4/1 (2006), 111–130
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Teaching Mathematics and Computer Science

A mathematical and didactical analysis of the concept of orientation

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Abstract. The development of spatial ability, in particular the development of spatial orientation is one of the aims of mathematics education.

In my work, I examine the concept of orientation, especially concepts of between, left, right, below, above, front, back, clockwise and anticlockwise. I analyze answers given for a simple orientation task prepared for elementary school pupils. I would like to call attention to the difficulties pupils have even in case of solving simple orientation problems.

We have different ways to know more about the crucial points of a concept, especially of the concept of orientation. In this study I bring out one of them. I analyze and make some didactical conclusions about the origin and the axiomatic structure of orientation.

Key words and phrases: spatial ability, orientation, topographical context, linear and cyclic order.

ZDM Subject Classification: A32, D72, G12.

1. Introduction

The development of spatial ability, in particular the development of spatial orientation is one of the aims of mathematics education.

Spatial orientation describes the visualization of a spatial arrangement in which the observer is part of the situation. Students must be able to use concepts of position, direction and orientation to describe the physical world and to solve problems.

The problem of orientation often arises in everyday life (e.g. in transportation), too. The left-right problem illustrates the difficulties well. There are even adults, who in case of a stressful situation become uncertain of left-right relations. There are many areas of everyday life, and of different school subjects as well, where the questions of orientation appear. In addition to mathematics, science, art, physical education and technology uses the concept of orientation.

Furthermore the weak spatial orientation often leads to different learning disabilities, e.g. dyscalculia.

Studying recent textbooks and curricula in Hungary, and speaking to teachers in elementary school, we can realize that there are only a few tasks on this topic. Pupils only learn about the orientation at the beginning of the first school year, and nevermore. We can not observe the systematic treatment of this topic.

In our work we examine the concept of *orientation*, especially concepts of *between*, *left*, *right*, *below*, *above*, *front*, *back*, *clockwise* and *anticlockwise*, which are sub concepts of *orientation* in the hierarchical system.

The aims of our article are:

- Calling the attention to the difficulties pupils have even in case of solving simple orientation problems.
- Giving a mathematical analysis of the concept of orientation.

We make use to the mathematical analysis of the axiomatic structure of the Euclidean geometry, and to the didactical analysis principally the theory of the didactical phenomenology elaborated by Hans Freudenthal.

2. A simple example to illustrate the difficulties by the uses of the concept of orientation

Should we deal with the problem of *orientation* in schools, or does it have a rich context in everyday life, for students to learn spontaneously and directly?

We want to illustrate a little part of the problem with a simple task.

We prepared a test for pupils connected to the term of *orientation.* The task that we analyze now was a part of an experiment with pupils of grade 1–4. We chose three different elementary schools in Debrecen, in Hungary. The first was the practice school of the teacher training college, the second was a school in a housing estate, and the third was a school in the city. The pupils in the practice school had very good abilities; they had been accepted to the school after a selection. We can say that average pupils attend the school mentioned secondly, and in the third school there are pupils whose abilities are average or below average, and whose social backgrounds are not optimal. With the composition of the pupils participating in my experiment, we tried to represent the real situation in the grades 1–4.

The test item, presented below was made for pupils' grade 1–2.

We asked preservice students to answer the question too, and make a short note about the possible difficulties of pupils solving this task. I was interested in how well they know the concepts mentioned above, and what their opinion is about the typical mistakes made by pupils. When they solved the tasks they had already completed one year general and one year didactical mathematic course.

What is the position of the bird in relation to the chair? Make connections between the pictures and the adequate words! (Figure 1)

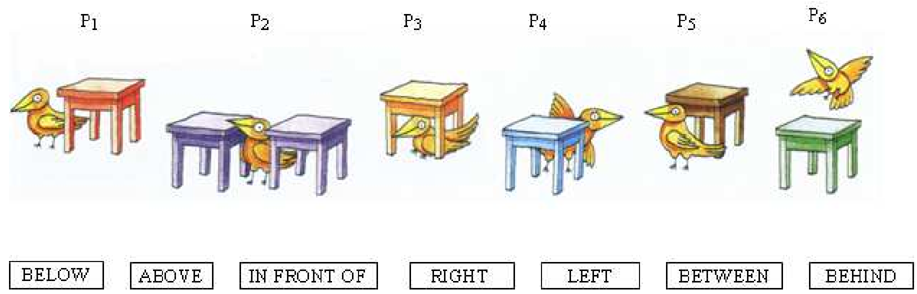

Figure 1

The pupils should be familiar with such kinds of tasks (either with real objects or only drawings on paper), because before entering the school they dealt with so called *words of relations* in the kindergarten and they met the requirements of the aptitude tests for the school. The task is an iconic representation of well-known situations, so the first step is for the pupil to imagine that based on their earlier experiences and only the second step is to analyze the situations. So this task is presumably more difficult then the task with real chairs and toys.

