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Teaching Mathematics and Computer Science

On four-dimensional crystallographic groups

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Abstract. In his paper [12] S. S. Ryshkov gave the group of integral automorphisms of some quadratic forms (according to Dade [6]). These groups can be considered as maximal point groups of some four-dimensional translation lattices in E^4. The maximal reflection group of each point group, its fundamental domain, then the reflection group in the whole symmetry group of the lattice and its fundamental domain will be discussed. This program will be carried out first on group T. G. Maxwell [9] raised the question whether group T was a reflection group. He conjectured that it was not. We proved that he had been right. We shall answer this question for other groups as well. Finally we shall give the location of the considered groups in the tables of monograph [4]. We hope that our elementary method will be useful in studying linear algebra and analytic geometry. Futhermore, 4-dimensional geometry with some visualisation helps in better understanding important concepts in higher-dimensional mathematics, in general.

Key words and phrases: four-dimensional crystallographic groups, fundamental domain, point groups, quadratic form, reflectiongroups.

ZDM Subject Classification: G40, G50, G70, H40, H60.

1. Introduction

The topic of geometric transformations creates connections between group theory, linear algebra and geometry. Geometric transformations form a group with successive application as the product operation. Reflections have high priority among the congruence transformations. In two- and three-dimensional space every congruence transformation can be produced as a product of some reflections. When we generalize the concept of affine space, vector space and crystal structure we can draw up multidimensional geometric problems applying methods of linear algebra. The question arises in higher dimensions if we can generate some congruence groups by reflections or not. In this paper we discuss a problem in four-dimensional space which throws light on the fact that there exist transformation groups which are not reflection groups.

S. S. Ryshkov [12] gave the maximal group of integral linear automorphisms of the following quadratic form:

$$\begin{aligned} q(\mathbf{x}) = 4x^1x^1 + 4x^2x^2 + 4x^3x^3 + 4x^4x^4 + 2x^1x^2 - 4x^1x^3 - \\ - 4x^1x^4 - 4x^2x^3 - 4x^2x^4 + 2x^3x^4. \end{aligned} \tag{T}$$

The similar problem was discussed by Dade [6] for:

$$\begin{aligned} q(\mathbf{x}) = 4x^1x^1 + 4x^2x^2 + 4x^3x^3 + 4x^4x^4 - 2x^1x^2 - 2x^1x^3 - \\ - 2x^1x^4 - 2x^2x^3 - 2x^2x^4 - 2x^3x^4, \end{aligned} \tag{P_4}$$

$$q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 - x^1x^2 - x^2x^3 - x^3x^4. \tag{S_4}$$

This latter is equivalent [4] with the quadratic form:

$$\begin{aligned} q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 + x^1x^2 + x^1x^3 + \\ + x^1x^4 + x^2x^3 + x^2x^4 + x^3x^4. \end{aligned} \tag{S_4}$$

Futhermore, we have the following forms:

$$q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 - x^1x^2 - x^3x^4, \tag{B}$$

$$q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 \tag{C_4}$$

$$\begin{aligned} q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 + x^1x^2 - x^1x^3 - \\ - x^1x^4 - x^2x^3 - x^2x^4 \ . \end{aligned} \tag{Q_4}$$

This is again equivalent with the quadratic form in [4]:

$$\begin{aligned} q(\mathbf{x}) = x^1x^1 + x^2x^2 + x^3x^3 + x^4x^4 + x^1x^2 + x^1x^4 + \\ + x^2x^3 - x^3x^4. \end{aligned} \tag{Q_4}$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the basis vectors of any above lattice $\mathbf{\Lambda}$. The quadratic form $q(\mathbf{x})$ expresses the length square of the vector $\mathbf{x} = x^1 \cdot \mathbf{e}_1 + x^2 \cdot \mathbf{e}_2 + + x^3 \cdot \mathbf{e}_3 + x^4 \cdot \mathbf{e}_4$. G. Maxwell [9] raised the question whether the group T above was a reflection group. In this paper we answer this question in the negative. Our method will be carried out in detail only for the quadratic form of T, however, in the tables of Section 6 we summarize the results for other cases as well.

