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Teaching
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Comparative geometry on plane and sphere

Didactical impressions

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Abstract. Description of experiences in teaching comparative geometry for prospective teachers of primary schools. We focus on examples that refer to changes in our students' thinking, in their mathematical knowledge and their learning and teaching attitudes. At the beginning, we expected from our students familiarity with the basics of the geographic coordinate system, such as North and South Poles, Equator, latitudes and longitudes. Spherical trigonometry was not dealt with in the whole project.

Key words and phrases: re-structuring geometric concepts; comparing different systems within the same subject; comparative geometry on plane and sphere; manipulative devices on plane and sphere.

ZDM Subject Classification: G10.

Introduction

The paper gives a selection of experiences in teaching about an educational project, at the Department of Mathematics, Faculty of Elementary and Nursery School Teachers' Training, Eötvös Loránd University, Budapest, Hungary. The first courses at our department started at 1996.

The rationale that underlies the method of comparative geometry can be formulated in several ways [12]. In a work of Kárteszi [7] we find three psychological-didactical postulates in the teaching of geometry:

- Comparing and contrasting properties of the plane with properties of another well-known surface leads to a deeper understanding of the concept of the plane.
- The concerted process of learning and teaching requires continuous comparison and contrast. Without these activities, comprehensive understanding and operative knowledge remain unattainable for the student.
- Illustration and manipulation are instrumental in the teaching of geometry, because they give way to a quicker, deeper and more effective understanding of the concept and its consequences.

In our approach to the theory of learning [9] we have found many connections with Hejný's ideas. He worked out an axiomatic system based on simple facts of plane geometry, and compared it with other geometric systems [2]. In addition, he formulated and tested his theory of learning based on 'Atomic analysis' ([3], [4]) that appears to be very close to our definition of 'elementary actions'.

It was also encouraging that our project has been studied, tested and developed further in Hungary and several countries of the world. We refer to Vásárhelyi's article [18], Jaakko Joki's book in Finland [6], and Zionice Garbelini Martos' dissertation in Brazil [14].

In content and style, we found Lakatos' book [8] very close to our theoretical considerations. We think that the goals that Lakatos reached by studying three-dimensional solids, can also be accomplished by comparative geometry, in a palpable and demonstrative manner.

Similarly to our project, Henderson's work [5] deals with comparative geometry on the Euclidean plane, on the spherical surface and on the hyperbolic plane. His book is mainly addressed to university students, but the clarity, the examples and models seem to be the closest to our own approach to geometry teaching.

The first surveys about the project date back to the eighties. During the last two decades, pupils of a wide range in age and ability took part in various courses that were adapted to the special needs of primary, secondary and higher education.

From the nineties, Julianna Szendrei [17] initiated the extension of the project to primary and even nursery teachers' training. In tertiary education, she pioneered to introduce this material out of the scope of prospective mathematics teachers.

The present paper originates from reconciled contributions of the two authors. On the one hand, the theoretical background, design of adequate manipulative tools and some earlier results were given in a paper of Lénárt [9]. On the other hand, Ágnes Makara [12] worked out the methods of connecting the material with our curriculum, and with the needs of our students to make use their own learning experiences in their future teaching.

Aims of teaching geometry in teacher training

Acquisition of knowledge in:

- Geometry
- Didactics of mathematics
- General knowledge

Development of abilities and skills:

- Development of problem solving abilities through mathematics
- Developing abilities of communication and empathy

Main objective of the project

Even among highly motivated students, we have only too often found self-controversial, misinterpreted concepts, fragmental and inoperative knowledge of two- and three-dimensional geometry.

Moreover, even in cases when students possess adequate knowledge about a concept, they often have trouble to put it into exact wording. We got uncertain or obscure answers to questions such as: "What is the difference between the straight line and other lines of the plane?" "What is the definition of perpendicular lines?" "What is the connection between different definitions of parallel lines?" "What is a polygon?" "What is an interior angle and an exterior angle of a triangle?" "What is the radius and the centre of a circle on a given surface?" These questions which might seem so simple, proved very difficult when the properties of different surfaces, different geometric systems had to be taken into consideration. For example, when defining the radius of a spherical circle, most of the students tried to apply segments of straight lines instead of arcs of great circles. Likewise, it was not easy for them to give a definition of polygons that includes biangles (or even unigons!) on the sphere.

