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Teaching Mathematics and Computer Science

Decomposition of triangles into isosceles triangles I

Let the students ask bravely

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Abstract. We report about working up an open geometric problem as a mathematical research with pupils of a mathematics camp. This paper shows the didactic aims and the methods we worked with, the didactic results. The second part of this paper gives a general solution of the problem, using pure mathematics and a computer programme.

Key words and phrases: triangle, equilateral triangle, isosceles triangle, decomposition, analogy, discovering mathematics, guided discovery.

ZDM Subject Classification: G43, K23, C73, D43.

1. Introduction

In the summer of 2000, two of the authors attended a week-long mathematics camp organized by an enthusiastic group of the alumni of Fazekas Mihály High School, Debrecen. The camp was aimed for children between the ages of 10 and 16. They worked in small groups and solved mathematical problems with the help and guidance of the teachers. This two-part article is the result of the work of an excellent group of 13 and 14-year-olds: it is based on their questions and ideas.

Every given problem and solution is the result of the collective effort of the group and the teacher. These are followed by some methodological notes. Apart from the beauty of the students’ solutions, we also would like to introduce an alternative method of teaching mathematics by them. It is the common belief of the authors that students acquire the most both about science and the world around them if they do their own research. Moreover, they can incorporate the factual knowledge they utilize a lot more effectively than when we make them practice theorems with the help of some obvious examples (usually only) right after proving them.

In our case, the students worked together in groups, and the teacher “only” summarized and systematized.

2. Didactic aims, theoretical basis

The group was made up of students who were interested in mathematics to a greater extent and had more extensive and wider mathematical knowledge than the average (we knew this in advance). Due to this, our preliminary **didactic aims** during the organization of the activities were the following:

(1) to develop problem-solving skills;

(2) to demonstrate and teach mathematical research methods;

(3) to teach new materials, to approach existing knowledge from new aspects by *guided discovery*.

2.1. The developement of problem-solving strategies

Eric Ch. Wittmann gave ten primary conditions in order to improve problem-solving skills ([6] pp. 101–102, [5] pp. 108–109):

(1) Aquiring knowledge by ***“discovery” learning and teaching***.

(2) Urging students to use ***“diverse thinking”*** (using various wordings, approaching the same problem from different directions, connecting different fields of mathematics, mixing methods).

(3) Pushing the exclusive application of ***automatical sequences*** of ideas ***into the background***.

(4) ***Examining open problems*** (there are no direct questions, there are different ways of putting up questions, and possibilties of research).

(5) Urging students to ***raise questions*** and identify problems themselves.

(6) Shaping a “language” that makes it possible for students to ***express their thoughts***.

(7) ***Urging intuitive reasonings and conjectures*** (a small but independent step is worth more than the picturing of a demonstrated chain of ideas).

(8) ***Learning "heuristic" strategies***.

(9) Shaping a ***constructive attitude towards mistakes***.

(10) Urging ***discussions, reflections, and argumentations***.

This served as basis in the course of our work.

2.2. The demonstration and teaching of mathematical research methods

György Pólya writes the following in his [2] II volume:

"The teaching of mathematics should acquaint the students with all aspects of mathematical activity as far as possible. Especially it should give opportunity to the students for independent creative work – as far as possible."

Ragarding this as an "ars poetics" we took the following points of view into consideration when **selecting the basic problem**:

The problem, due to its nature, should be *open* and should produce opportunity:

- should perform experiments, observations from different points of view, and structures;
- should percieve regularities setting out from examining special cases (*induction*);
- should investigate the validity of regularities that have already been perceived in other cases, it should perceive and apply *analogies*;
- should *draw up* and check *common conjectures* in special cases;
- should *prove common conjectures.*

We strove for the followings when planning our **teaching method**:

- directing the common "research" towards the students' ideas, thoughts, questions (even if it comes to a "dead end"; dead ends can be very instructive);
- summing up and making it clear for students where we are;
- formulating questions, if necessary, whose answers can help students to proceed on their own way;
- drawing attention to necessary materials and recall the background knowledge that is required for solving the problem.