2. Basic concepts and notation

E: real affin and Euclidean space of finite dimension.

V: the vector space of translations in E.

B: symmetric, non-degenerate, positive definite bilinear form on V. B makes E above a Euclidean space. $B(\mathbf{v}, \mathbf{v}')$ expresses the scalar product of vectors $\mathbf{v}, \mathbf{v}' \in \mathbf{V}$.

q: quadratic form on V. For $\mathbf{v} \in V$ put $q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$.

$|\mathbf{v}|$: defines of the length of vector $\mathbf{v}$. $|\mathbf{v}| = q(\mathbf{v})^{\frac{1}{2}}$.

H: hyperplane of E.

Definition 1. If the vector $\mathbf{n} \neq \mathbf{0}$ in V is orthogonal to H with respect to B, then the transformation:

$$s(\mathbf{x}) = \mathbf{x} - 2 \cdot \frac{B(\mathbf{x}, \mathbf{n})}{B(\mathbf{n}, \mathbf{n})} \cdot \mathbf{n}, \quad \mathbf{x} \in V$$

is called the orthogonal reflection in H.

The orthogonal reflection s preserves B and it is identity on H.

Definition 2. Groups generated by orthogonal reflections in some hyperplanes are called reflection groups or Coxeter groups.

Definition 3. Let $\mathscr{H}$ be a locally finite set of hyperplanes of E. W denotes the group generated by the orthogonal reflections in the hyperplanes of $\mathscr{H}$. A (closed) fundamental domain of the group W is a subset C of E if:

a) For every $\mathbf{x} \in E$, there exists $\omega \in W$ such that $\omega(x) \in C$.

b) If $\mathbf{x}, \mathbf{y} \in C$ and $\omega \in W$ are such that $\mathbf{y} = \omega(\mathbf{x})$, then $\mathbf{y} = \mathbf{x}$.

Definition 4. If W is a group generated by the set S of orthogonal reflections in the hyperplanes (walls) of a fundamental domain, then the relations of types

$$s_i^2 = 1 \text{ and } (s_i \cdot s_j)^{m(s_i, s_j)} = 1, \quad s_i, s_j \in S$$

determine the group. These are the defining relations of the group.

Definition 5. Let C be a fundamental domain. S denotes the set of orthogonal reflections in the walls of the fundamental domain C. The vertices of the

Coxeter graph represent the elements of S. If $m(s_i, s_j) = 2$, then s_i and s_j are not connected, if $m(s_i, s_j) \neq 2$, then $m(s_i, s_j)$ is written on the edge connecting s_i and s_j. When $m(s_i, s_j) = 3$, then we can omit 3 from the edge.

3. The maximal reflection group of the point group T

Further on we use the former quadratic form T in the form

$$\begin{aligned} q(\mathbf{x}) = {} & (x^1 + x^2 - x^3 - x^4)^2 + (x^1 - x^3)^2 + (x^1 - x^4)^2 + \\ & + (x^2 - x^3)^2 + (x^2 - x^4)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2. \end{aligned} \tag{1}$$

The appropriate bilinear form is:

$$\begin{aligned} B(\mathbf{x}, \mathbf{y}) = {} & 4x^1y^1 + 4x^2y^2 + 4x^3y^3 + 4x^4y^4 + x^1y^2 + x^2y^1 - 2x^1y^3 - \\ & - 2x^3y^1 - 2x^1y^4 - 2x^4y^1 - 2x^2y^3 - 2x^3y^2 - 2x^2y^4 - \\ & - 2x^4y^2 + x^3y^4 + x^4y^3. \end{aligned} \tag{2}$$

First we determine those hyperplane reflections that transform the lattice onto itself. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ be the basis vectors of the lattice and let O be any point of E as origin, $\overrightarrow{OE_i} := \mathbf{e}_i$ $(i = 1, 2, 3, 4)$. In the following we simply write plane instead of hyperplane.