The result of the test:

The question was answered by 69 first and 79 second grader pupils with different abilities, and by 36 students who have completed 4 semesters.

Before analyzing the answers we must note, that there were 6 pictures and 7 words, so there were some pupils who had difficulties with this fact, and wanted all of the words to connect to the pictures. This may be one of the reasons why they connected more words to the same picture.

First take a look at the global results of the task (Figure 2).

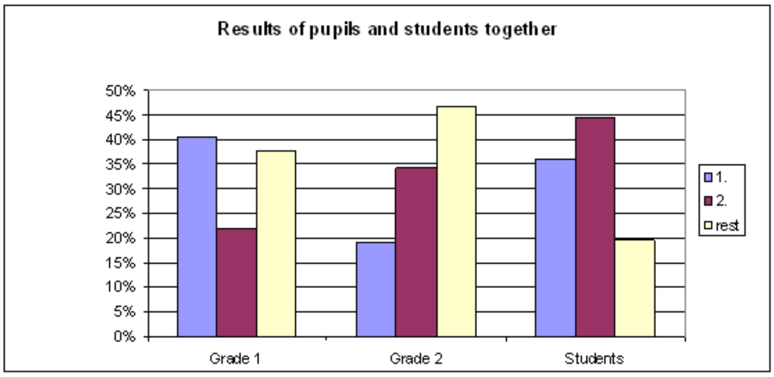

Figure 2

The two most frequent answers were, in order of pictures:

1. P_1: left, P_2: between, P_3: below, P_4: behind, P_5: in front of, P_6: above
2. P_1: right, P_2: between, P_3: below, P_4: behind, P_5: in front of, P_6: above

We can nevertheless see that the ratio of other answers is quite high. It is decreased by students, but it is still high. We can not highlight any more concrete answers; they showed a very great deviation.

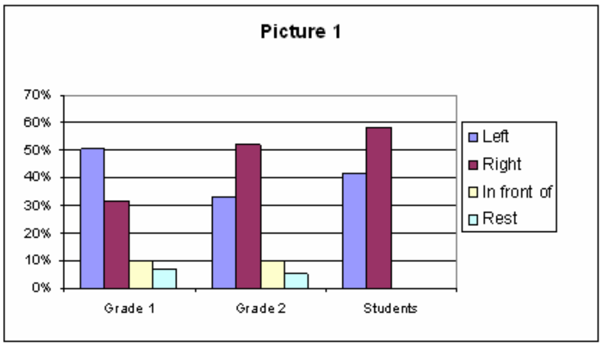

Figure 3

Neither the pupils nor the students had problems with **Picture 2**, **3** and **6**. More than 92% of the pupils and students gave the correct answer: *between* for **Picture 2**, *below* for **Picture 3** and *above* for **Picture 6**.

Picture 1, **Picture 4** and **Picture 5** were not so easy for them.

Let us see in more details **Picture 1** (Figure 3).

We can see that even in case of students there are no significant differences between answers *right* and *left*. The answer in front of is about 10% regarding pupils, and there are other answers too.

Picture 4 and **Picture 5** show us not so big differences, but we can see more hesitancy then in case of **Picture 3** and **6**. We can find just the same answers without differences regarding 1 and 2. grade pupils or students. The “wrong” answer was mainly *left* or *right* (Figure 4, Figure 5).