2.3. Teaching new material, placing the existing knowledge into other perspective by *guided discovery*

Since the available time in the camp (4 occasions – 90 minutes each) made it possible and the composition of the group (talented and inquiring students) and the nature of the camp (talent spotting) called for it, we used the method of guided teaching and discussion. We strongly believe, and our observations back up this idea, that *guided discovery* can be used at all levels of the teaching mathematics. Though this method is linked to problem solving in our essay, we still think that aquiring new materials (concepts, definitions, theorems, proofs, algorithms) is more effective for the students' part if it is discovered by themselves either to some degree or on the whole. Pólya writes the following about this [2] in his II volume:

“What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the student's mind and the teacher should act only as midwife. ... ... The principle is: let the students discover by themselves as much as feasible under the given circumstances.”

András Ambrus ([5] p. 147) analyses the teachers' methods that have been applied in case of the traditional way of teaching and the “guided discovery” way of teaching. He compares the roles of the teacher in the processes. Further on, relying on and completing this analysis, we highlight some characteristics of guided discovery that we used in the problem-solving process that is being presented below. The teacher:

- brings up open problems and problematic situations, recommends these to the students, along which they can start off in different ways;
- makes the students draw up the particular parts of the problem in connection with the general and open problem he or she has brought up (this has a serious effect on motivation, since students treat the problem as it was theirs);
- helps students during the process of problem-solving so that they will be able to help themselves (role of the “catalyst”);
- highlights and makes students understand the methods, strategies, algorithms and techniques that can be applied to other problems generally;
- takes the risk that his or her “discovering algorithm” changes either to some extent or on the whole due to the ideas and questions of students (it came about during our work in the camp);

- takes the risk of receiving questions from students, which he or she did not think of during planning, and does not even know the answer to it (this also came about at the camp; the second part of our essay is on its results).

3. The Basic Problem

In one of the first sessions the children were given the following widely-known problem.

For what values of n ($n \geq 2$) can an equilateral triangle be decomposed into n equilateral triangles with no common inner point?

Solution

For $n = 2$ there is no such decomposition since the three vertices of the original triangle must be in three different triangles.

For $n = 3$ there is no such decomposition, either, because since no two of the three triangles (each containing a vertex of the original triangle) can have a common point there will be an area that is left out of the decomposition. This area can be triangular, quadrangular, pentagonal or hexagonal (Figure 1).

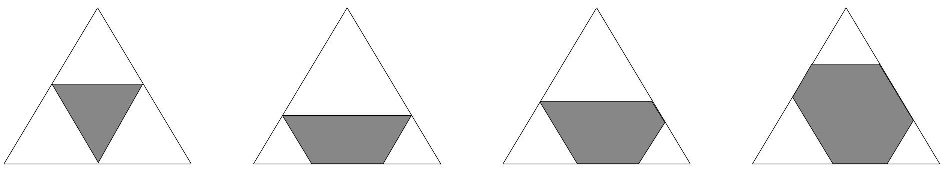

Figure 1.

For $n = 4$, we can get a correct decomposition by drawing the medians of the original triangle (Figure 2).

So any equilateral triangle can be decomposed into 4 equilateral triangles. It follows that if an equilateral triangle can be decomposed into n equilateral triangles, it can be decomposed into $n+3$ equilateral triangles as well. All we need to do is to decompose one of the smaller triangles into 4 even smaller triangles as seen in Figure 2 so the number of the triangle in the decomposition grows by 3. (Children discovered this analogy by themselves.) This means that for any $n = 3k+1$ ($k \in N^+$) an equilateral triangle can be decomposed into n equilateral triangles.

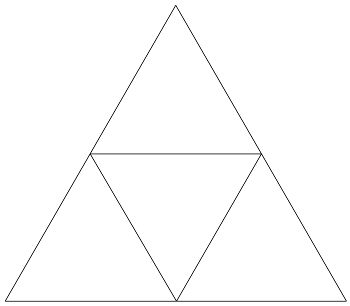

Figure 2.

This meant no problem for this group of talented children whose intelligence is above average, but it is important to point out that even though now we have proved the possibility of decomposing for $n = 4$ and therefore infinitely many n's, it does not follow that the decomposition can be made for any n. This is supported by the exploration of the case when $n = 5$. It was quite easy for the students after solving the problem for $n = 3$.

The part left out after drawing the three triangles containing the vertices of the original triangle can only be a isosceles trapezoid (Figure 1/b) as an equilateral triangle (as well as a pentagon or hexagon) cannot be decomposed into two equilateral triangles. But isosceles trapezoids with angles of 60 and 120 degrees cannot be decomposed into 2 equilateral triangles either, since the diagonal divides one of the 60-degree angles. Therefore there is no decomposition for $n = 5$.