Our aim was to give our students self-confidence, clearer understanding of geometric concepts, direct experience in mathematical discovery, and joy and satisfaction in their mathematical studies.

Basic idea

To attain our main objective, we compare the geometry of the plane with the geometry of the spherical surface. Sometimes we adventure into 3-D space, and at the end of the course we give a very brief outlook of the Bolyai–Lobachevskian hyperbolic geometry, based on the hemispherical Poincaré model.

The project is built on elementary notions of synthetic geometry and on the four arithmetical operations. Spherical trigonometry is not included, although the project greatly helps understanding in this field, too.

Manipulatives

In the plane: sheets of paper & traditional ruler and compass; in some cases, the computer screen.

On the sphere: Lénárt sphere & accessories. A football-sized, transparent, plastic sphere, with a supporting torus, spherical 'draft papers', that is, hemispherical plastic transparencies that can be fit onto the sphere, and marked and wiped; spherical ruler and compass; and a map projection to create a draw – on globe out of the sphere. The surfaces can be marked with OH-markers and pens, and the drawings can be corrected or wiped off with a damp cloth or alcoholic wad.

Students and circumstances

At the courses, the total number of students was about one hundred, mostly women in their early twenties. The ratio of females/males at such colleges in Hungary is about ten to one, although the trend is recently changing for the males’ favour. About 80% of our students came to the college with ‘excellent’ or ‘good’ marks in mathematics at their secondary school maturity exams. They took up mathematics at our college as a compulsory subject. They are intelligent young people, interested in maths, natural sciences and humanities – and fashion and dancing and politics and many other topics.

After finishing their studies they will become qualified teachers of all subjects in the first four grades of primary school, and also teachers of maths at the fifth and sixth grades. Math takes up about one-sixth of their lessons. One-fifth of their math lessons deal with geometry.

They attended two consecutive semesters in geometry, with a total time of about 60×45 minutes.

Our department is well equipped with computers, software materials and manipulative devices. Students could have worked individually, with one sphere per one student, but they chose to work in pairs, with one sphere kit for each pair. The atmosphere of the lessons allowed them to communicate freely between each other, not only within a pair, but within the whole group.

In each group, the first lesson of the course was devoted to a survey on the level of geometric knowledge of our students. The test papers of this survey were [12] always written anonymously. We did not require mathematically exact definitions; and we only asked for the outlines of the proofs, without detailed discussion.

In these preliminary tests, the performance of students was about 15–20%. In problem solving exercises, where mathematical content was hardly above the level of upper elementary school, the output was about a mere 10%.

Goals

- Brush up secondary school knowledge in geometry.
- Create new knowledge embedded in the old context, mainly for clearing up and re-structuring geometric concepts.
- Develop abilities of problem solving.

- Apply old and new tools in geometry learning and teaching.
- Connect geometry with geography and other subjects, with emphasis on applications and real-life situations.
- Arouse interest in the history of science.
- Develop abilities of communication, empathy and creative debate.
- Learn about methodology of geometry teaching for 6–12-year-olds, with emphasis on the role of demonstration and manipulation.

Summary of the curriculum of the project

Historical aspects of geometry (Points of interest selected from Babylonian, Egyptian and Greek mathematics, such as the golden section, constructing regular pentagon, the theorem of Pythagoras, the origins of Euclidean geometry, etc.)

Basic concepts of geometry in plane, space and sphere

Mutual position of basic geometric objects in space

Measurement in plane and on sphere (Concept of measurement. Measurement of distance and angle)

Classification of shapes in plane and on sphere (circle; polygons)

Concept and measurement of area in plane and on sphere

Geometry of the triangle in plane and on sphere

Geometric transformations in space, in plane and on sphere. Similarity and congruence

Geometric constructions in plane and on sphere

Space geometry (Classification of solids; regular solids, theorem of Euler about vertices, faces and edges; measurement of surface and volume of solids; connection between regular solids and the sphere)

Axiomatic foundation of geometry (Euclides, Bolyai, Lobachevsky, Riemann, Hilbert)

The list above demonstrates our intention to select topics that are just as important for traditional plane geometry as for spherical or any other non-Euclidean geometry.