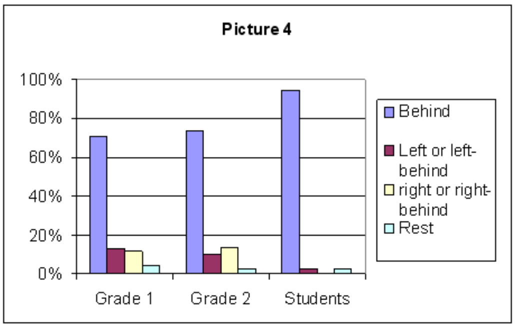

Figure 4

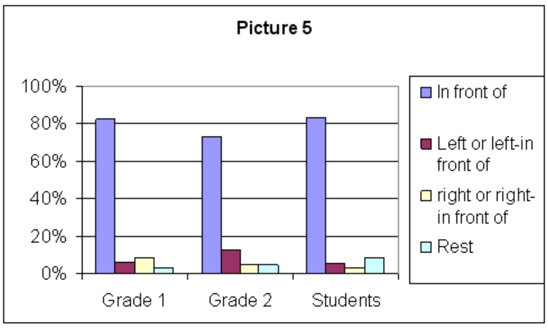

Figure 5

What is the reason that the task shows even between the students discrepancy?

First of all we must see the problem of relation. The polarities *above–below*, *in front of–behind*, *left–right* we use in a topographical context. The polarity *above–below* is clear thanks to gravitation. There are many objects (for example chairs) which posses an *above–below*, which keep its value independent of their situation in space. In addition, in our case the situation of the chair matches the space polarity. The relation *between* was without problem too. This relation is the basis of the other, so we can assume, that it becomes before the others in the individual developmental process. If a pupil did not understand the meaning of the word *between*, he or she can not understand for example the meaning of *the one or the other sides* either.

How can we determine the *left side* of a chair? When do we say that the bird is standing on the *left* of a chair? The distinction of *left side–right side* can be transferred from one's own body to other bodies and objects. With people we don't have difficulties, but with non-living objects we behave less consequently. We often say that the *front side* of a cupboard or a chair in front of us is the side that faces us, but its *right* or *left side* is considered related to us and sometimes related to the object.

The question "What is the position of the bird in relation to the chair?" has at least two correct answers.

If pupils see the situation as a mirror image, the *front–back* direction is opposite to them, the direction of the *above–below* is the same, so the correct answer to **Picture 1** is *right*. If we consider the chair to be a non-living object without its own *front–back* direction, we compare the situation to us; the correct answer as such for **Picture 1** is *left*.

The polarity of *right–left* derives from the human body. Other objects can be assigned a *right–left* by comparison. If we say the *above–below* and the *front–back* of a chair, then its *right–left* as seen from the chair is determined. The problem is that pupils did not consider how they should see the situation, from the position of the chair or from their own position.

The problem remains when determining the *front–back* of the chair too. We can either see the picture as a mirror image, or not. What is the direction of *front–back*? Is the *front* closer or farther to the observer? (Notwithstanding should we note, that the answer *in front of* instead of *behind*, or opposite, has not frequently encountered in this task.)

Analyzing the answers we can consider that the task was undetermined. We must determine the position of the observer in every case if we want to get an unambiguous answer.

With respect to **Pictures 4** and **5** we can speak about the position of the bird related to the chair as well.

Whereas the chair hasn't got any faces, the observer can relate to the face of the bird. The *below–above* direction is unambiguous, but the other two directions aren't. The bird (a living object with a face) determines the direction of *front–back*. There were many answers for these pictures *left* or *right* and *in front of* for **Picture 1**. We could not find any consistencies: There was no answer for **Pictures 4** and **5** double *right* or double *left*, whereas the bird was on the *left side* of the chair supposing that the *front* is the direction of the bird's face. We suppose that the instruction *. . . the position of the bird relating to the chair* is easy to confuse with instruction *. . . the position of the chair relating to the bird.*

The trouble in this topic is shown by the fact, that there are many answers which were *behind* for **Picture 4** and not the *front* for **Picture 5**, and opposite. So it seems that the two opposite directions are also not automatic in grade 1 and 2.

Now we will mention some remarks of the students about the possible causes of the pupil's difficulties.

There were 8 invaluable answers. 10 students wrote, that the pupil does not know the meaning of the words, 8 of them thought that the problem was the position of the observer, by the opinion of 5 students pupils mixed directions *left–right*, 1 student thought about the mixing directions *below–above* or *front–back.* Only 8 students, the 22% of the group felt the most probable reason for the difficulties. They thought about the position of the observer.