At this point the students were ready with the construction in Figure 3 for $n = 6$. (P, Q, R and S are appropriate trisecting points; F is the midpoint of segment SR.)

The students worked in their own exercise books during the activities, but had the chance to discuss their ideas and thoughts with each other. In case of this part of the problem (in case $n = 6$ *construction) we asked them to work individually for a short time. There were students who found solution to the problem quickly of course, but there were some who got to the right construction more slowly after several attempts. It is characteristic of the “strength” of the group that everyone found the construction that is seen in Figure* 3*. The fastest student told us that he had known what to do immediately, because he remembered an exercise in which a symmetrical trapezoid had to be made from five congruent regular triangles, so he had* met a similar problem *in other words.*

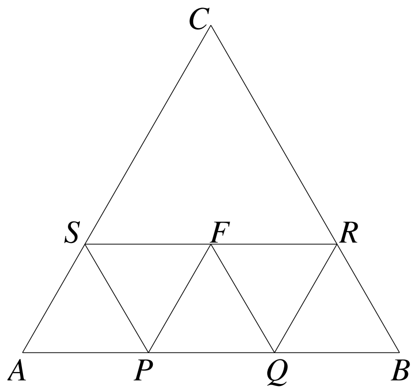

Figure 3.

However, with the help of the method applied before, we can show that a decomposition exists for any $n = 3k$ ($k \geq 2$ integer), since any triangle can be decomposed into four smaller triangles. *We can observe that the students used analogical thinking, since they performed the induction just like they did in the case of generalizing from* $n = 4$ *to* $n = 3k + 1$.

At this point they were asked the following question: For what values of n should we prove the existence of a decomposition so that we completely solve the problem?

By this question, the teacher requires the conscious application of our method (the decomposition of 1 triangle into 4 triangles) and the synchronic analogical thinking.

The answer came quickly: only for $n = 8$, since if there is such a decomposition we have proved the existence of a decomposition for any $n = 3k + 2$ ($k \geq 2$ integer). By this we would have an answer for any n in question. Students easily constructed the decomposition for $n = 8$, similarly to what they did for $n = 6$ as it is shown in Figure 4. (Here P, Q, R, S, T are appropriate quartering points; M and N are the trisecting points of line segment TS.)

To sum up, an equilateral triangle cannot be decomposed into 2, 3 or 5 equilateral triangles, but it can be decomposed into 4 or any $n \geq 6$.

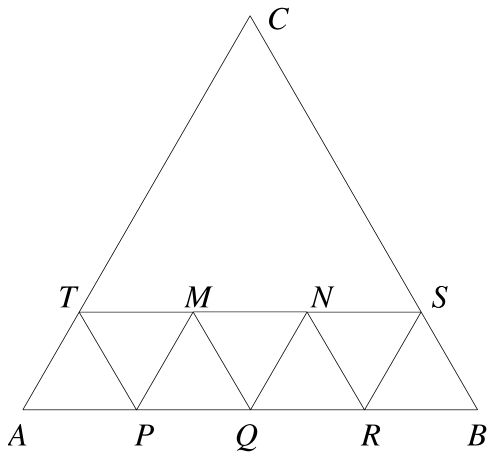

Figure 4.

4. Let's weaken the condition

After the solution of our first problem one of the students pointed out that though an equilateral triangle cannot be decomposed into 3 equilateral triangles, it can be decomposed into 3 isosceles triangles by connecting the midpoint of the triangle with the vertices (Figure 5).

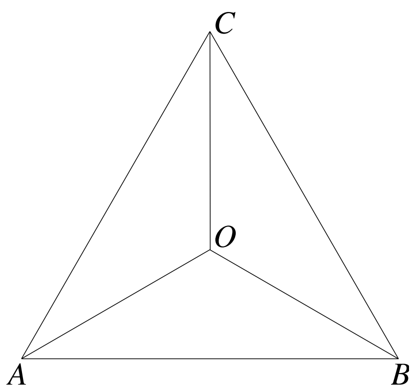

Figure 5.

In the following we will see that such a seemingly unimportant comment is sometimes worth being taken seriously.

This observation gave us the idea of weakening the conditions in cases where we cannot decomposition into equilateral triangles, that is let’s examine the following question:

Is it possible to decompose an equilateral triangle into a) 2; b) 5 isosceles triangles? (For 3 we have already seen the decomposition.)