Let H_1 be a possible reflection plane and let the reflection in H_1 map the origin O to O_1. The vector $\overrightarrow{OO_1}$ belongs to the lattice $\mathbf{\Lambda}$. The translation by vector $\frac{1}{2}\overrightarrow{O_1O}$ maps H_1 onto the plane H through O. Then the reflection in H is equal to the product of the translation by vector $\overrightarrow{OO_1}$ and the reflection in H_1. If the translation by $\frac{1}{2}\overrightarrow{OO_1}$ maps H_1 onto the plane H_2, then H_2 obviously is a possible reflection plane, too. Thus all the suitable reflection planes can be obtained in the following steps:

- we determine the reflection planes through the origin O,
- then we determine all the reflection planes by translations from the former planes.

The latter happens as follows: Let H be a reflection plane through O and let $\mathbf{n} = (n^1, n^2, n^3, n^4)$ be a normal vector of H with respect to the scalar product derived by (2), where the greatest common divisor $(n^1, n^2, n^3, n^4) = 1$. Clearly, by translations of vectors $\frac{k}{2} \cdot \mathbf{n}$ (k is an arbitrary integer) we can get all the possible reflection planes.

Thus we determine all the reflection planes through the origin O. Let H be one of these planes. Reflecting E_1 in the plane H we get E_1'. For E_1' it holds

$|\overrightarrow{OE_1}| = |\overrightarrow{OE_1'}|$, that is, by the quadratic form $|\overrightarrow{OE_i}| = |\mathbf{e}_i| = 2$. If $\overrightarrow{OE_1'} = \mathbf{e}_1' = = k^1 \cdot \mathbf{e}_1 + k^2 \cdot \mathbf{e}_2 + k^3 \cdot \mathbf{e}_3 + k^4 \cdot \mathbf{e}_4$, then we have:

$$\begin{gathered}(k^1 + k^2 - k^3 - k^4)^2 + (k^1 - k^3)^2 + (k^1 - k^4)^2 + (k^2 - k^3)^2 + \\ + (k^2 - k^4)^2 + (k^1)^2 + (k^2)^2 + (k^3)^2 + (k^4)^2 = 4.\end{gathered} \tag{3}$$

The sum of 9 square numbers can be equal to 4 if and only if:

- eitheir one of them is equal to 4, the others are equal to 0;
- or four of the 9 numbers are equal to 1, the others are equal to 0.

All of $(k^1)^2$, $(k^2)^2$, $(k^3)^2$, $(k^4)^2$ cannot be 0. If one $(k^i)^2 = 4$, then the right-hand side of (3) would be 16. Hence only the second case is possible. Thus we get the values for k^i in Table 1.

Table 1. $\mathbf{e}_1' = k^1 \cdot \mathbf{e}_1 + k^2 \cdot \mathbf{e}_2 + k^3 \cdot \mathbf{e}_3 + k^4 \cdot \mathbf{e}_4$

k^1	1	−1	1	−1	0	0	0	0	0	0	0	0	0	0	1	−1	1	−1
k^2	1	−1	0	0	1	−1	0	0	0	0	1	−1	1	−1	0	0	0	0
k^3	1	−1	0	0	0	0	1	−1	0	0	0	0	1	−1	0	0	1	−1
k^4	1	−1	0	0	0	0	0	0	1	−1	1	−1	0	0	1	−1	0	0

Then a normal vector of plane H (if $E_1 \neq E_1'$) will be:

$$\overrightarrow{E_1 E_1{}'} = (k^1 - 1) \cdot \mathbf{e}_1 + k^2 \cdot \mathbf{e}_2 + k^3 \cdot \mathbf{e}_3 + k^4 \cdot \mathbf{e}_4.$$

We determine the normal vector $\mathbf{n} = n^1 \cdot \mathbf{e}_1 + n^2 \cdot \mathbf{e}_2 + n^3 \cdot \mathbf{e}_3 + n^4 \cdot \mathbf{e}_4$, where $(n^1, n^2, n^3, n^4) = 1$.

