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tmcs@inf.unideb.hu
http://tmcs.math.klte.hu
Teaching Mathematics and Computer Science

Miscellaneous topics in finite geometry

In memory of Professor Dr. Ferenc Kárteszi (1907–1989)

Herbert Zeitler

Abstract. The article starts with a short introduction to finite (K, L)-geometry. Then a lot of counting propositions is given and proved. Finally the famous theorem of Miquel is investigated in classical and in finite geometry. At the end of the article there is a call to all readers: Don't forget (finite) geometry and don't forget the outstanding geometer Prof. Dr. F. Kárteszi!

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ZDM Subject Classification: G10.

In 2007 Prof. Dr. Ferenc Kárteszi would have been 100 years old. During his last years I had the opportunity to correspond with him. Doing so, I learned to appreciate his exceptional mathematical qualities and his natural didactical talent. All these motivated me to remember this outstanding mathematician.

Ferenc Kárteszi was born on 13 February 1907 at Cegléd. He studied at the University of Budapest and here he earned the degree "doctor of sciences". During the years 1931–1940 he was a teacher at a high school in Győr. Later he always kept strong connections with the school. So, for instance, he was a coworker of a journal for pupils. He succeeded the jump to university – a very exceptional event. In 1950 he took over a chair for geometry (projective and descriptive geometry) and didactics at the University of Budapest. This position he hold up to his retirement in 1977. His main mathematical subject was finite

Figure 1. Professor Dr. Ferenc Kárteszi

geometry. In a worldwide appreciated book he presented this special field in a remarkable way and proved with this book his extraordinary didactical talent. He also was working in other fields as classical geometry, combinatorics, descriptive geometry and graph theory. A lot of books and many different research papers were published. Some years he spent as guest professor at the University of Bologna and other Italian universities. After a life full of work he died on 9 May 1989 in Budapest.

The mathematical world had lost an enthusiastic and inspiring geometer.

1. What you should already know

1.1. Some algebra

Starting with a field K, a quadratic extension field L may be constructed by adjoining a suitable element.

A well-known example:

$K = \mathbb{R}$, field of real numbers; $f(x) = x^2 + 1$ polynomial irreducible in $\mathbb{R}$; $\mathbb{C} = \{x_1 + i\,x_2 \mid x_1, x_2 \in \mathbb{R}\}$ with $i^2 = -1$; the element i is adjoined; $\mathbb{C}$ is the field of complex numbers. Mostly we restrict ourselves in this paper to the finite case and start with a finite field (Galois field):

$$K = GF(q) = GF(p^e), \quad p \text{ prime}, p > 2, \quad e \in \mathbb{N}.$$

(The condition $p > 2$ is sometimes denoted by Char $K \neq 2$. It means $1 + 1 \neq 0$, where 1 is the neutral element of the multiplication.) There always exists a polynomial $f(x) = x^2 + b$ irreducible in K. Then it turns out that $L = \{x_1 + \varepsilon x_2 \mid x_1, x_2 \in K\}$ with $\varepsilon^2 = -b \notin (K^*)^2$ and $K^* = K \setminus \{0\}$ is a quadratic separable extension field $L = GF(q^2)$. The element ε is adjoined.

Definition 1. $\bar{X} = x_1 - \varepsilon x_2$ is the conjugate element of $X = x_1 + \varepsilon x_2$ and $N(X) = X\bar{X} = x_1^2 + b\,x_2^2$ is the norm of X.

1.2. Some geometry

In complete analogy to the geometry over $\mathbb{C}$ – more precisely over the pair $(\mathbb{R}, \mathbb{C})$ – we now develop a geometry over the pair (K, L), the so-called (K, L)-geometry – emphasizing always the finite case. We do not give any proofs. They can be found in the book [1, 2].

Points: $\mathfrak{P} = L \cup \{\infty\}$, $|\mathfrak{P}| = q^2 + 1$.

Lines: $\mathfrak{G} = \left\{X \in L \mid X\bar{M} + \bar{X}M + d = 0\right\} \cup \{\infty\}$ with $M = m_1 + \varepsilon m_2$, $X = x_1 + \varepsilon x_2$

$$= \left\{x_1, x_2 \in K \mid x_1 m_1 + b\,x_2 m_2 + \frac{1}{2}d = 0\right\} \cup \{\infty\}.$$

We have $M \in L^* = L \setminus \{0\}$, $d \in K$. Multiplication with $a \in K^*$ yields a new equation but the same line (equivalence property).

Elementary properties concerning lines:

Two distinct points $A, B \in L$ determine exactly one line $g(A, B)$; each line has exactly $q + 1$ points; number of lines $|\mathfrak{G}| = q(q + 1)$; $P \in L$ is an element of $g(A, B)$ with $A, B \in L$ if and only if $\frac{A-P}{B-P} \in K$; in any point $P \in L$ there exist $q + 1$ lines.

$$
\begin{aligned}
\textit{Circles:}\quad \mathfrak{K} &= \{X \in L \mid N(X-M) = c\} \\
&= \bigl\{X \in L \mid X\bar{X} - X\bar{M} - \bar{X}M + M\bar{M} = c\bigr\} \\
&\quad \text{with } M = m_1 + \varepsilon m_2,\ X = x_1 + \varepsilon x_2 \\
&= \bigl\{x_1, x_2 \in K \mid (x_1 - m_1)^2 + b\,(x_2 - m_2)^2 = c\bigr\}.
\end{aligned}
$$

We have $M \in L$, $c \in K^*$.