On the other hand, most of the concepts connected with spherical geometry are part of traditional Geography curriculum of upper elementary and middle schools around the world.

Findings, reactions and explanations

Following are examples of our findings, with attempts to throw some light on them. We are well aware that our attempts of explanation are not by any means the only possible ones. Our aim here is to give an outline of the fundamental structure of our project and an impression about the atmosphere of the lessons.

Examples when the difference between plane and sphere helps understand the concept

Example 1.

Exercise: Find properties of the great circle that are common with the properties of the straight line.

- Teacher's expectation: Make a list of physical observations, like drop of water running down on the surface or a string stretched taut; or of mathematical concepts like the shortest distance, or the line determined by two points.
- Observation: About 90% of the students feel it necessary to experiment with a piece of taut string. They discuss at length the properties of the line that is created in this way. They also perform the experiment with a drop of water running down on the tilted plane or on the sphere. One of us, teachers, with the best intentions of helping them, gives an advice: 'Do not drop the water right onto the top of the sphere. Keep the sphere *tilted*!' Surprisingly, they understand the illogical advice, and put the drop where we expect them to put.

  About 60% of the cases, a discussion of the following type takes place among our students: One of them says: 'This line on the sphere shows the shortest distance, so it must be a straight line.' Another student replies: 'It is impossible, because there are no straight lines on the sphere.' Again, the first student: 'But what is a straight line in the plane?'
- Explanation: Taking into account that our students had learnt in middle and high school about the Equator and the longitudes on the earth-globe, we were surprised to see how much they insisted on physical experimentation about

the great circle and its properties. It was direct experience that converted an empty definition into operational knowledge.

A 'tilted sphere' is mathematically inconceivable; but, as one of our reviewers remarked, our students interpreted its meaning in the gravitational field where the experiment took place. In this context, the sphere does have a 'top' point, and 'the tilted sphere' refers to a point that is different from this top point. At any rate, we teachers used this term incorrectly, so we admit that to err is human – even for the teacher.

The discussion among students as quoted above, indicates a very important moment in the development of their thinking. It might be the first occasion to examine in depth the concept of straight line. Their new experiences with the spherical great circle urged them to reconsider a rote-learned definition of the straight line in the plane.

Example 2.

Exercise: Draw a great circle that is parallel to a given great circle.

- Teacher's expectation: Discover that no such line exists.
- Observation: Students try to apply various methods of constructing a parallel straight line to the case of spherical great circles. These methods are known from their secondary school studies (see detailed didactical reference in [12]). About 30% of the students try out several positions of the ruler to construct a non-intersecting great circle. About 60% start from the assumption that an equidistant line from a great circle must be another great circle. 5–10% make experiments with drawing a 'second perpendicular', that is, try to construct a parallel by erecting a perpendicular to the original great circle, then erecting another perpendicular to the first perpendicular. After discovering the non-existence of two different parallel great circles, 10–20% begin to debate the question whether a great circle should be called parallel with itself. Usually, they end up with better understanding, but without a compromise on this issue.
- Explanation: For the overwhelming majority, the concept of parallelism is strictly connected with the geometric figure of Euclidean straight lines in the plane. Therefore they think that any property that guarantees parallelism in the plane will have the same effect on great circles of the sphere. It proves very demonstrative and helpful to discover how and why these attributes fail to produce parallel great circles. The discussions and debates within the group indicate how the rote-learned definition of parallelism is transformed

into a well-understood concept. A video recording that was taken during a lesson on this topic shows the stages of perception. Some of the students cry out loudly when meeting the AHA experience [1].

Considering the question of a line being parallel with itself, about 50% rule out self-parallelism, because they state: 'Parallel great circles have no common points.' The other half of the group say, 'Great circles are parallel if the points of the first great circle are all equidistant from the second great circle'. They conclude that a great circle can only be parallel with itself. At any rate, they must rethink the precise definition of parallelism.

