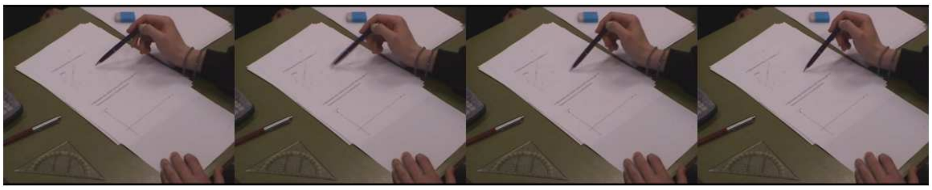

Figure 3. Daniel draws a small tangent in the air in task 5f.

On the basis of these observations he sketched the acceleration-time graph (Figure 4). The "tangents in the air" representation is dynamic because it represents how the tangents are changing along the graph. Using this representation, Daniel perceives how the acceleration changes. Particularly, he related the zero point of the acceleration to a "straight" tangent, by which he obviously meant a horizontal tangent. In task 5c he already considered the tangent and gestured drawing a tangent but did this only at one point. As in the other tasks, also in this task Daniel made perceptions of the acceleration sequentially following the graph. Similarly to task 2, he proceeded from right to left as he considered how the acceleration changes from point $t = 10$ to $t = 0$: "Here the acceleration is on the x-axis [sic $a = 0$] and then time decreases, the acceleration increases all the time.”

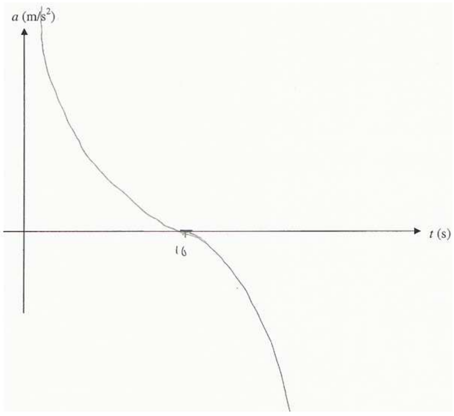

Figure 4. Daniel’s acceleration-time graph in task 5f.

6. Conclusions and discussion

It was analyzed how Daniel worked in the embodied world, particularly, how he perceived the derivative from a graph of a function. Daniel made very impressive perceptions. Besides noticing in task 2, for example, the maximum and minimum points of the derivative, he even perceived how the derivative changed. This aspect is not directly connected with the physical features of the graph because in the teaching-learning sequence no methods (e.g., concavity) for seeing where the derivative is increasing and where decreasing were discussed. He could also estimate the value of the derivative of the function $f(x) = 2^x$ at a point without making any inscriptions. In the embodied world Daniel used the steepness, the increase, and the horizontalness of a graph as well as the slope of a tangent to a graph for thinking about the derivative. The increase/decrease of a graph was related to the sign of the derivative. The steepness and the slope of a tangent were tools for thinking about the magnitude of the derivative. The steepness was accompanied with a gesture presented in Figure 1, and the tangent was accompanied with sketching a tangent in the air. In task 5 Daniel also used a new kind of representation which included drawing small tangents in the air along the graph of the velocity. This representation was not discussed in the teaching-learning sequence. Thus, it was his invention. This dynamic representation seemed to help Daniel to see how the acceleration changes.

Daniel seemed to work sequentially along the graph noticing how the derivative changes. Therefore, his working resembles that described by Tall and Watson [32]. This way of working with the graph is very different from obtaining first information about different properties and intervals and then coordinating these, see Baker et al. [2]. The sequential way of observing the graph may help students to develop their reasoning with graphs instead of only memorizing procedures. For example, Daniel seemed to coordinate the first and second derivative over the same interval when he reasoned in task 2 how the derivative changed. This kind of coordination was reported to be especially difficult for students in the study of Baker et al. [2]. It should also be noted that students may proceed sequentially to different directions. For example, Daniel's perceptions of the increase/decrease of the derivative in task 2 would have been thought to be incorrect without noticing that he proceeded from right to left.

Daniel's work in the embodied world was very impressive, taking into account that he was at a very early stage in learning the derivative. In the embodied world, he already had knowledge of the most important applications of the derivative, such as finding the extreme values of a function. He also seemed to consider the derivative as an object which had some properties. Therefore, this case study gives support to Tall's [29, 30] and Gray and Tall's [5] claim that learning may begin also by perceiving the concept as an object.

Compared to Susanna [10], Daniel seemed to perceive the derivative more transparently. They used similar representations of the steepness, the increase, the horizontalness and the slope of a tangent, but Susanna focused more on these representations and the physical appearance of the graph than on the derivative. On the contrary, for Daniel, these representations seemed to be invisible so that he could focus on the derivative. He even compared the value of the derivative at several points. This difference was perhaps expected, as Daniel had been more successful in mathematics before this course. Nevertheless, it seems that the transparency concept may be used to characterize students' learning in the embodied world and it has some similarities to the action-effect development described by Tall [30], Poynter [19] and Pegg and Tall [18]. For both students, gestures were an essential part of the use of the representations. Therefore, it is reasonable to consider these gestures as visible sides of the representations and not just separated gestures. Although Daniel did not use gestures as often as

Susanna, the gestures when used seemed to be essential to his thinking and helped him to make the derivative concept visible in the same way as Roth and Welzel [24] reported in their study. Especially, in task 2, Daniel first determined the minimum point of the derivative without gesturing, but to clarify his reasoning, he made the gesture shown in Figure 1. Despite their differences, both students demonstrated good reasoning and conceptual knowledge in the embodied world. Therefore, these studies emphasize how working in the embodied world at the beginning of the learning of the derivative may benefit both advanced and less advanced students.

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