Elementary properties concerning circles:

Three distinct points $A, B, C \in L$ determine exactly one circle $k(A, B, C)$; each circle has exactly $q + 1$ points; number of circles $|\mathfrak{K}| = q^2(q-1)$; $P \in L$ is an element of $k(A, B, C)$ with $A, B, C \in L$ if and only if $\frac{A-P}{B-P} : \frac{A-C}{B-C} \in K$ and $\frac{A-C}{B-C} \notin K$; in any point $P \in L$ there exist $q^2 - 1$ circles.

Cycles: $\mathfrak{Z} = \mathfrak{G} \cup \mathfrak{K}$, $|\mathfrak{Z}| = q(q^2+1)$

Cycle preserving mappings

Homographies	Antihomographies
$\mathfrak{H}\qquad X' = \dfrac{SX+T}{UX+V}$	$\bar{\mathfrak{H}}\qquad X' = \dfrac{S\bar{X}+T}{U\bar{X}+V}$

We have $S, T, U, V \in L$ and $\det = SV - TU \neq 0$.

These mapping transform the set of all cycles onto itself. We have $|\mathfrak{H}| = |\bar{\mathfrak{H}}| = q^2(q^4-1)$. The reflections in lines and circles are antihomographies.

The (K, L)-geometries are called Möbius-planes or inversive planes [1].

1.3. An example

$K = GF(3)$, $f(x) = x^2 + 1$ irreducible polynomial, $L = \{x_1 + \varepsilon x_2 \mid x_1, x_2 \in K\}$ with $\varepsilon^2 = -1 = 2$. Points

$$
\begin{aligned}
\mathfrak{P} &= \{\,(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\,\} \cup \{\infty\} \\
&= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \cup \{10\}.
\end{aligned}
$$

In the first line all pairs (x_1, x_2) are notated and in the second line we have simply numerated one after the other.

Lines

equation	points
$x_1 = 0$	1, 2, 3, 10
$x_2 = 0$	1, 4, 7, 10
$x_1 + x_2 = 0$	1, 6, 8, 10
$2x_1 + x_2 = 0$	1, 5, 9, 10
$2x_2 + 1 = 0$	2, 5, 8, 10
$x_2 + 1 = 0$	3, 6, 9, 10
$2x_1 + 1 = 0$	4, 5, 6, 10
$2x_1 + 2x_2 + 1 = 0$	2, 4, 9, 10
$2x_1 + x_2 + 1 = 0$	3, 4, 8, 10
$x_1 + 1 = 0$	7, 8, 9, 10
$x_1 + 2x_2 + 1 = 0$	2, 6, 7, 10
$x_1 + x_2 + 1 = 0$	3, 5, 7, 10

$|\mathfrak{G}| = q(q+1) = 12.$

Circles

$c = 1$

m_1	m_2	equation	points
0	0	$x_1^2 + x_2^2 + 2 = 0$	2, 3, 4, 7
0	1	$x_1^2 + x_2^2 + x_2 = 0$	1, 3, 5, 8
0	2	$x_1^2 + x_2^2 + 2x_2 = 0$	1, 2, 6, 9
1	0	$x_1^2 + x_2^2 + x_1 = 0$	1, 5, 6, 7
1	1	$x_1^2 + x_2^2 + x_1 + x_2 + 1 = 0$	2, 4, 6, 8
1	2	$x_1^2 + x_2^2 + x_1 + 2x_2 + 1 = 0$	3, 4, 5, 9
2	0	$x_1^2 + x_2^2 + 2x_1 = 0$	1, 4, 8, 9
2	1	$x_1^2 + x_2^2 + 2x_1 + x_2 + 1 = 0$	2, 5, 7, 9
2	2	$x_1^2 + x_2^2 + 2x_1 + 2x_2 + 1 = 0$	3, 6, 7, 8

$c = 2$

m_1	m_2	equation	points
0	0	$x_1^2 + x_2^2 + 1 = 0$	5, 6, 8, 9
0	1	$x_1^2 + x_2^2 + x_2 + 2 = 0$	4, 6, 7, 9
0	2	$x_1^2 + x_2^2 + 2x_2 + 2 = 0$	4, 5, 7, 8
1	0	$x_1^2 + x_2^2 + x_1 + 2 = 0$	2, 3, 8, 9
1	1	$x_1^2 + x_2^2 + x_1 + x_2 = 0$	1, 3, 7, 9
1	2	$x_1^2 + x_2^2 + x_1 + 2x_2 = 0$	1, 2, 7, 8
2	0	$x_1^2 + x_2^2 + 2x_1 + 2 = 0$	2, 3, 5, 6
2	1	$x_1^2 + x_2^2 + 2x_1 + x_2 = 0$	1, 3, 4, 6
2	2	$x_1^2 + x_2^2 + 2x_1 + 2x_2 = 0$	1, 2, 4, 5

$|\mathfrak{K}| = q^2(q-1) = 18.$

2. Lines and points with respect to a circle

Now we prove some elementary theorems and give definitions in our finite geometry. Mostly we have only to count.