Solution (this is also the result of a joint effort by the students)

(1) To divide a triangle into two triangles it has to be cut up by a line going through one of its vertices. At the intersection of this line and the side of the triangle opposite to the above vertex two angles are formed the sum of which is 180 degrees. Therefore one of the angles is at least a right angle (Figure 6).

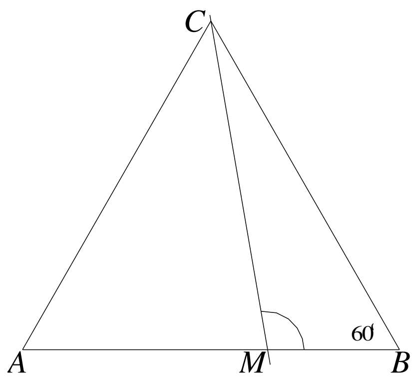

Figure 6.

So triangle BCM can only be isosceles if $CM = MB$. However, this is impossible, since $BCM\angle < MBC\angle = 60°$. So an equilateral triangle cannot be decomposed into two isosceles triangles. *In the heat of the research the group did not always go for the simplest solution. Here, for example, it would have been enough to point out that there is no isosceles triangle with exactly one 60-degree angle.*

(2) For the case of 5 isosceles triangles the children found two correct decompositions (Figures 7, 8).

The decomposition in Figure 7 can be found by dividing the triangle in the way we have already seen it into 3 isosceles triangles, and repeat the process for one of the new triangles. (Points D and E are the intersections of side AB and the perpendicular bisectors of AO and OB, respectively.)

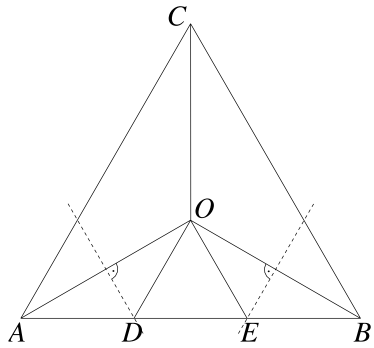

Figure 7.

The decomposition in Figure 8 is the result of connecting the centre of the circumscribed circle of the symmetrical trapezoid left after cutting off the equilateral triangle CDE and the vertices of the trapezoid: A, B, E and D. (If triangle CDE is chosen “small enough”, that is point D is closer to C than to A, then the centre of the trapezoid’s circumscribed circle is assured to fall within triangle ABC.

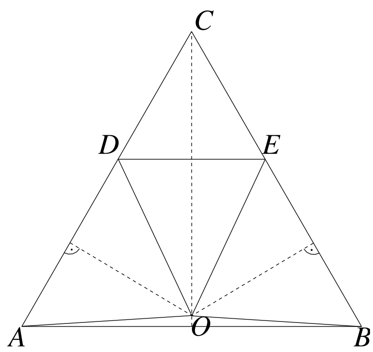

Figure 8.

To sum up, an equilateral triangle can be decomposed into n isosceles triangles for any $n \geq 3$.

5. For equilateral triangle the answer is "no". For which triangles will be the answer "yes"?

So far we proved that an equilateral triangle cannot be decomposed into 2 isosceles triangles. The teacher leading the workshop encouraged students to ask questions in connection with this fact.

The reader is also encouraged to formulate and answer similar questions. It is an interesting experiment to see in which direction and how far one gets with her own explorations.

Several good questions were asked. They could be divided into two groups. Some of the questions were concerned with the possibilities of dividing an equilateral triangle into two, somehow special objects. Here are some examples.

(1) Is it possible to divide an equilateral triangle into two congruent triangles? (The answer is yes. It is fairly trivial.)

(2) Is it possible to divide an equilateral triangle into two quadrangles? (It is also easy to answer, see Figure 9)

(3) Is it possible to divide an equilateral triangle into two convex quadrangles?

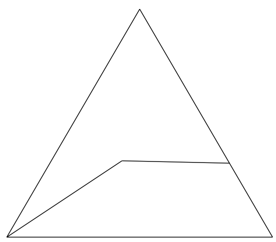

Figure 9.

The rest of the questions were about what kind of special plane figures can be divided into two parts with certain given properties. We chose the question most closely connected to our chain of thought together from this group.

Which triangles can be divided into 2 isosceles triangles?

