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Teaching Mathematics and Computer Science

Dynamic geometry systems in teaching geometry

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"Geometry is the art of correct reasoning on incorrect figures."
George Polya, writing on the "traditional" mathematics professor
in his book *How to solve it: A new aspect of mathematical method*

Abstract. Computer drawing programs opened up new opportunities in the teaching of geometry: they make it possible to create a multitude of drawings quickly, accurately and with flexibly changing the input data, and thus make the discovery of geometry an easier process. The objective of this paper is to demonstrate the application possibilities of dynamic geometric systems in primary and secondary schools, as well as in distance education. A general characteristic feature of these systems is that they store the steps of the construction, and can also execute those steps after a change is made to the input data. For the demonstration of the applications, we chose the Cinderella program. We had an opportunity to test some parts of the present paper in an eighth grade primary school.

Key words and phrases: comparative geometry, computer drawing programs, dynamic geometric systems.

ZDM Subject Classification: G12,B42,U52.

1. Introduction

In the United States it turned out as early as the mid-1970s that students lag far behind the previous generation as far as their level of knowledge and education is concerned. Pedagogical experts were looking for ways how learning could be made more effective than before. The appearance and quick spread of computers gave a significant impetus to such research. The emergence of computer programs in education forces people to rethink their views, as we have the reconsider what we regard as important. Formal knowledge is devaluated (since the computer can perform such tasks faster and better), and a higher value is associated with the capability for problem solving.

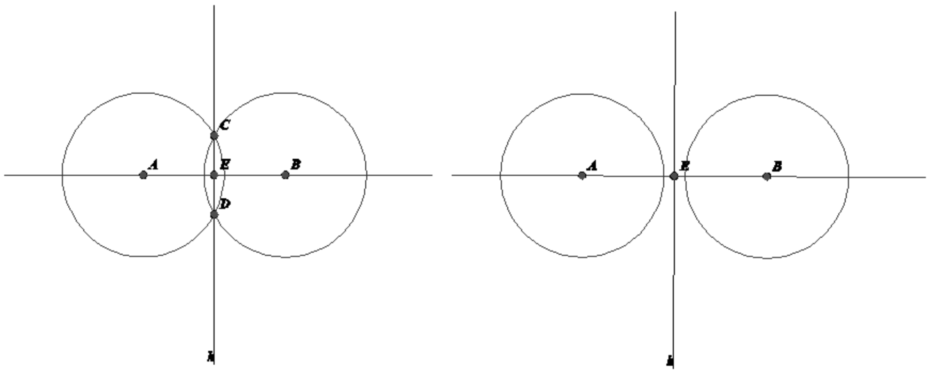
The way to problem solving in geometry starts with drawings, since we need accurate drawings to arrive at the right conclusions. As an example, let us consider the well-known theorem that in any triangle the altitudes intersect on another in the same point. In order to convince the students of the validity of this statement, we may ask a student to construct a drawing. Unfortunately, in practice, the three lines will rarely intersect in one point. Just a little inaccuracy will result in the altitudes to be off a little bit. Instead of being educational, the drawing will confuse the students. Therefore, accurate drawings are an absolute necessity in education and practice. Computer-based drawing programs provide assistance with regard to this objective, opening up new opportunities in the teaching of geometry. They make it possible to create a multitude of drawings quickly, accurately and with flexibly changing the input data, and thus make the discovery of geometry an easier process. Conclusion: to problem solution process the dynamic geometric systems (DGS) are good match.

The objective of this paper is to demonstrate the application possibilities of dynamic geometric systems in primary and secondary schools. For the purposes of demonstration we chose the Cinderella program, which of course is not the only possible choice (Euklides, Cabri, Euklid, etc.).

2. The main characteristic features of dynamic geometric systems

A dynamic geometric system is not just a computer-aided drawing tool that makes the preparation of a concrete, static drawings possible; rather, it regards as dynamic units. A general feature of these systems is that they store the steps of the construction. Using the example mentioned in the introduction, we can easily check the fact that the three altitudes go through the same point, and not only for one specific triangle, but for a rather large set of different triangles. This way we can avoid the situation of having obvious interrelationships between various parts of the drawing that the original conditions of the problem did not call for. The requirements toward a DGS raise a very important question: to what extent is the construction executable after a modification of the input data. Cinderella also provides an answer to this question.