The possible values for n^i are in Table 2.

Table 2. $\mathbf{n} = n^1 \cdot \mathbf{e}_1 + n^2 \cdot \mathbf{e}_2 + n^3 \cdot \mathbf{e}_3 + n^4 \cdot \mathbf{e}_4$

n^1	0	2	1	−1	1	−1	1	−1	1	−1	1	−1	1	0	2	0	2
n^2	1	1	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0
n^3	1	1	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1
n^4	1	1	0	0	0	0	0	1	1	1	1	0	0	1	1	0	0

We look for those coordinates n^i, for which vectors $\mathbf{e}_i'$ $(i = 1, 2, 3, 4)$ can be expressed as an integral linear combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ in the reflection formula (4) below:

$$\mathbf{e}_i' = \mathbf{e}_i - \frac{2 \cdot B(\mathbf{e}_i, \mathbf{n})}{B(\mathbf{n}, \mathbf{n})} \cdot \mathbf{n}. \tag{4}$$

Thus

$$\frac{2 \cdot B(\mathbf{e}_i, \mathbf{n})}{B(\mathbf{n}, \mathbf{n})} \cdot n^j \tag{5}$$

must be integer for all $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

By using the bilinear form (2) we shall see there exist four such $\mathbf{n}$ vectors (Table 3).

Table 3. $\mathbf{n} = n^1 \cdot \mathbf{e}_1 + n^2 \cdot \mathbf{e}_2 + n^3 \cdot \mathbf{e}_3 + n^4 \cdot \mathbf{e}_4$

n^1	0	−1	1	1
n^2	1	1	1	1
n^3	1	0	0	1
n^4	1	0	1	0

We still would have to examine those reflections which map E_1 onto itself.

Then we apply this method for each of the other points E_i $(i = 2, 3, 4)$. So we get two more possible normal vectors. The suitable coordinates are collected in Table 4.

Table 4. $\mathbf{n} = n^1 \cdot \mathbf{e}_1 + n^2 \cdot \mathbf{e}_2 + n^3 \cdot \mathbf{e}_3 + n^4 \cdot \mathbf{e}_4$

n^1	0	1	1	1	−1	0
n^2	1	0	1	1	1	0
n^3	1	1	0	1	0	−1
n^4	1	1	1	0	0	1

Finally we have obtained six reflections:

σ_1: the reflection plane is H_1, its normal vector is $\mathbf{n}_1 = (0, 1, 1, 1)$,

σ_2: the reflection plane is H_2, its normal vector is $\mathbf{n}_2 = (1, 0, 1, 1)$,

σ_3: the reflection plane is H_3, its normal vector is $\mathbf{n}_3 = (1, 1, 0, 1)$,

σ_4: the reflection plane is H_4, its normal vector is $\mathbf{n}_4 = (1, 1, 1, 0)$,

σ_5: the reflection plane is H_5, its normal vector is $\mathbf{n}_5 = (-1, 1, 0, 0)$,

σ_6: the reflection plane is H_6, its normal vector is $\mathbf{n}_6 = (0, 0, -1, 1)$.

At the reflection σ_1:

$$\begin{aligned}
\mathbf{e}_1' &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\\
\mathbf{e}_2' &= \mathbf{e}_2\\
\mathbf{e}_3' &= -\mathbf{e}_2 - \mathbf{e}_4\\
\mathbf{e}_4' &= -\mathbf{e}_2 - \mathbf{e}_3,
\end{aligned}$$

or by row-column multiplication:

$$\begin{pmatrix}\mathbf{e}_1'\\ \mathbf{e}_2'\\ \mathbf{e}_3'\\ \mathbf{e}_4'\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & -1 & 0 & -1\\ 0 & -1 & -1 & 0\end{pmatrix} \cdot \begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\\ \mathbf{e}_4\end{pmatrix}$$

by Coxeter's convention [5], or in transpose form:

$$\begin{pmatrix}\mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3' & \mathbf{e}_4'\end{pmatrix} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & -1 & -1\\ 1 & 0 & 0 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}$$

or simply:

$$\sigma_1\colon \begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & -1 & -1\\ 1 & 0 & 0 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}$$

as in tables of [4].