With respect to the result of the analysis of the task, we can establish that there are many unambiguous situations, many disappearing problems by solving “simple”, well-known tasks. Therefore we must speak about the topic of *orientation* in schools. Dealing with the question of *orientation* is not only the task of teaching mathematics, but of other subjects too. Here we only mention science, physical education, art, and technology.

How can we teach this topic in mathematics lessons? Which are the crucial points of the concept of orientation, and what are the linked concepts that are necessary to understand the substance of orientation, and which concepts are based on orientation-problems?

The answer is quite difficult. In our opinion one of the possibilities of solution is to turn to mathematics and investigate the axiomatic structure of the concept of orientation.

3. Some mathematicdidactical theoretical backgrounds

> *"Our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world. Phenomenology of a mathematical concept, structure, or idea means describing it in relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is didactical phenomenology, a way to show the teacher the places where the learner might step into the learning process of mankind."*
>
> (Freudenthal, 1983, p. ix)

Our aim is to apply this conception for the mathematical concept of orientation.

The first step – said Freudenthal – is to constitute the *mental* objects, and the second step (if relevant any) is to *mathematise* them as a concept. *Mathematising*, by the interpretation of Freudenthal, is the act of doing mathematics. He holds that pupils should rather be involved in the process of creating a mathematical system, and then confronted with it as a finished product.

There are many mathematical concepts that have been clear enough in the minds of students not to need precise definition. So there are many *mental objects*, which do not demand the concept formation in schools. Most of the topological concepts, especially the problem of *orientation* belongs to that category. We know that terms such as *back–front*, *below–above*, *left–right*, *left turn–right turn*, *left side–right side*, etc. are almost not being learned, let alone taught in different everyday situations in a rich context. The teacher's task is to look for such everyday life situations and create problems, which help them to survey and develop the actual knowledge of students on this topic.

To constitute the *mental objects* of the term *orientation*, Freudenthal emphasizes the importance of the topographical context. This means the places of objects and perceivers in their mutual physical and mental relations. In this context we can speak about *polarities*: *above–below*, *front–back*, *inside–outside*, *left–right*, *left turn–right turn* etc. These *polarities* are connected to the human body, and to the topographical context.

When planning tasks for the polarities we can not ignore the representation theory of Bruner. Bruner suggested three ways of transforming experiences into a model of the world: the enactive, the iconic and the symbolic representation. The thinking process can go

- enactively: through concrete actions with concrete objects,
- iconically: through pictures, diagrams, imagined situations, or
- symbolically: through mathematical symbolisms or words.

In primary school the enactive phase is very important. In case of polarities it means, that pupils must confront with real situations not only at home but in the classroom too. If they have enough experience, then we can deal with the iconic representations, such as my above mentioned task. The symbolic representation of the problem is the subject of the next chapter.

We have different ways to know more about the crucial points of the concept of *orientation*. In this study we bring out one of them. We will analyze the origin and the axiomatic structure of the concept of orientation.

Studying the history of a concept helps us to understand the difficulties that mathematicians go through while constituting a definition.

Studying the works of different mathematicians we can compare them and observe the similarities and differences in building up the concept.

"No mathematical idea has ever been published in the way it was discovered." (Freudenthal, 1983, p. ix)

We agree with this statement of Freudenthal's, however reading the axiomatic presentation of a topic of mathematics we try to identify the structure and the relations between the basic notions.

If we want to emphasize the structural aspects of mathematics in teaching mathematics we must do it very carefully. It is not the case that we want to teach mathematical concepts in an axiomatic way in elementary school, but the importance of axiomatization in mathematics and its implications for teaching cannot be ignored. Studying the axiomatic structure of a concept is useful if we want to know more about it. The propositions and concepts are logically linked together in a deductive manner, and it explicitly shows the logical relationships between the axioms, theorems and definitions.

There are several logically equivalent definitions for the same concept. All of them are rich in numerous links and relationships.

Last but not least, axiomatization and systematization lead to an integrated understanding of a topic, and provide a certain global perspective or broad overview, which is necessary for teaching.

4. Mathematical and mathematical-historical analysis of the concept of orientation

The starting point of our investigation is the axiomatic structure of Euclidean geometry.

We refer to 3 different works, 3 possibilities to build up the concept of orientation. We emphasize not only the differences between the basic approaches of these studies, but the similarities between them, which bring us closer to the core of the concept.