It is worth remarking that the problem of parallel lines inspires students to discover a great number of concepts in non-Euclidean geometries, and get familiar with great ideas and personalities in the history of mathematics.

Example 3.

Exercise: Construct two opposite points.

- Teacher's expectation: Find different methods of construction, preferably on the spherical surface.
- Observation: At the beginning, almost all of them step out into three dimensions. 'Do we know the centre of the sphere? If so, then take a straight line through a spherical point and through the centre. The opposite point will be where this straight line pierces the sphere on the other side.' 'Let us look through the centre, and we see the opposite point on the opposite side.' 'Cut the sphere into halves, and erect perpendiculars at the centre of the section circles.'

  About 30% applies the spherical ruler to fit the top of the saddle to the point, draw the equatorial great circle, and turn the ruler to the opposite side. The top point of the saddle gives the opposite point in this position. About 10% make use of the fact already known that two great circles intersect in two opposite points. Even in this case, a number of misunderstandings pop up. Some of the students think that the two intersecting great circles must be perpendicular to yield the correct solution.

  One of the students draws two great circles with two points of intersection, and we ask: 'How is a point of intersection located with respect to the other?' She correctly puts her fingers on the opposite points, looks up with a worried face, and says: 'I do not know!' We tell her: 'But your fingers already know!' She looks down on her fingers, studies them for a moment, then looks up again, and happily says: 'Oh yes, they are opposite!'

- Explanation: However simple the existence of opposite points might seem, and however familiar the North and South Poles might appear from geography, this phenomenon is so unusual in the context of Euclidean geometry of the plane that it is quite a challenge to change the fixed way of perception. This is an example of the fact that even the simplest experience is not at all simple when a negative anticipation is already present and must be overcome by experience [16].

Examples when similar concepts between plane and sphere helps understanding

Here we can refer to a number of concepts, such as: concentric circles, perpendicular lines, vertical angles, bisectors and altitudes in a triangle, regular polygons, etc.

General observation: Similarities between planar and spherical concepts often prove helpful for the students to make a construction or grasp a theorem on the sphere. However, after several disappointments regarding one-to-one correspondence between planar and spherical concepts, students become increasingly suspicious of such coincidences. They scrutinize even more carefully the ‘similar’ cases than the ‘different’ ones.

Explanation: Although the project is based upon comparison and contrast, it is extremely useful to create a proper mixture of differences and likenesses between plane and sphere. A monotonous repetition of differences would make the curriculum boring. It is the effect of unpredictability and surprise that develops cautious thinking and logical reasoning.

Examples when the analogy between geography and spherical geometry influences students’ thinking

Example 1.

- Observation: When talking about parallel lines on the sphere, 20% of the students refer to the Equator and the Tropic of Cancer; or the Equator and the Tropic of Capricorn; but never to the Tropic of Cancer and the Tropic of Capricorn together; not they refer to the Arctic Circle or the Antarctic Circle.
- Explanation: Most of their knowledge of spherical geometry prior to this course originates from their earlier studies about the geographic coordinate

system. However, in our opinion, these studies lack sufficient foundation of experimenting on the sphere. Therefore, in the context of synthetic spherical geometry, their knowledge often proves inoperative. In the present case, they pick out one of the characteristics of parallel straight lines, namely, equidistance, and apply this property onto the sphere. Even this false analogue is closely stuck to the visual image. The Equator and the Tropic of Cancer remind them of the planar parallels; but the Tropic of Cancer and the Tropic of Capricorn are, so to speak, too far away from each other to reinforce this image, let alone the much smaller Arctic or Antarctic Circles. It was interesting for us to see how much they needed clarification of concepts in the geographical coordinate system. However, this clarification helped them to useful discussions about the fundamental concepts of Euclidean and non-Euclidean geometries.

Example 2.

Exercise: Draw a family of concentric circles on the sphere, and find the concept of polarity between poles and polars.