THEOREM 1. *Let k be a circle and g a line, then we have $|k \cap g| \in \{0, 1, 2\}$.*

PROOF. We have to do some calculations.
Without loss of generality (performing a suitable homography) we start with

$$k\colon x_1^2 + bx_2^2 = 1, \quad g\colon x_1 m_1 + bx_2 m_2 + \frac{1}{2}d = 0.$$

Because of the equivalence property we can choose $m_1 = 1$ (or $m_2 = 1$). Substitution yields

$$\left(bx_2 m_2 + \frac{1}{2}d\right)^2 + bx_2^2 = 1$$

and further

$$\begin{aligned} x_2 &= \frac{1}{2(b^2 m_2^2 + b)} \left[-bdm_2 \pm \sqrt{4b^2 m_2^2 + 4b - d^2 b}\right] \\ &= \frac{1}{2(b^2 m_2^2 + b)} \left[-bdm_2 \pm \sqrt{D}\right]. \end{aligned}$$

With $2(b^2 m_2^2 + b) \neq 0$ we distinguish three cases

$$D \begin{cases} \in (K^*)^2 & \text{two intersection points,} \\ = 0 & \text{one intersection point,} \\ \notin (K^*)^2 & \text{no intersection point.} \end{cases}$$

□

This result leads to the following definition.

DEFINITION 2. With respect to a given circle k a line g is denoted by

$$\left.\begin{array}{l} \textit{secant} \text{ (intersecting line)} \\ \textit{tangent} \text{ (touching line)} \\ \textit{passant} \text{ (avoiding line)} \end{array}\right\} \text{if and only if } |k \cap g| = \begin{cases} 2 \\ 1 \\ 0. \end{cases}$$

THEOREM 2. *At each point of a circle k there exists exactly one tangent.*

Proof. We have only to count.

Let P be a point of the circle k. Connecting P with all the other points of k we obtain exactly q secants. Since there exist totally $q+1$ lines in P there remains exactly one. Due to Definition 2 this is a tangent. $\square$

Theorem 3. *Concerning the number of lines in different classes in respect of a circle k we have*

- *tangents:* $q+1$,
- *secant:* $\frac{1}{2}q(q+1)$,
- *passants:* $\frac{1}{2}(q+1)(q-2)$.

Proof. The number of tangents follows immediately from Theorem 2 and Subsection 1.2. This circle k contains $q+1$ points and therefore $\binom{q+1}{2}$ pairs of points. So the number of secants is $\frac{1}{2}q(q+1)$. Due to Subsection 1.2 there exist $q(q+1)$ lines. Then there remain

$$q(q+1)-(q+1)-\frac{1}{2}q(q+1)=\frac{1}{2}(q+1)(q-2)$$

passants. $\square$

2.1. Mixed items, concerning parallels

Definition 3. Two lines g, h are parallel if and only if they are either equal or they have exactly the point ∞ in common. As usual we write:

$$g \parallel h \Longleftrightarrow g=h \text{ or } g\cap h=\{\infty\}.$$

The set of all lines parallel to a line g is denoted as pencil of parallels.

Theorem 4. *Let g be a line and P a point $P\notin g$. Then in P there exists exactly one line h parallel to g.*

Proof. Connecting P with any point $X\in g$, $X\neq\infty$ we get exactly q lines. As we know (Subsection 1.2) there are exactly $q+1$ lines in P. So one line h remains with $g\cap h=\{\infty\}$. $\square$

Remark 1. The set of all points different from ∞, together with the set of all lines forms the so-called "derived (in ∞)" plane. Because of Theorem 4 and some other qualities mentioned in [1] it turns out that this is an affine plane.

Theorem 5. *In each pencil of lines we have exactly q lines and totally we get $q+1$ such pencils.*

Proof. Without loss of generality we start with the line $g\colon x_1 = 0$, the x_2-axes. Because of Theorem 4 the set of lines $x_1 = c$ with $c \in K$ forms the corresponding pencil of lines. With $|K| = q$ the first part of our theorem is already proved. The second part follows immediately from the fact $|\mathfrak{G}| = q(q+1)$. □

Theorem 6. *Let t_1 be a tangent to the circle k. Then there exists exactly one other tangent t_2, with $t_1 \neq t_2$, $t_1 \parallel t_2$. The pencil determined by t_1 and t_2 contains exactly $\frac{1}{2}(q-1)$ secants and $\frac{1}{2}(q-3)$ passants.*

Proof. We are working with a very special configuration. This means no loss of generality – always the same trick.

Pencil of parallels: $x_1 = c$ with $c \in K$.

Circle k: $x_1^2 + bx_2^2 = 1$.

Substituting we get $x_2^2 = \frac{1-c^2}{b}$.

We are looking for tangents, for touching points. Therefore we get $1-c^2 = 0$, $c = \pm 1$.

So there exist exactly two parallel tangents t_1, t_2: $x_1 = \pm 1$ and two touching points $A(1,0)$, $B(-1,0)$.

All the other lines of the pencil are either secants or passants with respect of the circle k. Besides A, B the circle contains $q-1$ points and that's why we have exactly $\frac{1}{2}(q-1)$ secants in the pencil. Subtraction finally yields the number of passants in the pencil:

$$q - \frac{1}{2}(q-1) - 2 = \frac{1}{2}(q-3).$$

□

Definition 4. With respect to a given circle k we divide the set of points in three classes as follows:

- *on-points:* all points on k,
- *ex-points:* all points on tangents without the corresponding touching points,
- *in-points:* all the remaining points.

Theorem 7. *If X is an ex-point of a circle k, then in X there exist exactly two tangents of k, through X.*

PROOF. Let t be a tangent to k, P the corresponding touching point and $X \in t$, $X \neq P$ (Figure 2).