First of all we have to talk about the Greeks; who made geometry abstract and deductive. They introduced such idealized concepts as the *point*, the *line*, etc. While abstract, these concepts are connected to reality and appear as archetypes of physical things. The Greeks obtained information about this idealized world by deduction. In ancient mathematicians way of thinking we can find intuitive elements and axiomatic features too. Greeks wanted to know the structure of the real world, which is recognizable, perceivable.

Their geometry was static, so the concept of orientation, which was intuitively clear, was not necessary.

Euclid mentioned in his famous work, Elements, the extension of a line segment over the one or the other endpoint, which would be a possibility of introducing the concept of orientation of the straight line, but he did not think in this direction. There is not any indication of the orientation of the plane too.

The symbolic-formalist school of the 19th century changed the direction of mathematical philosophy by eliminating the intuitive aspects of the foundations of geometry.

One of the first, modern axiomatic treatments of Euclidean geometry was made by **David Hilbert**. His book, "On the foundation of geometry", appeared in 1899 [9].

Hilbert introduced six undefined terms: the *point*, the *line*, the *plane*, *incidence*, *between* and *congruence*. To build the concept of orientation he needs the *axioms of incidence and order* (Hilbert, 1899, p. 3–5). The axioms of order describe the relation *"between"*, and make the ordering of points of a line, plane or space possible.

The work of the Hungarian mathematician, **Béla Kerékjártó**, "On the foundation of geometry I.", appeared in 1937 [11].

His treatment of the foundation of geometry is similar to that of Hilbert. The axioms are nearly the same as in Hilbert's work. Kerékjártó followed two basic principles in his book. First, he defined the concepts and proved the theorems using only a minimal number of axioms, so he showed the place of a concept or theorem in the system of geometry. Second, he adhered to the expansion of the number of dimension. He showed the possibilities of analogies and generalizations by building up geometry of line, plane and space. This method, furthermore the use of algebra and topology gave the possibility to build up not only the Euclidean, but the Non-Euclidean geometries too.

The so called **Weyl**-axioms show us an alternative approach to building up Euclidean geometry. In this treatment the point, the vector and the operations of vectors are the undefined terms. The axioms for a vector space and for the Euclidean space are reachable from the term of vector equipped with binary operations: addition, scalar multiplication and dot products. The properties of these operations are the axioms of the vector space [10, 14].

Since the vector concept is the basis of this theory, we must investigate it.

There are many ways to construct the concept of vector. One of the well-known definitions is that a vector is an oriented line segment, but there exist a more intuitive definition too, for example a vector is definable as a special instruction to move.

Geometers arrived at the vector concept only in the mid-nineteenth century. Earlier the algebraic notation and symbolism which make vector methods powerful was not accessible. The dynamic properties of geometry were also unknown.

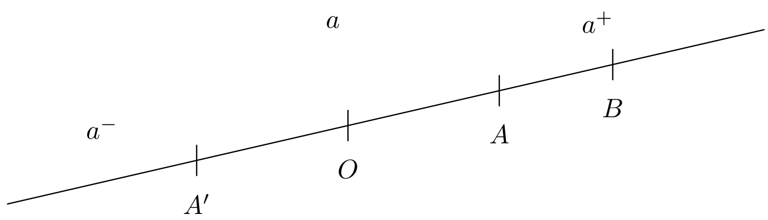

Figure 6

Hilbert and Kerékjártó defined the orientation of a line with the term *between*. If A, A', O, B are distinct points on line a, and point O is between A and A', but not between A and B then we say that A and B lie on the *same side* of O, and A and A' lie on the *opposite side* of O (Figure 6). We say that point O separates the line into two half-lines (a^-, a^+).

We can say furthermore, that A *precedes* B $(A < B)$, if A is *between* O and B, O precedes the points of a^+, the points of a^- precede O and the points of a^- precede the points of a^+.

In this binary relation *"precede"* the points of line a are in linear ordered, and line a is called an *oriented line*.

The vector, as an oriented line segment or an instruction to move, obviously gives an orientation to the line.

We can see, that the relation *between* is the basis of the concept of the *same and opposite side*. This fact explain that the word between was less problematic then the other in our "bird-chair" task.