- Teacher's expectations: Taking into account their familiarity with the geographic coordinate system, and the connection between the Poles and the Equator of the earth-globe, we expected an easy access to the concept of polarity.
- Observation: About 70% of them have the false impression at the beginning that there is only one example of polarity on the sphere, namely, the fixed North Pole, South Pole, and the Equator. Clearly, their former experience in geography interferes here. In addition, this type of correspondence between points and lines is very unusual for the Euclidean way of thinking. Although the students accept the fact, they need much time to make real use of this experience in various tasks and exercises. A good idea to help them is to mark out a geographic name, and ask for locating its opposite point and equator on the earth-globe. Because of the oceans prevailing on the surface of our planet, it is fairly difficult to find well-known opposite places. For example, Ho Si Minh City in Viet Nam and Lima in Peru are approximately opposite. We can make this task even more challenging if we give the name Ho Si Minh City, and ask our students to make a guess at its opposite place.
- Explanation: Clearly, their former experience in geography interferes here. In addition, this type of simple correspondence between points and lines is very unusual for the Euclidean way of thinking. Although the students accept the

fact that polarity in this simple form exists on the sphere, they need a great deal of time to make real use of this experience in various tasks and exercises, to attain operative knowledge.

Examples when the use of construction tools influences students' thinking

Example 1.

Exercise: Draw freely 5–6 points in the plane and on the sphere.

- Observation: In the plane, they draw points evenly distributed on the whole sheet. On the sphere, about 80% of the cases, all the points are jam-packed within a small region that can be covered by a palm of a hand.
- Explanation: Three alternatives:
  (1) Students are much more familiar with the plane than with the sphere. Therefore, they only make use of only a small part of the spherical surface that closely resembles the plane, as with the apparently 'parallel' Equator and Tropic of Cancer.
  (2) They consider the sheet of paper as a very small part of the infinite plane. By analogy, they restrict themselves to a relatively small region of the spherical surface (proposition of reviewer).
  (3) They are not used to move the paper while marking the points on it. So they do not move the sphere, either, just mark some points on the top of it in the original position.

Example 2.

Exercise: Draw long closed lines, for example great circles, on the sphere.

- Teacher's expectation: As with the points, they will behave as conveniently as possible, and draw the line with as little effort as possible.
- Observation: They stand up, and lean over the sphere, with their heads down, in an extremely clumsy and inconvenient manner, instead of simply turning the whole sphere with a movement of their wrist.
- Explanation: For the beginners, the continuity of their own movement represents the continuity of the line. It is not only the visual impression of the line that makes its imprint on the mind, but also the movement of the hands and muscles – as described in connection with the infant's learning [15]. It was truly surprising for us to see how closely our grown – up students followed the route of the child when they encountered new and unusual circumstances, as

with the spherical surface instead of the well-known plane. We found similar phenomena when we asked them to draw a great circle through two spherical points with the help of the ruler. It took surprisingly long time for them to find the right position of the ruler. Again, they must learn to concert the visual image with the muscular movement.

Description of students' work on a given exercise

Exercise: Determine the sum of exterior angles of a triangle in the plane and on the sphere.

- Teacher's expectation: The sum of interior angles in the plane is $180°$, so the sum of exterior angles must be $540° - 180° = 360°$. On the sphere, this sum lies between $180°$ and $540°$, so the sum of exterior angles must be between $540° - 180° = 360°$ and $540° - 540° = 0°$.

- Observation: Almost all the solutions were faultless in the plane. On the sphere, we found six main categories:

  (1) Transfers the result from plane to sphere without any reasoning (6%).

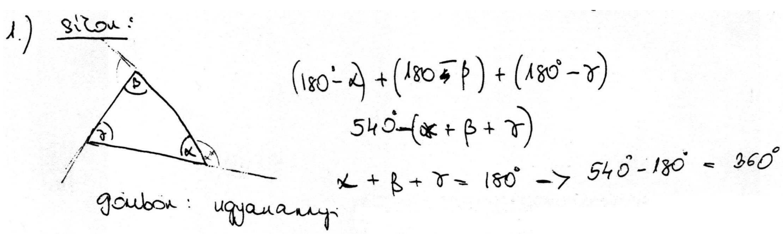

Figure 1

(Figure 1. Translation of the Hungarian text: 'In the plane'; 'The result is the same on the sphere'.)