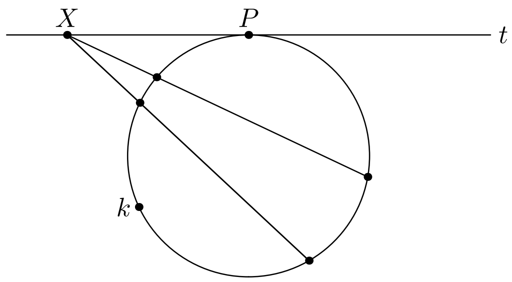

Figure 2. Tangents from ex-points

Besides P we have q points on k. It cannot happen that all the connecting lines of these points with X are secants, because q is an odd number. Therefore in X exist *at least* two tangents to k. This holds for all points $X \in t$, $X \neq P$. Together with t in this way we have *at least* $q+1$ tangents. Due to Theorem 3 there exist exactly $q+1$ tangents. That's why in each point $X \in t$ we have exactly two tangents.

This proof is working for all ex-points of k.

(Naturally Figure 2 and also Figures 5, 6, 7 are only visualizations of algebraic facts.) □

THEOREM 8. *Concerning the number of points in different classes with respect to a circle k we have*

- *on-points:* $q+1$,
- *ex-points:* $\frac{1}{2}(q^2+1)$,
- *in-points:* $\frac{1}{2}(q^2-2q-1)$.

PROOF. According to Definition 4 the number of on-points is immediately to see.

The ex-point ∞ is a very special point. It contains more than two tangents with respect to k. Therefore the point ∞ is at first deleted.

Besides ∞ and the corresponding touching point each tangent possesses exactly $q-1$ ex-points. There exist $q+1$ tangents and in respect of Theorem 7 each ex-point is counted twice. For the number of ex-points we get $\frac{1}{2}(q+1)(q-1) = \frac{1}{2}(q^2-1)$. Finally we must add the point ∞ again and it follows $\frac{1}{2}(q^2-1)+1=\frac{1}{2}(q^2+1)$. Totally there exist q^2+1 points. With this, subtraction yields the number of in-points

$$(q^2+1)-(q+1)-\frac{1}{2}(q^2+1)=\frac{1}{2}(q^2-2q-1).$$

□

THEOREM 9. *Concerning the number of lines of different classes through points of different classes – all with respect to a circle k – we have*

points	*tangents T*	*secants S*	*passants P*
on	1	q	0
ex $\neq\infty$	2	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
in	0	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$

PROOF. In the case of on-points everything is clear.

One ex-point $\neq\infty$ contains exactly two tangents (Theorem 7) and therefore $\frac{1}{2}(q-1)$ secants. By subtraction this yields $(q+1)-\frac{1}{2}(q-1)-2=\frac{1}{2}(q-1)$ passants.

An in-point never lies on a tangent. That's why we obtain $\frac{1}{2}(q+1)$ secants and as many passants.

If we consider the ex-point ∞, then we encounter quite another, a very strange situation. Each pair (Theorem 6) of parallel tangents determines a pencil of parallel lines. In this way we get $\frac{1}{2}(q+1)$ pencils with two tangents and $\frac{1}{2}(q-1)$ secants each. The remaining $q-\frac{1}{2}(q-1)-2=\frac{1}{2}(q-3)$ lines in each pencil are passants. □

THEOREM 10. *Concerning the number of points of different classes on lines of different classes – all with respect to a circle k – we have*

Lines	*on*	*ex*	*in*
Tangents T	1	q	0
Secants S			
S_1	2	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q-3)$
S_2	2	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
Passants P			
P_1	0	$\frac{1}{2}(q+3)$	$\frac{1}{2}(q-1)$
P_2	0	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)$

PROOF.

T *Tangents:* This case is clear.

P *Passants p:* We distinguish two cases.

P_1 *No tangent is parallel to p*

This means that any tangent intersects the passants p in a point $\neq \infty$. Let w be the number of ex-points on p different from ∞. Then it follows $w = \frac{1}{2}(q+1)$. Point ∞ is also an ex-point on p. Then it turns out that the total number x of ex-points lying on p is $x = w + 1 = \frac{1}{2}(q+1) + 1 = \frac{1}{2}(q+3)$. Subtracting finally yields the number y of in-points on p

$$y = (q+1) - x = (q+1) - \frac{1}{2}(q+3) = \frac{1}{2}(q-1).$$

The notations x, y, w are used in the same meaning in all the following proofs of Section 2.

P_2 *One tangent is parallel to p*

Due to Theorem 6 in this case we have even a second tangent parallel to p. All other tangents intersect p in an ex-point $\neq \infty$.
So it follows $w = \frac{1}{2}(q-1)$ and $x = w + 1 = \frac{1}{2}(q+1)$.
Subtracting yields $y = (q+1) - \frac{1}{2}(q+1) = \frac{1}{2}(q+1)$.

S *Secants s with $s \cap k = \{A, B\}$*

S_1 *No tangent is parallel to s*

Every tangent intersects s in a point. Considering the special situation of the tangents in A and B it turns out, that $w = \frac{1}{2}(q-1)$ and $x = w + 1 = \frac{1}{2}(q+1)$. A and B are on-points. Subtraction yields $y = (q+1) - \frac{1}{2}(q+1) - 2 = \frac{1}{2}(q-3)$.

S_2 *One tangent is parallel to s*

We then have two tangents parallel to s. They intersect s in ∞. Using the same ideas as in the last cases we obtain $w = \frac{1}{2}(q-3)$, $x = w + 1 = \frac{1}{2}(q-1)$. We take into account, that A, B are on-points. Then subtraction yields

$$y = (q+1) - \frac{1}{2}(q-1) - 2 = \frac{1}{2}(q-1).$$

□

Naturally we now are interested how often the different special cases in Theorem 10 really occur.