In the case of the plane we can discuss about three different treatments. Hilbert generalized the concept of the *same side* and *opposite side* on the following way:

Any line on a plane divides it in two parts. We say, that points A and A' lie on the *same side* of line a, if the segment AA' has no common point with a, and points A, B lie on *opposite sides* of a, if the segment AB has a common point with a (Figure 7).

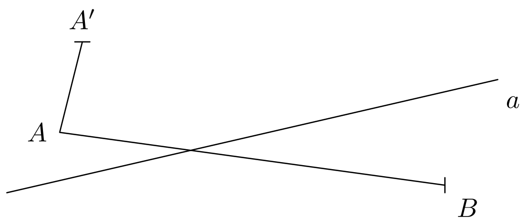

Figure 7

If a is an oriented line on a plane, we can distinguish the two sides of the line as *right* and *left sides*.

If we define the *right* (*left*) side of an oriented line on the plane, we can identify the *right* (*left*) side of any other oriented lines on the plane.

For example if the half-line OB lies on the right side of the oriented line OA, then the right side of the oriented line OB is the part of the plane which does not contain the half-line OA (Figure 8).

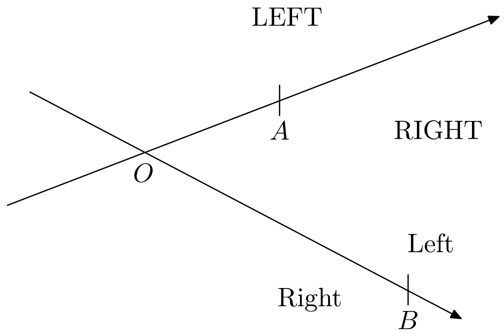

Figure 8

Once we define the internal and external points of a polygon [9], we can define the orientation (*circulation sense*) of a triangle. We say that the triangle ABC has *positive* (*anticlockwise*) *orientation*, if the internal points of the triangle are on the left side of the oriented lines AB, BC, CA, or are on the right side of BA, AC, CB (Figure 9). The triangle has *negative* (*clockwise*) *orientation* if the internal points of the triangle are on the right side of the oriented lines AB, BC, CA, or are on the left side of BA, AC, CB.

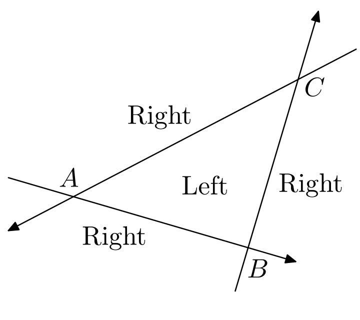

Figure 9

Accordingly we can fix the positive orientation of a plane by defining the positive side of an oriented line.

The starting concept of Kerékjártó's work was the *cyclic order*. We say, that half-lines a, b, c, d on the plane with a common start point O possess the *cyclic order* $(abcd)$, if half-lines a, b divide the plane into two parts, and c and d don't belong to the same part of the plane (Figure 10).

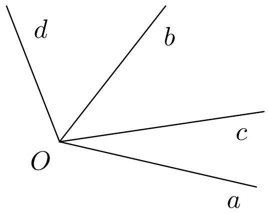

Figure 10

If a, b, c are three half-lines with a common start point O on the plane, we say that the *cyclic orientation* of a, b, c is a function, which orders to them one of their *permutations*.

Let (abc) denote the cyclic orientation determined by this permutation. The cyclic orientations (abc), (bca), (cab) are the same (Figure 11), and (cba), (bac), (acb) are opposite orientations.

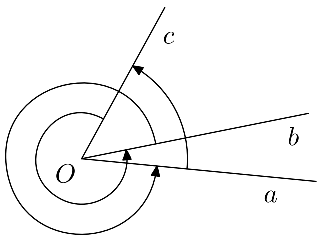

Figure 11

One of the two cyclic orders corresponds to one of the two *directions of rotation* around point O.

We call the two directions of rotation around point O negative and positive, or clockwise and anticlockwise.

We can order to every point of the plane a direction of rotation in a way that the directions of the rotation ordered to any two points are the same. This means that Euclidean planes are orientable. So the orientation of the plane is determined by the cyclic order of the vertices of a triangle.

In this case we can fix a positive orientation by defining the positive direction of the rotation around a point of the plane.