  (2) Explores a good idea, but does not give the final result (8%).

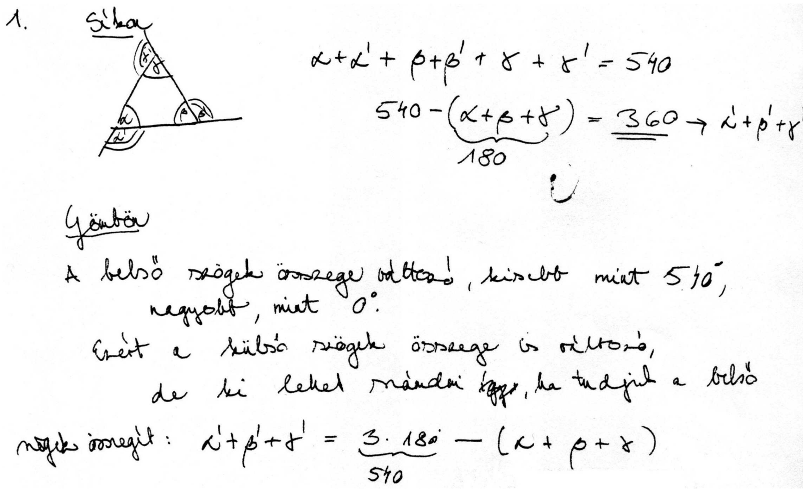

Figure 2

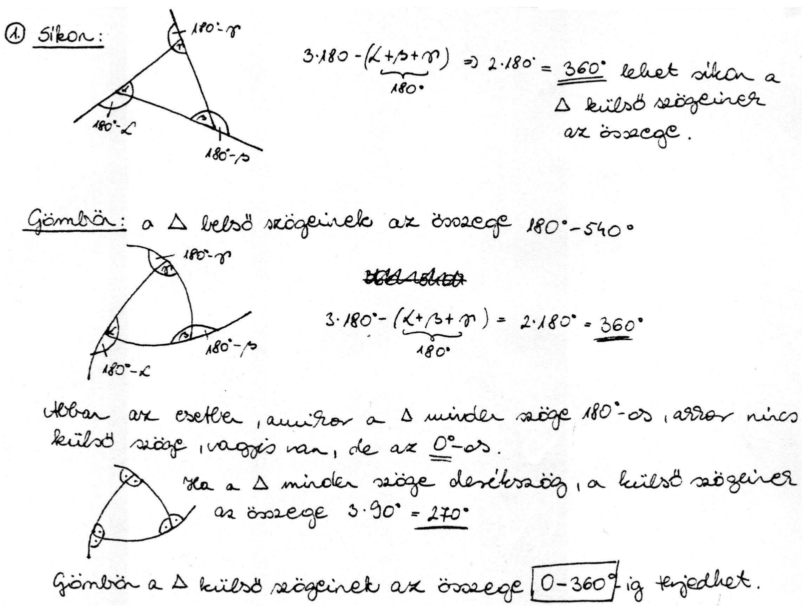

Figure 3

(Figure 2. Translation of the Hungarian text: 'In the plane'; 'On the sphere: the sum of interior angles is not a constant, but smaller than 540°, and greater than 0°. Therefore, the sum of exterior angles is not a constant, either, but can be computed if we know the sum of the interior angles'.)