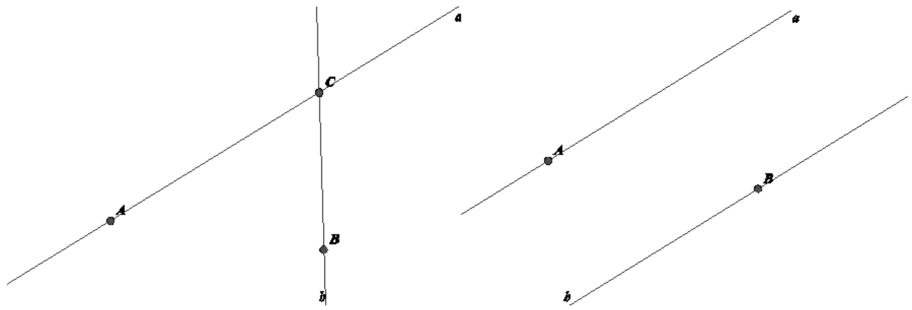
Consider the following situation: You draw two circles that intersect each other in two points. Next you join these two intersection points by a line, the so-called radical axis of the two circles. By moving the circles you can move this line. It will always perpendicular to the line connecting the centers of the circles and at a distance to each center that reflects the ratio between the radii. What should happen in a DGS if you move the circles apart? The two intersections disappear, and so does the radical axis? Or even worse: you are not allowed to move the circles in this way? Cinderella takes the following approach: recall that the two crossings are the common solutions of two quadratic polynomials, which happen to be real when there are two visible intersection points, and which are complex otherwise. Thus carrying out the calculations in the field of complex numbers always gives us two complex intersections whose connecting line happens to be real, and this is exactly the radical axis (Figure 1). This small example shows that the introduction of complex number calculations greatly simplifies geometric constructions by eliminating "vanishing" intersections.



<table>
<tr><td>C0</td><td>Circle(A;0)</td><td>(x + 3.36)2 + (y - 1.48)2 = 7.712</td></tr>
<tr><td>C1</td><td>Circle(B;0)</td><td>(x - 14.06)2 + (y - 1.4)2 = 7.722</td></tr>
<tr><td>C</td><td>Intersection(C0;C1)</td><td>(5.35 + i*-0.02|1.44 + i*-4.06)</td></tr>
<tr><td>D</td><td>Intersection(C0;C1)</td><td>(5.35 + i*0.02|1.44 + i*4.06)</td></tr>
<tr><td>h</td><td>Join(C;D)</td><td>y = 218x - 1165.39</td></tr>
<tr><td>E</td><td>Meet(a;h)</td><td>(5.35|1.44)</td></tr>
</table>

Figure 1. Construction radical axis

On an Euclidean plane, two lines intersect one another if they are not parallel. Cinderella performs the calculations on a projective plane, and therefore, it will be able to interpret the point of intersection of the two parallel lines at infinity with homogeneous co-ordinates. It will draw the third line parallel with these, across the given point at infinity (Figure 2).



A	Point(-1.72|-0.80)	(-1.72|-0.80)
B	Point(9.32|-2.28)	(9.32|-2.28)
a	LineThrough(A;31.430)	y = 0.61x + 0.25
b	Parallel(a;B)	y = 0.61x - 7.98
C	Meet(a;b)	(1:0.61:0)

Figure 2. Intersecting lines on Euclidean and projective planes

From the perspective of the user it is important to note that CINDERELLA performs the calculations on the complex projective plane. From an educational point of view, this may be perplexing (the execution of the previous construction may even case problems to a secondary school student). The teacher must take this into consideration when designing the application, and avoid any emerging problems this way.

3. Application possibilities

We have examined the following application possibilities of CINDERELLA in the teaching of geometry:

- Searching for sets of points (simple search for sets; the method of omitting conditions)
- Discussion, examination of limit values
- Theorem checking
- Restriction on construction tools used
- Comparative geometry

All application possibilities have been illustrated by one worked out examples.

4. Some means of developing creativity

The development of students' creativity is one of the most teacher dependent areas of mathematical education. In this area, Cinderella may be useful in improving students' skills.