Some of the students were very quick to announce some examples of such triangles: right triangles (Figure 10) can be indeed divided in such a way. It is a consequence of Thales' theorem.

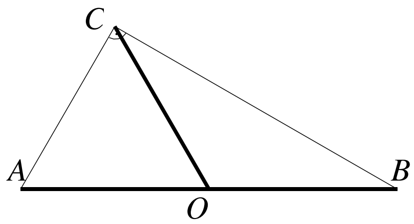

Figure 10.

There was a student who knew the special property of the isosceles triangle with degrees 36, 72 and 72, so this triangle also came up (Figure 11).

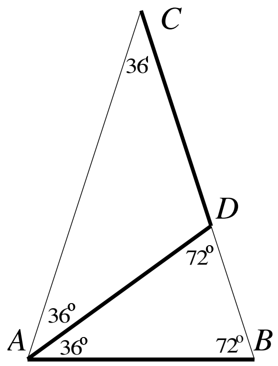

Figure 11.

After these examples the authors would like to point out that they find it very important that certain major theorems come up quite naturally unlike in the usual, direct applications.

After these examples we clarified that since we intended to find all such triangles, we needed to examine the problem in a systematic manner. In the following the course and results of these systematic explorations is presented.

When dealing with the equilateral triangle, it has already been shown that a triangle can only be divided into two triangles with a line passing through one of its vertices (Figure 12).

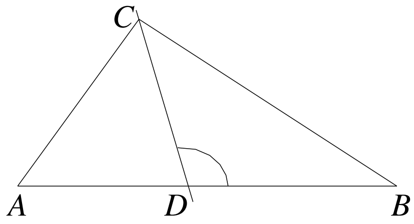

Figure 12.

In this case, however, at least one of the angles ADC and CDB is at least a right angle. Let $CDB\angle \geq 90°$. Triangle CDB can be isosceles only if $CD = DB$. For triangle ADC, however, there are three different possibilities.

(1) $AD = DC$. In this case D is the midpoint of side AC and D is equidistant from all three vertices of the triangle, therefore the triangle is right-angled as a result of Thales' theorem (Figure 13). Since Thales' theorem can be conversed, it follows that any right triangle can be divided into two isosceles triangles. (So one of our original examples has already been found by our systematic solution.)

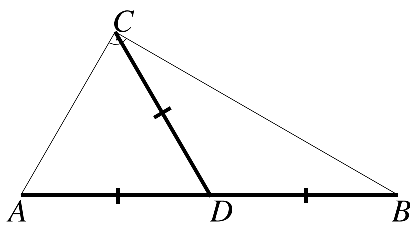

Figure 13.

(2) $AC = CD$. Let β denote the inner angle at vertex B. Then

$$DBC\angle = BCD\angle = \beta$$

and from the theorem on the outer angles of a triangle

$$ADC\angle = \beta.$$

From our assumption about the sides it follows that

$$CAD\angle = ADC\angle = 2\beta.$$

(See Figure 14.)

If $ACD\angle = \delta$, then for the sum of the inner angles of triangle ABC

$$180^\circ = 2\beta + \beta + \beta + \delta = 4\beta + \delta > 4\beta,$$

from which $\beta < 45^\circ$ follows.

It is quite obvious that our chain of thought can be reversed, that is any triangle satisfying the above condition can be divided into two isosceles triangle, so it is also a sufficient condition.

Thus it follows that any triangle with an inner angle less than 45 degrees and another inner angle which is exactly twice the previous one can be divided into two isosceles triangle as seen in Figure 14.

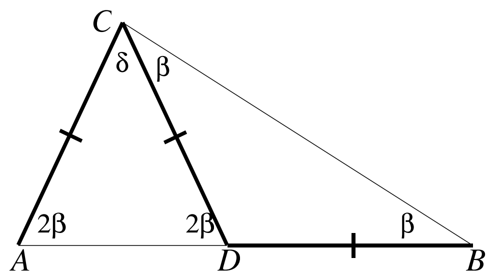

Figure 14.

Comment: A special case of such triangles is the isosceles triangle with angles 36, 72 and 72 degrees.