(3) Gives the correct solution for the general case, then describes concrete cases, such as the triangle with three right angles (16%)

(Figure 3. Translation of the Hungarian text: 'In the plane, the sum of exterior angles can only be 360°. On the sphere, the sum of interior angles is between 180°°–540°. In case when all the three interior angles are 180°, then there is no exterior angle, or rather, there is, but it is 0°. If all the three interior angles are 90°, then the sum of exterior angles is 270°. On the sphere, the sum of exterior angles is between 0°–360°')

(4) Examines degenerated cases, but does not reach the general solution (40%)

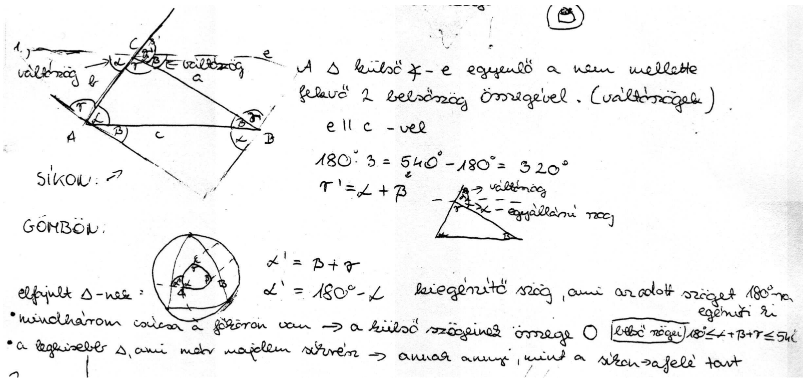

Figure 4

(Figure 4. Translation of the Hungarian text: 'In the plane, the measure of an exterior angle of the triangle is equal to the sum of the two remote interior angles (alternate angles). On the sphere, a degenerate triangle has all its vertices on a great circle, so the sum of exterior angles here is 0°. The sum of interior angles is between 180°–540°. The smaller the triangle, the closer to a planar region, so its properties tend to the properties of the plane'.)

(5) Correctly takes down the result, without any reasoning or proof (20%).

(6) Examines one single case, and incorrectly extends to the general case (6%).

- Explanation:
  (1) False analogy between plane and sphere.
  (2) Does not feel it necessary to give exact formulation of the final result; or in some cases is not able to do so.
  (3) Two alternatives:
    (a) Does not trust his own proof, and feels it necessary to examine concrete cases.
    (b) Is quite satisfied with the proof, and gives concrete examples as illustrations of their general result.
  (4) Is content with the solution for the extreme cases, and does not consider the 'in-between' cases-therefore fails to generalize the results.
  (5) Finds the exercise too simple to bother with detailed proof.
  (6) Lacks the ability to transfer from a particular case to the general solution.

One example of possible extension: Hyperbolic (Bolyai–Lobachevskian) geometry

The above experiences originate from comparing Euclidean plane geometry with spherical geometry. Euclidean geometry is modelled on a sheet of paper, spherical geometry on the surface of an eight-inch diameter plastic sphere. Luckily, there exists a hemispherical model of hyperbolic geometry, introduced by Poincaré, which can readily be connected with the geometry of the plane and the geometry of the sphere. This was displayed on a hemisphere of the same 8-inch diameter.

Hyperbolic geometry is not part of our present curriculum but we found it useful and demonstrative to pose questions that lead to hyperbolic geometry: 'What if the number of parallels is neither one (as in the plane) nor zero (as on the sphere), but something else?' 'What if the sum of angles in a triangle is neither 180 degree, nor more than 180 degree, but less than that?' 'What if the ratio of the perimeter of the circle to the length of the diameter is not π, nor less, but more than that?' When students already broke from their bonds to one fixed geometric system, they themselves ask questions of this type. At the end of the course we gave a very brief outlook to the hemispherical model of hyperbolic geometry. It was always welcomed, but not always fully understood..

The main steps for introducing the basic concepts in this manner, together with pictures of some useful models, are described in Lénárt [11].

Aspects of motivation

Among a number of favourable reactions on the part of the students, we found most encouraging their self-initiated discussions among each other, and, above all, their teaching each other with zest and vigour.

They often look at their 'official' recitation to the teacher as a pretended play, where the teacher is supposed to know all the answers, and students are only expected 'to please the teacher.' Teaching a peer is quite another matter, a real-life situation between two persons where one knows a little more and the other a bit less. In their informal talks between each other, our students explained a new concept, not only to their peers, but to themselves as well. We could follow the paths of understanding both with the explainer and with the listener. The old proverb 'Learn by teaching' has become everyday experience in our project.