Theorem 11. *Concerning the number of lines within the four classes S_1, S_2, P_1, P_2 we have*

$S_1:$	$\frac{1}{4}(q+1)^2$	$P_1:$	$\frac{1}{4}(q^2-1)$
$S_2:$	$\frac{1}{4}(q^2-1)$	$P_2:$	$\frac{1}{4}(q+1)(q-3)$
$S:$	$\frac{1}{2}q(q+1)$	$P:$	$\frac{1}{2}(q^2-q-2)$

Proof.

S *Secants* s

S_2 The two tangents and s determine (Theorem 6) a pencil with exactly $\frac{1}{2}(q-1)$ secants. Each of these secants is usable and leads to the case S_2. We have $\frac{1}{2}(q+1)$ pairs of parallel tangents and therefore the case S_2 occurs $\frac{1}{2}(q-1)\cdot\frac{1}{2}(q+1)=\frac{1}{4}(q^2-1)$ times.

S_1 Due to Theorem 3 there exist $\frac{1}{2}q(q+1)$ secants. We take away all the secants already used in the last case S_2. There remain $\frac{1}{2}q(q+1)-\frac{1}{4}(q^2-1)=\frac{1}{4}(q+1)^2$ secants, leading straightaway to S_1.

P *Passants* p*:* The proof is running exactly as with S.

P_2 The two tangents and p determine a pencil (Theorem 6) with exactly $\frac{1}{2}(q-3)$ passants. Each of these passants is usable and leads to case P_2. We have $\frac{1}{2}(q+1)$ pairs of parallel tangents and therefore the case P_2 occurs exactly $\frac{1}{2}(q-3)\cdot\frac{1}{2}(q+1)=\frac{1}{4}(q^2-2q-3)$ times.

P_1 Due to Theorem 3 there exist $\frac{1}{2}(q+1)(q-2)$ passants. We take away all the passants already used in the last case P_2. There remain $\frac{1}{2}(q+1)(q-2)-\frac{1}{4}(q+1)(q-3)=\frac{1}{4}(q^2-1)$.

□

Now it would be interesting to verify all our results using the special (K,L)-geometry with $q=3$ described in Subsection 1.3. For instance you could start with the circle $(2,3,4,7)$ and find out the corresponding special lines and points. With this it is possible to check all our counting results. We leave this as an exercise to the reader.

Counting, counting! For students this was boring. They did the work sometimes in a reluctant way. Therefore: Enough with counting! Let's go to geometry!

3. To play around with Miquel's theorem

3.1. The theorem in elementary geometry

In older books about classical elementary geometry you can find as a "fossil" the theorem of Miquel. Meanwhile this theorem has got fundamental importance within the foundations of geometry.

We start with a "basic configuration" which is easy to construct. It consists of four circles with eight intersection points as shown in Figure 3. The circles: (P,Q,A,B), (P,S,A,D), (Q,R,B,C), (R,S,C,D).

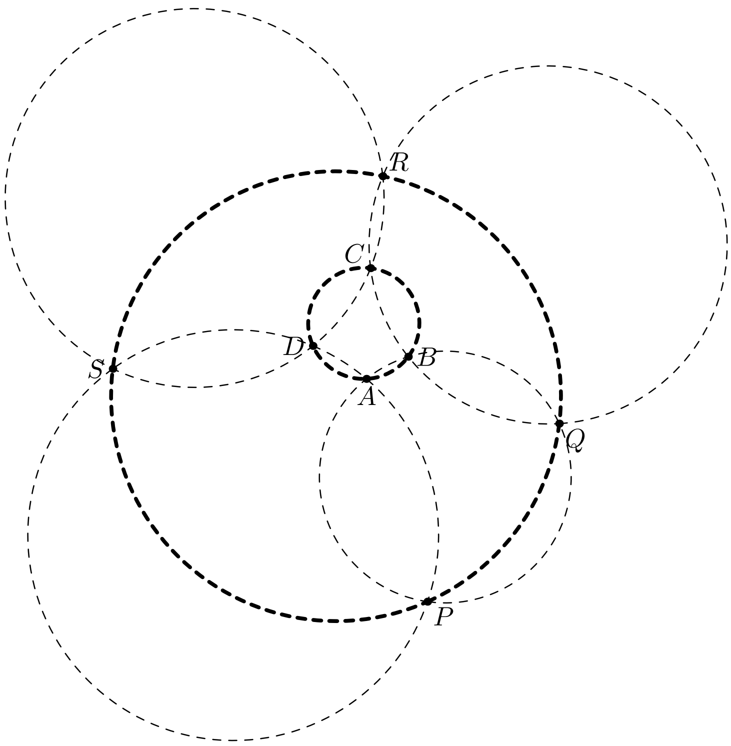

Figure 3. Theorem of Miquel

Theorem of Miquel. *If in the basic configuration the points P, Q, R, S are on a circle then the points A, B, C, D have the same property.*

Proof. Proving this theorem we use a well-known trick and perform a reflection in a circle with center P (inversion). Then the circles (P, Q, A, B), (P, S, A, D), (P, Q, R, S) are mapped into straight lines through the picture A' of A and the circles (Q, R, B, C), (R, S, C, D) again into circles. In this way we get a figure, often called as Pivot-configuration (Figure 4).