4.1. Searching for sets of points

4.1.1. Searching for sets

Cinderella provides an opportunity for locating sets of points even if it is not a circle or a line in question, since the program is capable of recognising even a conic section. We must define which point is to move on which object, and the movement of which point the program should draw as a consequence. Cinderella may be used for searching for sets of points in secondary school, since the students are aware of the concept of conic sections. Our objective is to help students arrive at a guess. If we want a dynamic drawing, we can also make animations and even save them in HTML format.

Example 1. M *is a point in the interior of a given circle. The vertex of a right angle is M and its arms intersect the circle at the points* A *and* B. *What is the locus of the midpoint of the line segment* AB *as the right angle is rotated around the point* M? (Source: KöMaL, November 2002, B 3588) (Figure 3)

4.1.2. The method of omitting conditions

The method of omitting conditions is a frequently used solution type in geometry. When we search for a set of points satisfying several conditions, then we first locate the sets of points satisfying the individual conditions separately, and then we take the common part of those sets.

A classical problem of Polya is the one concerning the "Gothic window". *(A triangular area is enclosed by a straight line* AB *and two circular arcs,* AC *and* BC. *The center of one circle is* A, *that of the other is* B, *and each circle passes through the center of the other. Inscribe into this triangular area a circle touching all three boundary lines.)* [6]

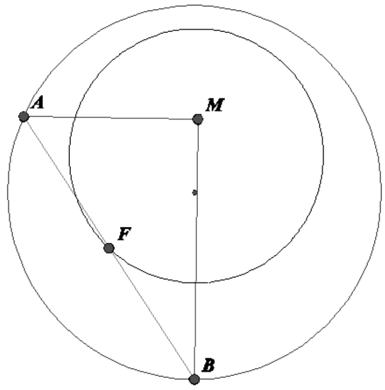


Figure 3. B 3588

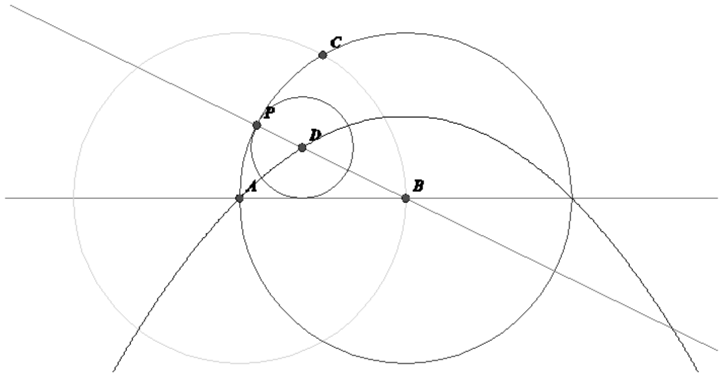
Using the method of omitting conditions, the task is to solve the interesting problem of finding a circle satisfying two conditions (one that is tangential to one of the circles as well as to the section). If we set the point of tangency, suppose on the circle, with the use of CINDERELLA we can construct the desired circle with central similarity.

Once the student has solved this problem, he or she can use the program to search for the set of solutions while the point of tangency is moving along the circle. It is not only on the basis of the resulting drawing that will suggest a parabola, but by displaying the construction list, the formula of the parabola is also recognisable (Figure 4). But of course not that way must solve this exercise, the Pythagoras-theorem gives the solution.

4.2. Discussion, the examination of limit cases

While solving a geometrical problem, we must also keep in mind that, depending on the initial data, the procedures of construction or even constructability itself may change. Therefore, the solution of constructing problems requires careful analysis. It is necessary to examine how the solution or the solvability of the problem varies if we consider not general, but special cases.

With the use of this "mouse-controlled", interactive, geometric program, a selected basic element may be moved by the mouse in any direction, and the whole construction will consistently change as a result. Therefore, the dynamic



D	Meet(c;g)	(4.02|-0.83)
C3	Circle(D;C)	(x - 4.02)2 + (y + 0.83)2 = 2.492
C4	Locus(C;C1;D)	0.01x2 - 0y2 + 0xy - 0.26x + 0.23y + 1 = 0

Figure 4. The "Gothic window" problem and the construction list

behaviour of the drawing becomes possible even from as early as eighth grade of primary school.