At the oral exams at the end of the semester, it happened quite often that students did not leave the room, even with the top mark already written in their lecture-book. Instead, they remained to discuss and analyse difficult or obscure parts of the material between each other, or with the teacher. We took their endurance for another sign of motivation and feeling of comfort.

Why do students find comparative geometry interesting

The most surprising result of our project was the fact that even those students who considered themselves to be unmotivated, sometimes hostile, to geometry, have shown deep and sincere interest in comparative geometry.

Following are some remarks concerning these results. We emphasize again that there are many other explanations possible.

In our opinion, the fundamental choice in teaching any school subject lies between teaching only one fixed system or various systems continuously compared and contrasted with each other. In social sciences, literature or language learning it has become generally accepted to introduce different aspects of historical events or literary works, or compare grammatical structures in different languages.

In like manner, our deepest conviction is that ***only one fixed system in any branch of mathematics is dead and unteachable***. Members of modern societies grow up in the right and duty of free choice in their public and private life. Any science that claims to be built on a fixed single system sharply contradicts everyday experience in other areas of students' life.

As a matter of fact, mathematics was among the first sciences to discover the possibility of different systems of axioms within the same subject. It is really striking for us to see that mathematics education is among the last to accept pluralistic approach within its boundaries!

Two hundred years ago, Gauss believed that the society of the early nineteenth century was not ripe enough to accept several systems within the same science. Was it true or not at that time, our society today is certainly ripe, even eager for experiencing different systems in history, politics, linguistics or mathematics. The appeal of comparative geometry for the students lies in offering different approaches to the same problem, as parallelism, angle, area, sum of angles, etc. A boring monologue of one system changes into an intriguing dialogue of two different ways of thinking.

To the best advantage, the tasks given to students should be just as simple and easy on the sphere as in the plane. This aim can best be achieved with the help of the physical sphere and related construction materials, contrasted with the traditional tools in the plane.

Other advantages of comparative geometry are described in the works of Hejný, Henderson, Martos, or Lénárt, as shown among the references.

Obstacles and difficulties

It is only too natural that a new idea, a new project inevitably meets the barriers that are well-known from the history of mathematics or didactics, even the history of sciences. We would like to emphasize three main points that we encountered in this regard.

It is not easy to get used to a new family of educational devices, to find the financial sources to supply a whole classroom, or to find storeroom in a crammed laboratory. Likewise, it is not easy to find enough time for a new topic, however attractive, in a crammed curriculum. Nevertheless, we have found – without any intention of offence – that the greatest obstacle is fear of the teacher from changing the role of the infallible master to the role of partner and companion of the students in searching the truth. We warn against this behaviour because we are convinced that mathematics as a school subject is to survive in general education if and only if it changes its fundamental attitude to a less autocratic and more humanistic approach.

Conclusion

The experiences of the project helped us to formulate our hypotheses, and to design, accomplish and evaluate further research from the mathematical, didactical and motivational points of view.

As we tried to show above, theoretical considerations and heuristic didactical considerations and accordant experimental evidence have supported our belief that the project provides an accessible and enjoyable route to geometry in particular and mathematics in general. We hope that this project fosters flexible and independent thinking and action, not only for the mathematically talented, but also for a much wider audience with average interest and knowledge in mathematics. Above all, it might be a vehicle for the most important messages of all education, namely, human communication, understanding and empathy.

We venture to think that comparative geometry may be part of a living curriculum, from middle school and up, for the mathematics classroom of the twenty-first century.

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There are thousands of Internet addresses on spherical and hyperbolic geometry. We give below some of them that bear a particular relevance to our material, and were accessible in March 2004:

`http://www.madeira.hccanet.org/staff/phelps/SPHERICAL GEOMETRY.html`
`http://www.esu.edu/math/sshema/proceed02/Iseri.doc`
`http://www.math.ohiou.edu/~connor/geometry`
`http://www.math.washington.edu/~king`
`http://www.mccallie.org/myates/spherical Geometry contents.htm`
`http://www.towson.edu/~gsarhang/Module for Spherical Geometry.doc`

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