The orientation of the plane by means of vector geometry [14]:

The isometry is a transformation that preserves distances. The Euclidean geometry can also be constructed from the linear algebra of the vector space, with special regard to the *dot product*. An isometry is a linear transformation and the dot product is its invariant.

A linear transformation preserves the dot product if and only if its matrix is orthogonal. A matrix is orthogonal, if it is invertible and its transpose is equal to its inverse. If L is an orthogonal matrix, then $\det(L)^2 = 1$.

The isometry is called direct if $\det(L) = 1$, and called indirect, if $\det(L) = -1$.

Isometries of the coordinate plane $\mathbb{R}^2$ have a 2×2 orthogonal matrix U. If $\det U = +1$, then there exists an angle θ in a way that

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{(rotation matrix)},$$

and if $\det U = -1$, then there exists an angle θ in a way that

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \quad \text{(reflection matrix)}.$$

The rotation matrix rotates all vectors through θ. The reflection matrix leaves one of the vector components unchanged, and reverses the other component.

An orientation of a Euclidean plane is an operation which assigns to each vector $\mathbf{u}$ a new vector $\mathbf{u}'$ having the following properties:

- the mapping is linear,
- the vector $\mathbf{u}'$ is perpendicular to $\mathbf{u}$ (the dot product is equal to zero),
- if $\mathbf{u}$ is a unit vector, then $\mathbf{u}'$ is a unit vector too.

If $\{\mathbf{i}, \mathbf{j}\}$ is an orthonormal basis, then there are only two possibilities: For $\mathbf{i}'$ must be equal either to $\mathbf{j}$ or to $-\mathbf{j}$. If $\mathbf{i}' = \mathbf{j}$, then $\mathbf{j}' = -\mathbf{i}$. Similarly, if $\mathbf{i}' = -\mathbf{j}$, then $\mathbf{j}' = \mathbf{i}$.

In the first case we call the basis *right-handed*, in the second case *left-handed*. If we choose one of these two possibilities, we call the plane an oriented plane.

An orientation specifies a positive direction of rotation.

The three kinds of treatments mentioned above, help us to clarify the problem of orientation on the plane and to construct tasks regarding to this problem. We have three ways to derivate the concept of left turn-right turn. First from the concept left side-right side, second from the cyclic permutation and third from the isometries, especially from the rotation.

Generalizing the case of planes we can give the orientation of a space.

By Hilbert's work, we can fix a positive orientation of the space by defining the positive side of an oriented plane. It means in our "bird-chair" task that we tell that the picture is a mirror image or not.

Kerékjártó generalized the concept of cyclic order of half-lines to cyclic order of half-planes (Figure 12).

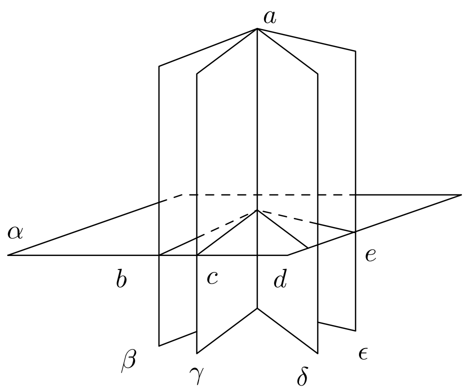

Figure 12

The direction of the rotation around line a is defined as a cyclic orientation of half-planes bordered by line a.

The orientation of the space is determined by an oriented line and a direction of a rotation around this line or by the permutations of the vertices of a tetrahedron.

In vector geometry the orientation of a Euclidean space is an operation which assigns to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ a new vector $\mathbf{u} \times \mathbf{v}$, their *vector product*.

Suppose that an orientation has been chosen, and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be an orthonormal basis. Then $\mathbf{i} \times \mathbf{j}$ must equal either $\mathbf{k}$ or $-\mathbf{k}$. In the first case we call the basis *right-handed*, in the second case *left-handed*.

We can characterize the orientation of a plane through the equation of the plane also.

Let O be an origin on the plane. Let $\mathbf{n}$ be a fixed nonzero vector and let k be a constant. If $\mathbf{r}$ is a position vector of a variable point P, then the equation $\mathbf{r} \cdot \mathbf{n} = k$ says that P lies on the plane.