Similarly, in this latter matrix forms:

$$\sigma_2\colon \begin{pmatrix}1 & 1 & -1 & -1\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & -1 & 0\end{pmatrix}$$

$$\sigma_3\colon \begin{pmatrix}0 & -1 & 1 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\\ -1 & -1 & 1 & 1\end{pmatrix}$$

$$\sigma_4 : \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\sigma_5 : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_6 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is easy to verify that:

$$\sigma_5 = \sigma_2 \cdot \sigma_1 \cdot \sigma_2$$
$$\sigma_6 = \sigma_4 \cdot \sigma_3 \cdot \sigma_4$$
$$(\sigma_1 \cdot \sigma_2)^3 = (\sigma_3 \cdot \sigma_4)^3 = I.$$

(I denotes the identity map).

Thus the reflection subgroup, mapping O onto itself, can be generated by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. The group generated by σ_1 and σ_2 above is denoted by A_2 (see the notation in [1]) and the same holds for σ_3 and σ_4 as well. Therefore the maximal reflection subgroup of the point group of T can be denoted as a direct product $A_2 \times A_2$, since $H_1, H_2 \perp H_3, H_4$. The Coxeter graph of this group is shown in Figure 1.

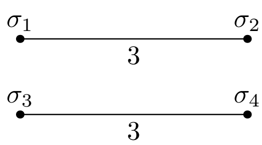

Figure 1. The Coxeter graph of the maximal reflection subgroup of the point group of T

4. The point group of T is not a reflection group

We determine the fundamental domain of $A_2 \times A_2$ above, that is $F_{A_2 \times A_2}$. This can be obtained as the intersection of suitable half spaces determined by planes H_1, H_2, H_3, H_4, respectively. The inequalities below determine the points $\mathbf{x} = (x^1, x^2, x^3, x^4)$ of the fundamental domain:

$$\begin{array}{lrl}
H_1^+ : & -x^1 + x^3 + x^4 \geq 0 & (B(\mathbf{x}, \mathbf{n}_1) \geq 0), \\
H_2^+ : & x^2 - x^3 - x^4 \geq 0 & (B(\mathbf{x}, -\mathbf{n}_2) \geq 0), \\
H_3^+ : & x^1 + x^2 - x^3 \geq 0 & (B(\mathbf{x}, \mathbf{n}_3) \geq 0), \\
H_4^+ : & -x^1 - x^2 + x^4 \geq 0 & (B(\mathbf{x}, -\mathbf{n}_4) \geq 0).
\end{array}$$

The intersection line of planes H_i, H_j, H_k will be denoted by f_{ijk}. The direction vectors of f_{123}, f_{124}, f_{134}, f_{234} are $(1,1,2,-1)$, $(1,1,-1,2)$, $(2,-1,1,1)$, $(-1,2,1,1)$, respectively. Using these we get the 2-dimensional faces of the fundamental domain. A_{ij} denotes the intersection of H_i and H_j (Figure 2).

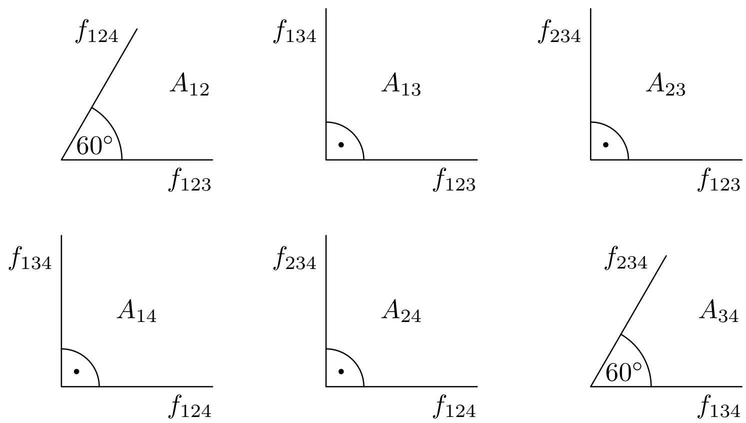

Figure 2. The 2-dimensional faces of the fundamental domain of the maximal reflection subgroup of the point group of T

