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Teaching
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On the models of the hyperbolic plane

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Dedicated to the 70th birthday of Professor Jenő Horváth

Abstract. We can see the most familiar models of the hyperbolic plane in one figure, which shows the connections between them by the help of projections.

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Introduction

Several geometric books introduce the models of the hyperbolic plane, the connections and the isomorphisms between them. This short article presents the connections of the most familiar models of the hyperbolic plane in the same figure. These connections are based on proper projections.

Let the Poincaré model ($\mathbb{G}$) be defined on the hemisphere $x^2+y^2+z^2=1$, $z>0$. In this model the lines are the half-circle sections whose planes are perpendicular to the xy-plane. (Detailed description in [BJ].) Let the Weierstrass model ($\mathbb{W}$) be on the sheet $z>0$ of the hyperboloid $x^2+y^2-z^2=1$ ([FR]). The lines of this model are the hyperbola-branch sections whose planes pass through the point $O(0,0,0)$. Let the Klein-Poincaré circle model ($\mathbb{K}$) be defined on the circle domain $x^2+y^2<1$ on the plane $z=0$, in this model the lines are the diameters and the circular arcs intersecting the base circle perpendicularly. The Cayley–Klein model ($\mathbb{C}$) is defined on the circle domain $x^2+y^2<1$ on the plane $z=1$, where the lines are the chords of the base circle ([BJ], [SZP]).

We build a less known model, the so-called polar-coordinate model ($\mathbb{P}$) of the hyperbolic plane on the complete plane $z = 1$. We assign the point with polar coordinates (r, φ) in the hyperbolic plane to the point with polar coordinates $(\sinh(r), \varphi)$ in the plane $z = 1$. (Detailed description in [WT].) In this model the lines are the Euclidean lines passing through the 'centre of the plane' N and the branches of hyperbolas whose asymptotes pass through the point N.

Let us consider a plane $y = c$ $(c \neq 1)$, that is perpendicular to the y-axis. On the half-plane $z > 0$ of this plane we can model the hyperbolic plane by the Poincaré's half plane model ($\mathbb{H}$). In this model the lines are the half circles whose centres are in the plane $z = 0$ and the Euclidean half lines parallel to the z-axis.

Projections

The above models can be derived from each other by orthogonal- and central projections. (The familiar projections are discussed in detail in several books, while the proofs of the isomorphisms between the $\mathbb{P}$-model and the other models can be found in [NL1].) Thus the $\mathbb{P}$-model does not require an analytic definition in higher dimensions either ([NL2]), it can be derived from the $\mathbb{W}$-model by an orthogonal projection. The connections between the models are the followings:

- The $\mathbb{P}$-model is the orthogonal projection of the $\mathbb{W}$-model onto the plane $z = 1$.
- The $\mathbb{C}$-model is the central projection of the $\mathbb{W}$-model from the point $O(0, 0, 0)$ onto the plane $z = 1$.
- The $\mathbb{G}$-model is the central projection of the $\mathbb{W}$-model from the point $S(0, 0, -1)$ onto the hemisphere $z > 0$ with center O and unit radius.
- The $\mathbb{K}$-model is the central projection of the $\mathbb{W}$-model from the point $S(0, 0, -1)$ onto the plane $z = 0$.
- The $\mathbb{P}$-model is the central projection of the $\mathbb{G}$-model from the point $O(0, 0, 0)$ onto the plane $z = 1$.
- The $\mathbb{K}$-model is the central projection of the $\mathbb{G}$-model from the point $S(0, 0, -1)$ onto the plane $z = 0$.
- The $\mathbb{C}$-model is the orthogonal projection of the $\mathbb{G}$-model onto the plane $z = 1$.
- The $\mathbb{H}$-model is the central projection of the $\mathbb{G}$-model from the point $M(0, 1, 0)$ onto a plane perpendicular to the y-axis.

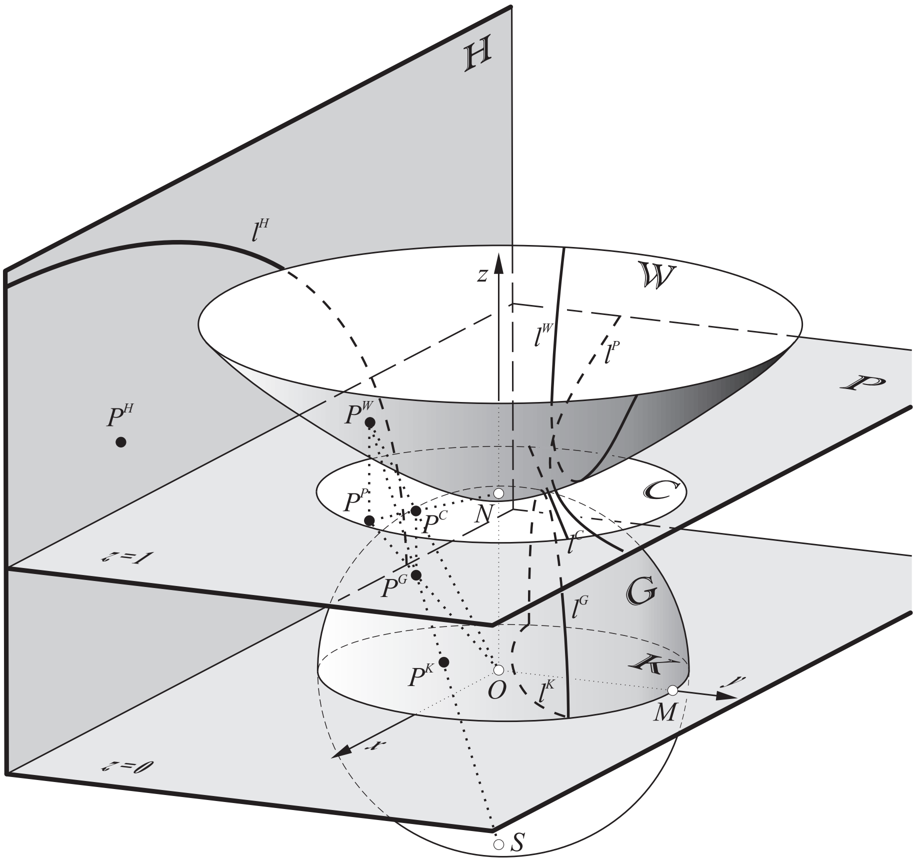

Figure 1. Connections of the models of the hyperbolic plane.

Figure 1 presents a point P and a line l in the different models, they are all associates of each other, respectively.

Remark 1. The connections between the models can be presented (possibly by a computer) in higher education, teachers' further training and special courses of secondary education. It can help with the better understanding of the hyperbolic plane geometry.

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