(3) $AC = CD$. Let β denote the inner angle at vertex B again. Since ADC is an outer angle of triangle CDB, we have

$$ACD\angle = ADC\angle = 2\beta.$$

Thus

$$BCA\angle = 3\beta.$$

If the inner angle at vertex A is denoted by α, then

$$180^\circ = \alpha + \beta + 3\beta = \alpha + 4\beta > 4\beta,$$

so $\beta < 45^\circ$. It is obvious that the above condition is also sufficient here.

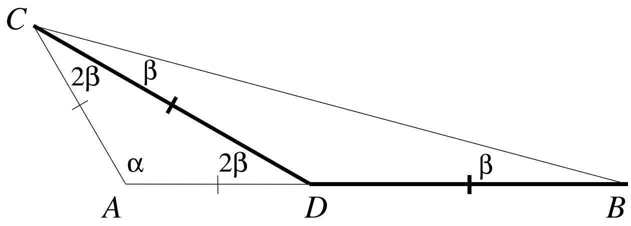

Figure 15.

Thus any triangle with an inner angle less than 45° and another inner angle three times the previous one can be divided into two isosceles triangles. (See Figure 15.)

We examined all possible cases, so we can summarize our results.

Any triangle that satisfies at least one of the following conditions can be divided into two isosceles triangles:

(1) **The triangle is right-angled.**

(2) **The triangle has an inner angle less than 45 degrees and another inner angle which is exactly twice the previous one ("$(\beta, 2\beta)$ type" triangles).**

(3) **The triangle has an inner angle less than 45° and another inner angle three times the previous one ("$(\beta, 3\beta)$ type" triangles).**

It is interesting that for the right triangle with a 30° angle all three conditions are satisfied. (And this is the only such triangle.)

6. Let's ask more questions

After successfully finding which triangles can be divided into two isosceles triangles the following question came quite naturally: which triangles can be divided into three isosceles triangles? ("Naturally" here means that students unanimously decided to work on this problem next.) Children felt that it was a more difficult problem, but certain partial solutions were found very quickly. (They are presented here in the order as they came up in the workshop.)

(1) Any acute triangle can be divided into three isosceles triangles by connecting the centre of its circumscribed circle with its vertices. (See Figure 16.)

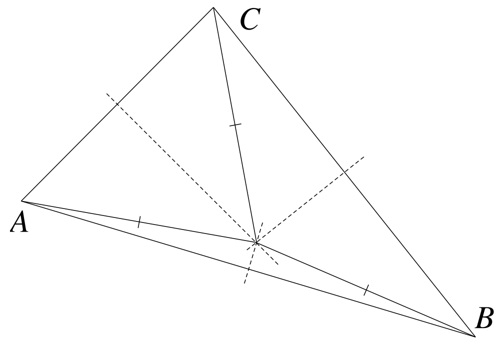

Figure 16.

(2) The isosceles right triangle can be divided into n isosceles triangles for any $n \geq 2$, thus into 3. (See Figure 17.) Here every small triangle is similar to the original one.

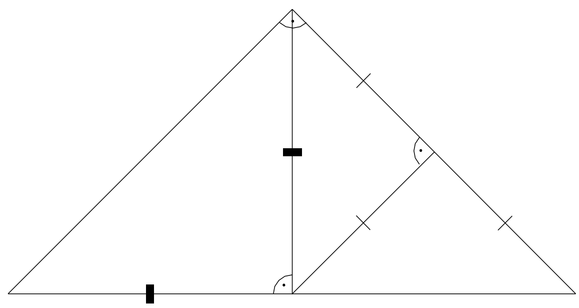

Figure 17.

(3) Any non-isosceles right triangle can be divided into three isosceles triangles in the following way (Figure 18). If the perpendicular bisector of hypotenuse AB intersects (the longer) leg BC then triangle ADB is isosceles and triangle ADC is right-angled. Divide the latter into two isosceles triangles in the familiar way.

(4) Isosceles obtuse triangles can also be divided into three isosceles triangles. The idea of decomposition is based on the decomposition in Figure 7. The feasibility of the decomposition is based on the fact that the centre of the circumscribed circle of an obtuse triangle is outside the triangle (Figure 19).

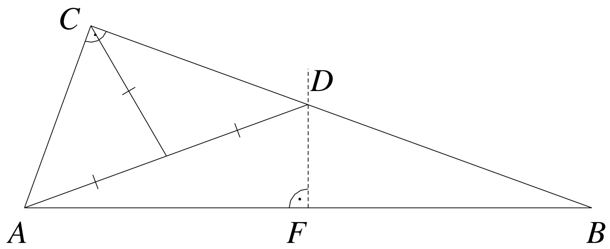

Figure 18.