We still have to detect whether the fundamental domain can be mapped onto itself by other symmetry operations or not. Of course, we look for those transformations which map the lattice onto itself as well.

Further on we give such transformations:

(1) Transformation f_1 maps the half space H_1^+ onto H_3^+, and H_2^+ onto H_4^+,

$$f_1\colon \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$f_1 \cdot f_1 = I$. f_1 is a reflection in a 2-dimensional plane.

(2) Transformation f_2 maps the half space H_1^+ onto H_4^+, and H_2^+ onto H_3^+,

$$f_2\colon \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$f_2 \cdot f_2 = I$. f_2 is a reflection in a 2-dimensional plane.

(3) Transformation $f_1 \cdot f_2 = f_2 \cdot f_1$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

By the Coxeter graph it would be possible to have a transformation which maps the half space H_1^+ into H_2^+ and the half spaces H_3^+ and H_4^+ are stable under this transformation. However, this would be a reflection in a three-dimensional plane bisecting the angle of H_1^+ and H_2^+. But we have already found all convenient reflections, it means that this reflection does not map the lattice onto itself. Similarly there is no such transformation which maps the half space H_3^+ into H_4^+ and leaves H_1^+ and H_2^+ invariant. After all there are no more such transformations.

Thus the elements of group $F = \{1, f_1, f_2, f_1 \cdot f_2\}$ in point group T map both the fundamental domain of the maximal reflection subgroup and the lattice onto itself. Clearly, all elements of T can uniquely be obtained as a product $f \cdot d$, where $f \in F$ and $d \in A_2 \times A_2$. The reason of this is the following: Let the transformation $t \in T$ map the point P into P'. Let P be a point of the fundamental domain F_1 and P' in the fundamental domain F_2. By product of some reflections we can get F_1 from F_2. Then the image of P' is the point P_1. Thus there is a transformation $f \in F$ which maps the point P onto P_1. Hence the transformation t is the product of f and some reflections. F has 4 elements, $A_2 \times A_2$ has $6 \cdot 6 = 36$ elements, so the order of the point group T is $4 \cdot 36 = 144$.

5. The maximal reflection subgroup

We shall determine the maximal reflection subgroup in the whole symmetry group of the lattice $\mathbf{\Lambda}$ of T as follows.

The fundamental domain F^* of the reflection subgroup will be a polyhedron which is a subset of $F_{A_2 \times A_2}$. The group will be generated by reflections in the 3-dimensional walls of F^*. Therefore we get the fundamental domain F^* in the following steps:

- We shift the plane H_i by the vector $\frac{1}{2}\underline{n}_i$ or $-\frac{1}{2}\underline{n}_i$ $(i = 1, 2, 3, 4, 5, 6)$, the image of H_i will be denoted by $H_{i'}$. We choose the plane that intersects the corner domain $F_{A_2 \times A_2}$.
- We choose the half spaces above that contain the origin O.
- The intersection of these half spaces and $F_{A_2 \times A_2}$ will be determined.