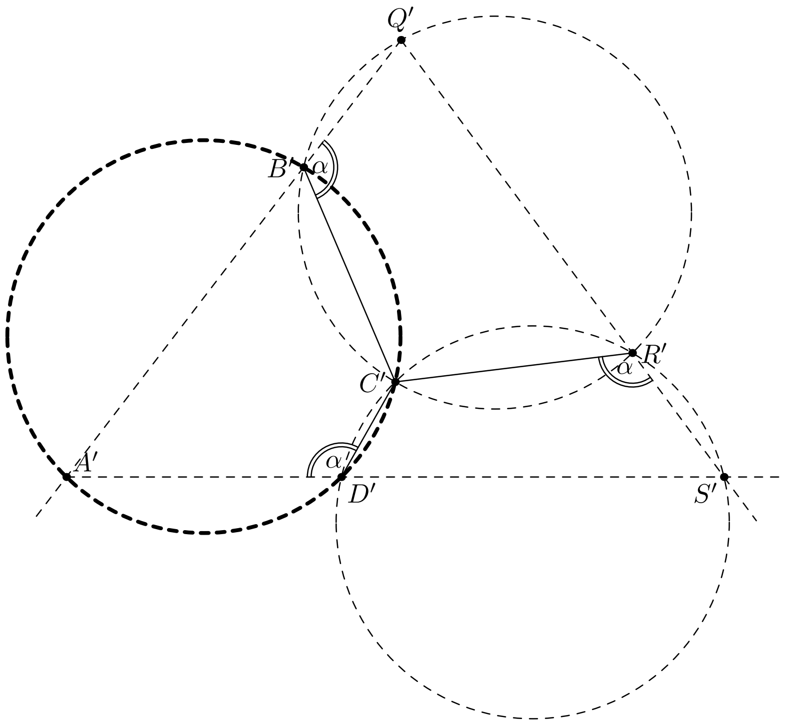

Figure 4. The Pivot-configuration

We prove our theorem within this new configuration. It is sufficient to show that the points A', B', C', D' are elements of a circle.

We need a theorem from school geometry. It consists of two parts.

(a) If a quadrangle is inscribed in a circle then the measures of its opposite angles add up to 180°.

Vice versa we have:

(b) If the measures of the opposite angles of a quadrangle add up to 180°, then its vertices are lying on a circle.

Sometimes the notion "chord-quadrangle" is used.

Now let us have a look in Figure 4. With $\sphericalangle(A'D'C') = \alpha$ we take from the chord-quadrangle $(D'S'R'C')$ the supplementary angle $\sphericalangle(C'D'S') = 180^\circ - \alpha$ and further with (a) $\sphericalangle(S'R'C') = \alpha$. In the same way the chord-quadrangle $(C'R'Q'B')$ yields $\sphericalangle(C'R'Q') = 180^\circ - \alpha$, $\sphericalangle(Q'B'C') = \alpha$ and finally $\sphericalangle(C'B'A') = 180^\circ - \alpha$. Because of (b) then $(A'D'C'B')$ is a chord-quadrangle – the four points A', B', C', D' are on a circle.

Now the complete Pivot-configuration is reflected backwards – and with this, the proof of Miquel's theorem is done. □

We explain once more the whole procedure:

The trick is to find a much simpler constellation using a suitable transformation (here a reflection in a circle). Then the new problem is solved. Finally all is – full of remorse – transformed backwards. Naturally, all the properties of the transformation must be known.

The operation is to compare with the translation from one language to another one.

Some remarks:

(1) Who was Miquel and who was Pivot? Where they are coming from? No people we asked had an answer.

(2) Naturally a lot of degenerate cases (coinciding points, point C' in Figure 4 is situated outside the triangle $(A'S'Q')$, ...) must be investigated. Using the notion of "oriented angle" a proof can be given, which includes all exceptions.

(3) In the literature of 19. and 20. century we find a lot of theorems – especially in the geometry of triangles – strongly connected with the Miquel theorem.

3.2. The theorem in (K, L)-geometry

The field K and the quadratic extension field L must not be finite in this section. We only require Char $K \neq 2$. We start immediately with the Pivot-configuration. This means that the existence of a "basic configuration" is assumed, we take it already for constructed. Because of writing technics we use – as you can see in Figure 5 – other notions for the points. With a translation (homography) we bring one point into the origin 0.

The classical theorems concerning the "chord-quadrangle" do not work here. So we use theorems from (K, L)-geometry already mentioned in Section 1. Here they are once more:

$P \in L$ is an element of a line $g(A, B)$ with $A, B \in L$ if and only if $\frac{A-P}{B-P} \in K$.

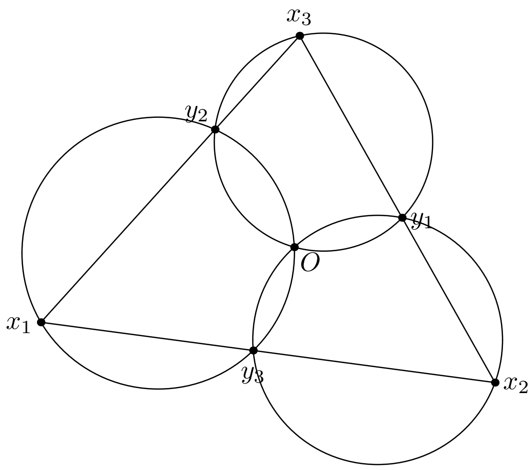

Figure 5. Again the Pivot-configuration

$P \in L$ is an element of a circle $k(A, B, C)$ with $A, B, C \in K$ if and only if $\frac{A-P}{B-P} : \frac{A-C}{B-C} \in K$ and $\frac{A-C}{B-C} \notin K$. From Figure 5 we take out the corresponding ratios respectively cross-ratios. Here they are:

Points on lines	Points on circles
$m_1 = \dfrac{y_3 - x_1}{y_3 - x_2}$	$n_1 = \dfrac{x_2 - y_3}{x_2 - y_1} : \dfrac{y_3 - 0}{y_1 - 0} = \dfrac{y_3 - x_2}{y_1 - x_2} \cdot \dfrac{y_1}{y_3}$
$m_2 = \dfrac{y_1 - x_2}{y_1 - x_3}$	$n_2 = \dfrac{x_3 - y_1}{x_3 - y_2} : \dfrac{y_1 - 0}{y_2 - 0} = \dfrac{y_1 - x_3}{y_2 - x_3} \cdot \dfrac{y_2}{y_1}$
$m_3 = \dfrac{y_2 - x_3}{y_2 - x_1}$	$n_3 = \dfrac{x_1 - y_2}{x_1 - y_3} : \dfrac{y_2 - 0}{y_3 - 0} = \dfrac{y_2 - x_1}{y_3 - x_1} \cdot \dfrac{y_3}{y_2}$

What exactly we have to show?

Using the two theorems cited above it is enough to prove the following: If five ratios are elements of K, then the sixth must have this property, too.

We multiply all six ratios.

$$n_1 \cdot n_2 \cdot n_3 \cdot m_1 \cdot m_2 \cdot m_3 =$$
$$= \frac{y_3 - x_1}{y_3 - x_2} \cdot \frac{y_1 - x_2}{y_1 - x_3} \cdot \frac{y_2 - x_3}{y_2 - x_1} \cdot \frac{y_3 - x_2}{y_1 - x_2} \cdot \frac{y_1}{y_3} \cdot \frac{y_1 - x_3}{y_2 - x_3} \cdot \frac{y_2}{y_1} \cdot \frac{y_3 - x_1}{y_3 - x_1} \cdot \frac{y_3}{y_2} = 1.$$

This result is very surprising and includes the proof of Miquel's theorem within the Pivot-configuration.

Transforming back as in Subsection 3.1 the general proof is also done.

3.3. The theorem in finite (K, L)-geometry

$K = GF(q) = GF(p^e)$, p prime, $p > 2$, $e \in \mathbb{N}$. In elementary geometry (Subsection 3.1) the existence of basic configurations can be shown by construction.

For (K, L)-geometries (Subsection 3.2) we had assumed such an existence in a very audacious way. With this requirement the theorem of Miquel then was proved.

In the case of finite qeometries we investigate the existence of basic configurations more precisely.

Theorem 12. *In finite geometries with $q > 3$ basic configurations exist.*

Proof. We go back to Section 2.

Given a circle k and one ex-point A. Due to 9 we have exactly two tangents in A and exactly $\frac{1}{2}(q-1)$ secants (Figure 6). In case $q = 3$ only one secant exists.

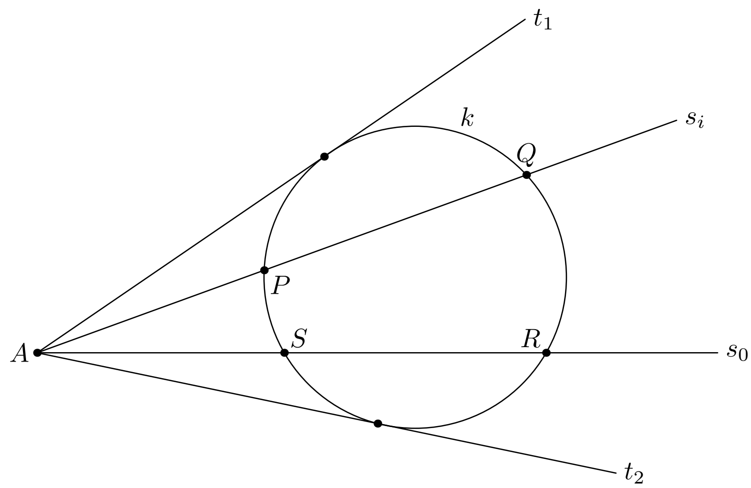

Figure 6. Lines in an ex-point

Now we perform a reflection σ_{s_0} (antihomographie) in a secant s_0. Using the properties of line reflections in (K, L)-geometry we obtain (with the notions in Figure 7) $\sigma_{s_0}(k) = k'$, $\sigma_{s_0}(s_i) = s_i'$, s_0 is a fixed point-line, $\sigma_{s_0}(P) = P'$, $\sigma_{s_0}(Q) = Q'$.

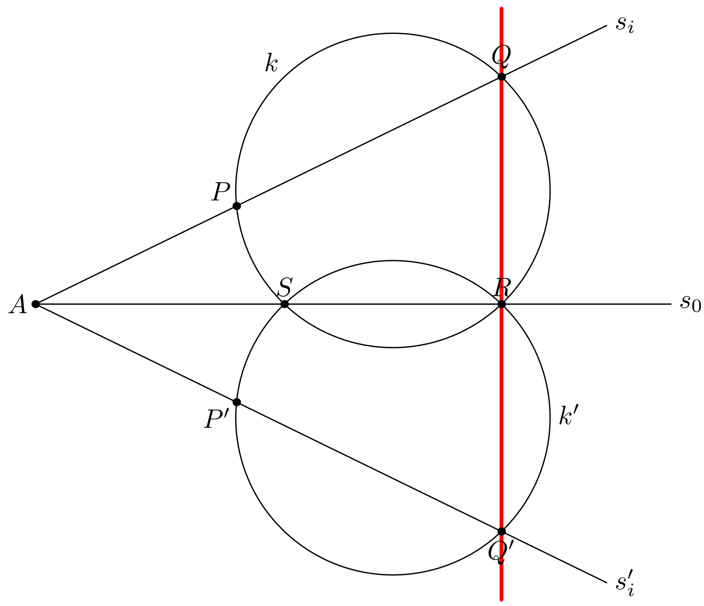

Figure 7. Reflection in a secant

The lines s_i, s_i' together with the circles k, k' form a base configuration within the Pivot-figure.