Example 2. *The point of intersection of mid-perpendiculars is the centre of a circle drawn around the triangle. How does the position of this centre depend on the largest angle of the triangle?* (8th grade) (Figure 5)

Discussion: If the triangle is acute-angled, then the centre of the circle around it is within the triangle; if the triangle is right-angled, then the centre of the circle around is the mid-point of the hypotenuse; while in case of obtuse-angled triangles, the centre is outside the triangle.

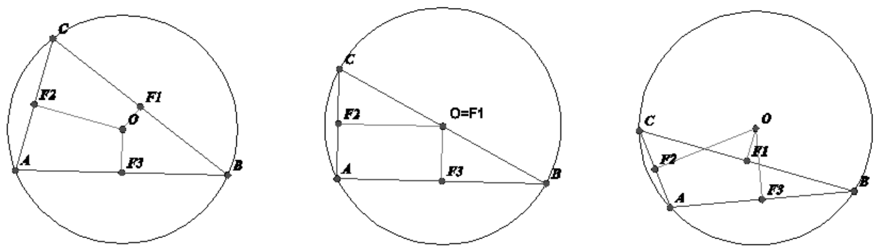


Figure 5. The position of the centre of the circle drawn around a triangle

4.3. Theorem checking

The proof in mathematics is strict deductive and formal method. But, in mathematical teaching must be enforce the proof's an another important characteristic the "understanding" character. Here the proof rather means reasons. The proof different levels are [1]:

- Experimental "proof"
- Summary-graphic proof
- Formal ("scientific") proof

At experimental "proof" the proof get verification with limited number of case. Of course this is not generally valid.

The summary-graphic active proof is making constructions and operations while is intuitively recognize, that the operations could adopt for a whole class and from that come certain conclusions.

With adoption of DGS the proofs circle increase too [1]:

- Dynamic invariance "proof": while the children were moving points they lived through certain situations. This proof level connect to experimental "proof".
- The visual-dynamic proof give answer to "Why is this in this way?" question and belong to summary-graphic proof.

The program may best be used to guess at the theorem, beginning from the primary school. (Section 5.)

4.4. Restriction on construction tools used

One good method of developing creativity is to approach known problems according to new kinds of rules. A good example for this in geometry is changing the rules of constructions, for example by putting restrictions on the construction tools that may be used. Let us consider three possibilities:

(a) using only rulers during the construction;

(b) executing old, well-known constructions using only compasses (Mascheroni-construction);

(c) a circle and its centre are given, and only rulers may be used (Steiner-construction).

Constructions to be made using only rulers are such that require points to be connected and intersection points of lines to be located. Using the latter two methods of constructions, all problems solvable by Euclidean construction can be solved, except, of course, that we cannot construct a line using the first method, or a circle using the second (Mohr–Mascheroni theorem; Poncelet–Steiner theorem).

When defining interactive problems, the teacher may control what construction tools are available in CINDERELLA, and thus creativity may be developed even from as early as eighth grade. Our objective is to execute the construction using the program.

EXAMPLE 3. *The Napoleon problem: construct a square inside a circle, using only compasses* [4] (Figure 6).

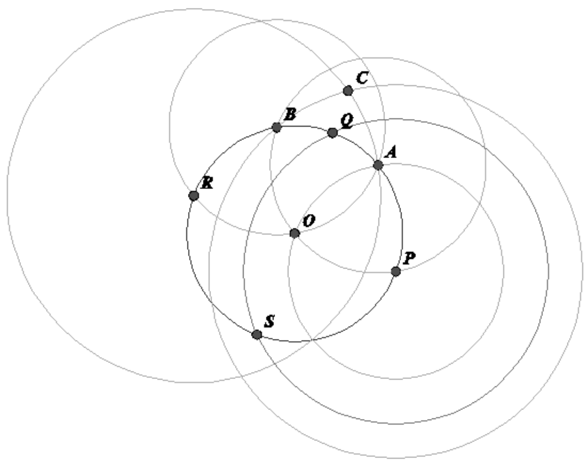


Figure 6. The Napoleon-problem

4.5. Comparative geometry

Comparative geometry aims at a multi-angle approach to mathematics. We also receive a more complete and accurate image of Euclidean geometry taught in school if we gain some insight into other systems of geometry. There have been educational experiments with comparative geometry in Hungary (conducted by Jenő Horváth, Attila Kálmán, István Lénárt, etc.) where the validity of Euclidean geometry was examined in spherical and hyperbolic geometry in regular classrooms as well as in study circles. The students responded favourably to the subjects discussed.