The vector $\mathbf{n}$ is called a normal vector to the plane. We can choose the constant k so, that $\mathbf{n}$ is a unit normal vector to the plane, and consider an operation that maps a vector $\mathbf{v}$ to $\mathbf{n} \times \mathbf{v}$. Therefore this operation defines an orientation in the two-dimensional sense on the plane. Thus the two possible choices of unit normal vector correspond to the two possible choices of orientation of the plane.

5. Didactical conclusions of the mathematical structure of the concept of orientation

Studying the three different ways of introducing the concept of orientation we can see, that they emphasize different properties as fundamental. These properties are worth summarizing.

(1) The treatment of Hilbert:

The starting point is the *linear order of the line*, and the *left-right side of the oriented line.*

Concepts and relations preceding the concept of orientation: *between*, *"precede"*, linear order, oriented line, and oriented plane, two sides of an oriented line or a plane.

Concepts based on or linked to orientation: *geometrical transformations* that preserve orientation or not.

Hilbert introduced the circulation sense of a triangle by the help of the concept of the left sides of oriented lines. The circulation sense is the basis of the orientation on a plane, and also that of characteristic of transformation i.e. preserving orientation or not.

(2) The treatment of Kerékjártó:

The starting point is that he assigns a *direction of rotation* to every point of the plane. Concepts and relations preceding the concept of orientation: the concept of *permutation* and *cyclic order*. He introduced the term of the clockwise and anticlockwise directions.

Concepts based on or linked to orientation: geometrical transformations, the concept of *oriented angle*.

The study of Kerékjártó's work highlights the link between cyclic permutation and orientation. It is worth thinking about that cyclic orders are probably early mental objects and arranging cyclically is an early mental activity in the individual learning process then linear order and arranging linearly. We refer to such kind of activities as sitting around a table, standing or dancing in a circle, counting out, etc.

(3) The treatment of Weyl:

The starting point in the vector geometry is naturally the *vector*. It is understandable as a motional instruction. The orientation of the plane is an *operation* which assigns a vector to the other vector.

Concepts and relations preceding the concept of orientation: *dot product*, *vector product*. We can introduce the concepts of *left- and right-handed basis*.

Concepts based on or linked to orientation: *Oriented angles*, classification of *isometries* of a plane.

The vector geometrical approach suggests the dynamic feature of the orientation. Following the instruction of a vector we can mentally travel on a line or a curve. By traveling in our own minds, we have to imagine the position of the traveler, so we have to relate to the other person who is moving.

Some remarks to the development of spatial orientation:

Since my aim was to deal with the problem how to teach the concept of orientation, or how to enhance the spatial orientation ability in its complex sense, the historical and mathematical analysis was rather useful in the following aspects:

- Studying the theory of orientation we can see, that this is a very complex concept, which was formed quite late in the history of mathematics. Hilbert was the first who gave an exact definition of oriented line and plane at the end of the 19th century. Because of this fact considering the results of the task analyzed in this article, we suggest a circumspect introduction of and practicing in the topic of orientation. We can not forget that the orientation is a complex geometrical and topological concept. It arises from everyday situations, but the precise definition as well as the curious understanding is quite difficult.
- The order of appearance of concepts in the history of mathematics or in the axiomatic treatment helps us to determine the difficulty level of a task related to these concepts.
- After identification of crucial points and linked concepts we can isolate and group the fundamental problems appearing by the development. It may be useful to deal more with the questions of *between*, *linear* and *cyclic order*, *permutation*, *vector*, etc. and give diversified tasks based on the three kinds of approaches closer to the every day situations. The logical links that are

known from the axiomatic treatment may help the children better understand the difficulties of the questions on this topic. Of course this would be the task not only of the first school year, but of the further school years too. On the basis of mathematical and historical analysis we may divide the relevant curriculum into 6 main parts:

  - Using words to describe spatial relations
  - Describing routes
  - Ordering cyclically
  - The coordinate system
  - Geometrical transformations
  - Objects from different viewpoints

- The three kinds of treatments of the same topic help as to solve problems which are arisen from the individual differences of pupils. If one of the approaches is inappropriate for a child, we can choose another.

Since the tasks elaborating for the elementary school pupils are complex and appear in every year in various manners, we think that to learn more about the concepts related to the term of orientation enhance not only the ability of spatial orientation, but other mathematical and nonmathematical abilities too.

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(Received October, 2005)