The inequalities below determine the fundamental domain F^*:

$$
\begin{aligned}
H_1^+ &: \quad -x^1 + x^3 + x^4 \geq 0, \\
H_2^+ &: \quad x^2 - x^3 - x^4 \geq 0, \\
H_3^+ &: \quad x^1 + x^2 - x^3 \geq 0, \\
H_4^+ &: \quad -x^1 - x^2 + x^4 \geq 0, \\
H_{5'}^+ &: \quad x^1 - x^2 + 1 \geq 0, \\
H_{6'}^+ &: \quad x^3 - x^4 + 1 \geq 0.
\end{aligned}
$$

A 2-dimensional axonometric projection of the fundamental domain is shown in Figure 4. B_{ijkl} denotes the common point of the corresponding planes H_i, H_j, H_k, H_l. The maximal reflection subgroup in the whole symmetry group of the lattice $\mathbf{\Lambda}$ is generated by reflections σ_1, σ_2, σ_3, σ_4, $\sigma_{5'}$, $\sigma_{6'}$ in planes $H_1, H_2, H_3, H_4, H_{5'}$, $H_{6'}$, respectively. The Coxeter graph of this group is shown in Figure 3. It is the group denoted by $\tilde{A}_2 \times \tilde{A}_2$ (see the notation in [1]).

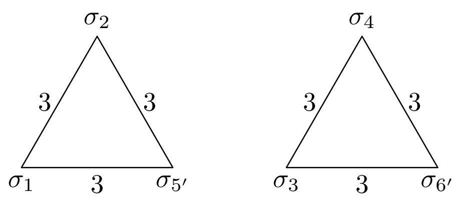

Figure 3. The Coxeter graph of the maximal reflection group of the whole symmetry group of the lattice $\mathbf{\Lambda}$

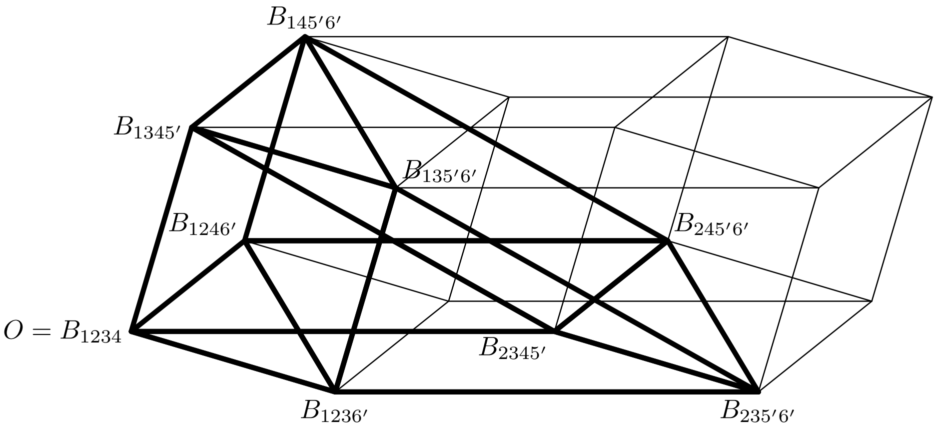

Figure 4. The 2-dimensional axonometric projection of the fundamental domain F^*

6. Results for other cases

We apply the method above for each quadratic form. Table 5 and Table 6 contain the data for all mentioned cases. The location in monograph [4] (see its tables) gives the crystal family, the crystal system, the Q-class, and the Z-class of the groups by their numberings.

Table 5. Results for the different cases I. (see [1, 4])

The group in Introduction	T	P_4	S_4
The maximal reflection group of point group	$A_2 \times A_2$	A_4	A_4
Is the group a pure reflection group?	NO	NO	NO
The location in the tables of monograph [4]	XXI.29/09/01	XXII.31/07/01	XXII.31/07/02
The reflection subgroup in the whole symmetry group	$\tilde{A}_2 \times \tilde{A}_2$	$\tilde{A}_4$	$\tilde{A}_4$

Table 6. Results for the different cases II. (see [1, 4])

The group in Introduction	B	C_4	Q_4
The maximal reflection group of point group	$G_2 \times G_2$	C_4	F_4
Is the group a pure reflection group?	NO	YES	YES
The location in the tables of monograph [4]	XXI.30/13/01	XXIII.32/21/01	XXIII.33/16/01
The reflection subgroup in the whole symmetry group	$\tilde{G}_2 \times \tilde{G}_2$	$\tilde{C}_4$	$\tilde{F}_4$

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