The Cinderella program provides an opportunity to make constructions also on a sphere (in simple elliptical geometry) and in the Poincare circle model (hyperbolic geometry), similarly to the above mentioned educational experiments, even from as early as eighth grade. Again, the objective is to help students arrive at a guess.

Example 4. *How many common points may two different lines have?* (Figure 7)

- In Euclidean geometry: max. 1.
- In hyperbolic geometry: max. 1.
- In (simple) elliptic geometry: 1.

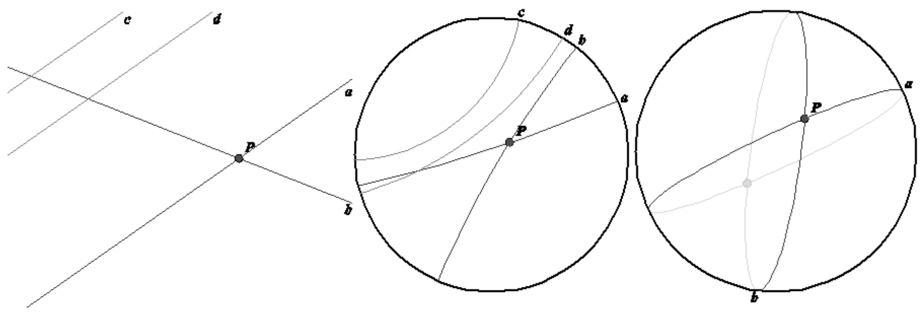


Figure 7. Common points of two different lines

5. Case study

Our experiment was carried out in Nagyrábé (Hungary) in Móricz Zsigmond Primary School on year 8 in the study circle of 7 pupils (Section 4.3). (The participation of the pupils in the study circle is not compulsory.) The pupils were happy to try out the mathematical exercises on computers and the conditions were also satisfactory since the school possessed the required quantity and quality of computers.

During the previous lesson to the experiment the pupils got familiar with the program. The goal of the preparatory lesson was dual. Both revising the previously learnt subject-matter and practising the usage of the program so the next time they would be able to use it independently without any problem. After looking through the most important tools for constructing they revised the cases of the triangle-construction. They made triangles, for all the cases when triangles can be uniquely determined by giving their special properties, using the computer program. At the end of the lesson they were allowed to make any kind of figures. As the children have been learning computer science for 2 years now – and they have already used a drawing program – this practice seemed to be efficient to get them to be confident of using this program.

During the experimental lesson the children worked mainly on their own. The type of the lesson was new subject acquiring and its title was: The altitudelines and orthocentre of the triangle. (In course of the conventional mathematics lesson this topic would be the subject of the next week’s lessons.) The children got familiar with the program during the previous lesson and they brushed up their geometric knowledge related to the topic, too. The goal of this lesson was not just the improvement of the geometric problem-solving ability and improving the pupils’ independent thinking but revealing the possible advantages and disadvantages of the dynamic geometric systems used during school lessons.

The goal of the first exercise is revising the previous knowledge (the construction of a triangle if 3 properties are specified), in the case of the other exercises the goal was acquiring the new topic (the orthocentre and its place). The instrument usage was mixed, they made the geometric constructions at first on paper and later on computer.

Exercise 1. *Construct a triangle if the lengths of the 3 sides are given.* $a = 3\,cm$, $b = 5\,cm$, $c = 6\,cm$. *Construct the altitudeline which belongs to vertex A.*

They made the drawing with compasses and straightedge in their exercise books. During the rest of the lesson they made the constructions on the computer and noted down just the commentary.

Exercise 2. *Construct acute triangle ABC and draw the altitudelines which belong to vertices A, B.*

After marking the point of intersection of the two altitudelines (M), they constructed the third altitudeline. The program automatically pointed out that point M is on the third altitudeline.