The perpendicular bisectors of sides AC and BC intersect base AB in points D and E, respectively, so $AD = DC = CE = EB$.

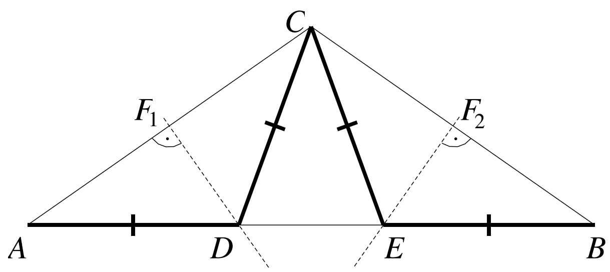

Figure 19.

After these examples some students made the conjecture that any triangle could be divided into three isosceles triangles, but it seemed to be equally difficult either to prove or disprove it. (At this point even the leader of the workshop could not think of a generalization.) We agreed that students were free to look for non-isosceles obtuse triangles that can be divided into three isosceles triangles in their free time. They actually brought some examples, but we were not able to completely generalize the problem during the camp. It is not very surprising, though, since – as the reader can see it for in the second part of our article – the authors finally found all possible solutions with the help of a computer.

7. A nice result based on previous ideas

The incompleteness of the $n = 3$ case motivated students to spend their afternoon looking for non-isosceles obtuse triangles that can be divided into three isosceles triangles. Moreover, on a workshop following an all-day excursion one of the students showed that some of them together proved the following:

Any triangle can be decomposed into n isosceles triangle if $n \geq 4$.

That is they found a mathematical theorem by themselves! They did the following.

(1) Any triangle can be divided into four isosceles triangles. This is the consequence of the following: (a) any triangle has a height that is inside the triangle, and this height divides the original triangle into two right triangles; (b) any right triangle can be divided into two isosceles triangles (Figure 20).

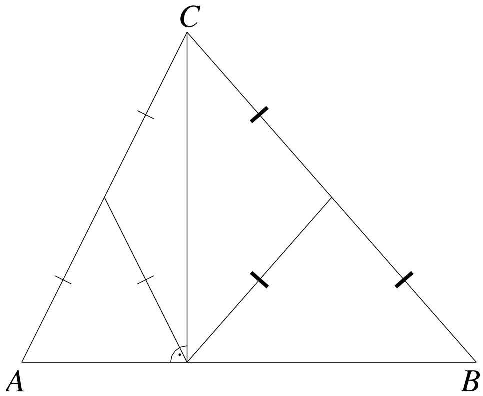

Figure 20.

From this it also follows that in any decomposition the number of isosceles triangles can be increased by 3 so that a small triangle can be divided into four parts as it can be seen in Figure 20. *(The same old idea again!)* So we know that any triangle can be divided into n isosceles triangles if $n = 3k + 1$, $k \in \mathbb{N}^+$.

(2) Any triangle can be divided into five isosceles triangles. (a) It has already been proved for equilateral triangles. (See Figures 7, 8.) (b) If a triangle is not equilateral, then it has two sides of different length. Measuring the

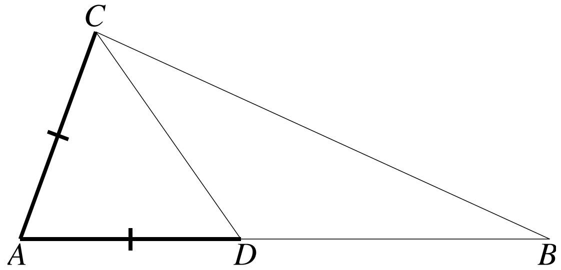

Figure 21.

shorter one on the longer one from their common vertex the original triangle can be divided into two triangles. One of these is isosceles (Figure 21). (c) Triangle BCD can be divided into four equilateral triangles (Figure 22). It follows that any triangle can be divided into n isosceles triangles if $n = 3k+2$, $k \in \mathbb{N}^+$.

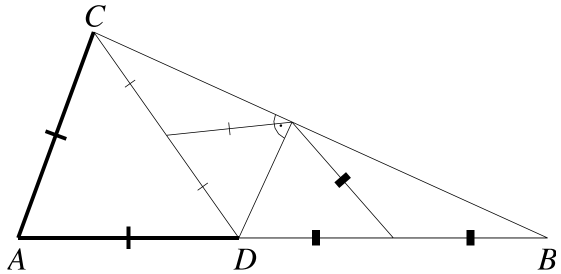

Figure 22.