Together with the proof in Subsection 3.2 the theorem of Miquel is proved in our finite geometry for all $q > 3$. □

In case $q = 3$ our construction does not work, because we have only one secant in A. There is a strong suspicion that then no base configuration exists. Indeed, we have the following theorem.

THEOREM 13. *In case $q = 3$ basic configurations can't exist.*

Therefore, the theorem of Miquel cannot hold. This is easy to understand. Because the geometry with $q = 3$ is too poor – there exist not enough points.

PROOF. It is advantageous to have a nice visualization for our basic configurations and the theorem of Miquel. The cube model seems to be the best.

The six circles of Miquel's theorem correspond the circumcircles of the six faces of a cube.

The opposite faces of this cube – for instance $(PQBA)$, $(SRCD)$ are parallel. They have no point in common, but they include all the eight vertices of the cube (Figure 8).

Now we like to complete these two corresponding circumcircles by two new ones. This should be performed such that a basic configuration develops. The missing circles must be embedded in parallel faces of the cube and they must contain all eight vertices of the cube. In our example we have $(QRCB)$, $(PSDA)$.

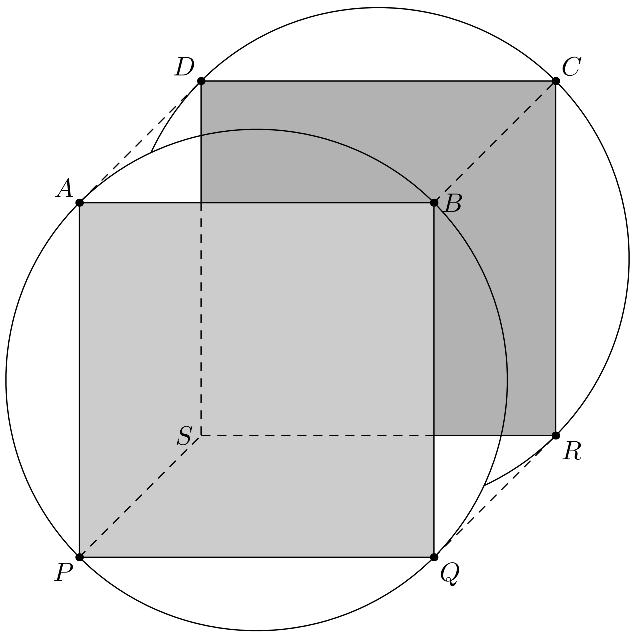

Figure 8. Cube model: basic configuration

Now we are going back to the (K, L)-geometry with $q = 3$, represented in the tables 1.3. Doing so, we have the cube model always at the back of our mind.

In a first step we sort out all pairs of avoiding circles. Transfering to the model this means to consider parallel faces of the cube. We discover, that besides ∞ for any such pair of circles remains a rest-point, not element of the concerning circles in the pair.

Pairs of circles	rest-points
$(2,3,4,7)$–$(5,6,8,9)$	1
$(1,3,5,8)$–$(4,6,7,9)$	2
$(1,2,6,9)$–$(4,5,7,8)$	3
$(1,5,6,7)$–$(2,3,8,9)$	4
$(2,4,6,8)$–$(1,3,7,9)$	5
$(3,4,5,9)$–$(1,2,7,8)$	6
$(1,4,8,9)$–$(2,3,5,6)$	7
$(2,5,7,9)$–$(1,3,4,6)$	8
$(3,6,7,8)$–$(1,2,4,5)$	9

We are coming to the next step. Given an avoiding pair – for instance (2347), (5689) with rest-point 1. This means that in the model we have the cube vertices 2, 3, 4, 5, 6, 7, 8, 9. Now we must find another pair of avoiding circles with exactly the same points. Parallel faces with the same vertices – therefore without the point 1. A look in the table shows that this is not possible. Each other pair contains 1.

This idea holds for all pairs of avoiding circles. □

4. Abstraction and application, an outlook

4.1. The road of abstraction

Geometry has developed step by step from visual, intuitive geometry to analytic geometry over the real or complex numbers and then to (K, L)-geometry. A certain summit of abstraction was reached with finite geometries.

A similar development is going on in painting. Starting with realism the road leads up to modern painting where only abstract structures and marks of colours are important.

4.2. The road of application

Students believe that finite geometry is a mere game with glass beads, played only in the brains of some elitist mathematicians. They would greatly be surprised if they learned that there exist applications in our real world. Here we restrict ourselves to name some fields of such applications. We give only catchwords: Cosmology, theory of elementary particles, creation of molecule clusters, geometry within the atomic nucleus, statistics, coding theory, ...

All these fields can no longer work without finite geometry.

Let me finish this paper with an appeal:

Do not forget geometry, especially finite geometry (as well in school as in university too)*!*

Do not forget the important work of Prof. Dr. Ferenc Kárteszi!

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HERBERT ZEITLER
MATHEMATISCHES INSTITUT
UNIVERSITÄT
D–95440 BAYREUTH
GERMANY
AND
AHORNWEG 5
D–95643 TIRSCHENREUTH
GERMANY

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