Exercise 3. *Move the vertex A to get a right triangle.*

Exercise 4. *Move the vertex A to get an obtuse triangle.*

They drew up a conclusion about where the orthocentre should be in an acute, a right, and an obtuse triangle, thereafter they put down the theorem in their exercise book.

Exercise 5. *Construct the triangle and the orthocentre if* $a = 5\,cm$*,* $b = 6\,cm$*,* $\gamma = 110^\circ$*.*

The students worked with great enthusiasm during the lesson. At the end of the lesson they printed the figures and glued them into their exercise books. (Figure 8)

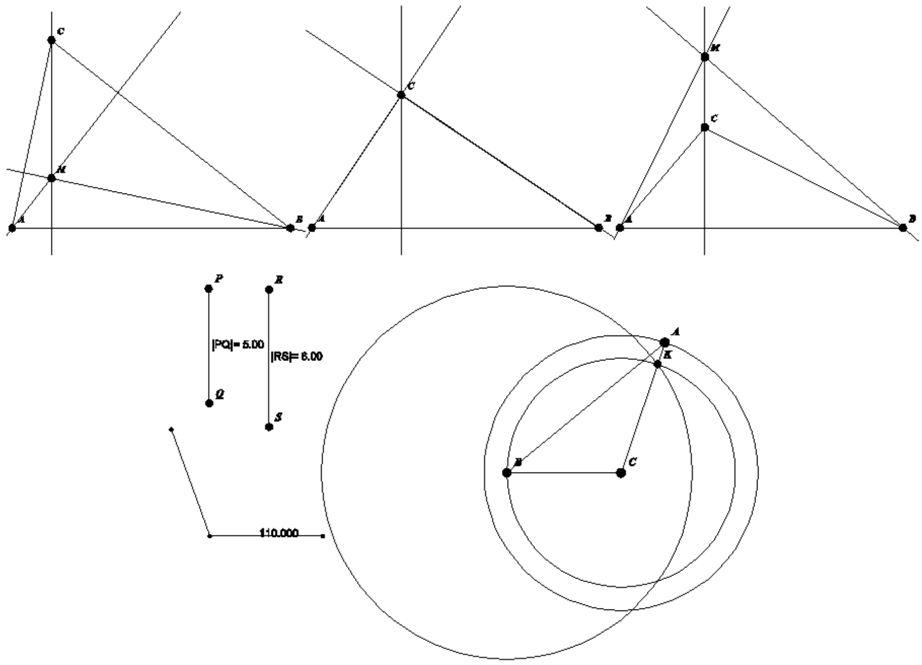


Figure 8. The figures fixed into the children’s exercise books

6. The effectiveness of the lesson, students’s and teacher’s evaluation

F. A. pupil: “At short intervals we checked lot of figure.”

B. L. pupil: “Mistakes in constructions, inaccurate constructions do not occur.”

G. L. pupil: “In the lesson we drew four triangles at short intervals, but we would have done more. When we moved the triangle’s vertex, then appeared lots of triangles too.”

B. J. pupil: “Using the program was not difficult. It is easier to construct with this program. In this way by means of icons it was easy to prosper.”

Teacher: “The children’s mathematical and computer technique knowledge stabilised by using program. We could make lots of illustrations, accurate illustration at short intervals. For children it is important to gain experience. We saw the orthocentre moving in plane.”

It is obvious that it is not only the organisation of computer-aided education, but also the products thus created require special expertise; however, evaluation in a computer-based learning environment is more life-like: it is not only the extent and accuracy of pupils knowledge that we find out about, but also whether this knowledge will be usable and retrievable when it is needed in life.

Inevitably, at the various demonstrations of study materials and education systems designed for computers relatively rarely are the possible disadvantages and pitfalls discussed. Experts usually agree that there is not much point in trying to find way of satisfying old objectives by new methods; instead, education should also strive to adjust to the requirements of the quickly changing world. The best way to decide in case of theoretical debates is an experimental test. Using them properly, mathematical software could be used in teaching real mathematical knowledge. With their highly accurate, beautiful and spectacular illustrative opportunities they can facilitate the understanding of the material (illustrating the material with hand-made drawings on the blackboard or paper is time consuming and inaccurate by comparison), so they can be well utilised in teaching mathematics.

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