(3) Any triangle can also be divided into six isosceles triangles. (a) The original triangle is divided into two right triangles by (one of the) heights which is inside the triangle. (b) One of the right-angled triangles is divided into two, the other one into four isosceles triangles (Figure 23). Since any small triangle can be divided into four isosceles triangles, it follows that any triangle can be divided into n isosceles triangles if $n = 3k$, $k \geq 2$.

It can be seen once more how the students were using analogical thinking, since here they completely "copied" the techniques used for equilateral triangles. In this case they ended up with a proof a bit more complicated than necessary, since they did not realize that with the trick applied when dividing

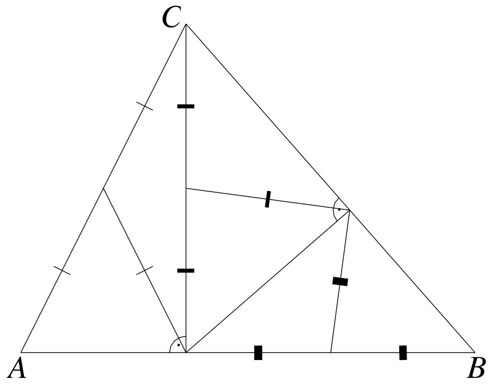

Figure 23.

into 5 isosceles triangles can be used to increase the number of the resulting isosceles triangles by one at a time.

This follows from the fact that the remaining part after drawing the first isosceles triangles is an obtuse triangle. So this triangle has two sides of different lengths, so we can apply the "trick" for this triangle again. We conclude the decomposition by dividing into 4 isosceles triangles (Figure 24). If we are to divide a non-equilateral triangle into n triangles ($n > 4$), after cutting off the first $n - 4$ isosceles triangles divide the remaining obtuse triangle into 4 isosceles triangles.

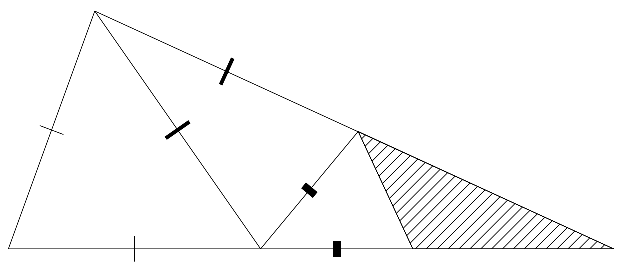

Figure 24.

The children, using their own ideas, had everything in hand for the complete proof after the first two steps. It is a very exciting and difficult job for the teacher to think it over at the spot and give feedback to the children immediately: is it a *good idea or not; does it help to solve the problem or not; or should we call the children's attention that their idea is more powerful than they thought (as was in our case).*

8. Final words (Epilogue)

To be honest, we only wanted to deal with the decomposition of equilateral triangles in the camp. It is due to the motivated and talented students in the group that we got this far. The mathematical results given here is nothing new for those familiar with the area. However, they were new for the students, and it is very important that they discovered these things themselves. They could experience what mathematical research is like, and this is a very important experience even though (or rather evidently) some questions were not answered, some problems were not solved. The joint effort with students was a crucial experience for us, teachers as well. And it seemed natural that as for the question left open (that is, which non-isosceles obtuse triangles can be decomposed into three isosceles triangles) we "investigated" further.

References

[1] G. Pólya, *How to Solve It*, Bibliotheca, Budapest, 1957 (in Hungarian).

[2] G. Pólya, *Mathematical Discovery I-II*, Tankönyvkiadó, Budapest, 1979, 1985 (in Hungarian).

[3] G. Pólya, *Mathematics and Plausibile Reasoning I-II*, Budapest, 1988, 1989 (in Hungarian).

[4] A. Soifer, How Does One Cut Triangle?, *Center for Excellence in Mathematical Education*, Colorado Springs, 1990.

[5] A. Ambrus, *Introduction to Didactics of Mathematics*, ELTE Eötvös Kiadó, Budapest, 1995 (in Hungarian).

[6] E. Ch. Wittmann, *Grundfragen des Mathematikunterrichts*, Vieweg, Braunschweig, 1981.

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