1/2 (2003), 271–296
tmcs@math.klte.hu
http://tmcs.math.klte.hu

Teaching
Mathematics and
Computer Science

From iteration to one – dimensional discrete dynamical systems using CAS

Csaba Sárvári and Mihály Klincsik

Abstract. In our paper we present the basic didactical framework and approaches of a course on one-dimensional discrete dynamical systems made with the help of Computer Algebra Systems (CAS) for students familiar with the fundamentals of calculus. First we review some didactical principles of teaching mathematics in general and write about the advantages of the modularization for CAS in referring to the constructivistic view of learning. Then we deal with our own development, a CAS-based collection of programs for teaching Newton's method for the calculation of roots of a real function. Included is the discussion of domains of attraction and chaotic behaviour of the iterations. We summarize our teaching experiences using CAS.

Key words and phrases: discrete dynamical system, CAS, multiple representation, modularization, constructivistic view of learning, Newton's method, domains of attraction, chaotic behaviour.

ZDM Subject Classification: C30, D30, I90, N80.

1. Introduction

In this paper we present how one can get from the exploratory discussion of iteration and especially Newton's iteration to the emergence of some basic concepts of the discrete dynamical systems. We deal only with the one-dimensional case. The whole complexity of discrete dynamical systems can be demonstrated in this special case.

Iteration figures in most of the calculus courses. The motivation for teaching these methods of finding roots has changed with the appearance of CAS. Using computers and especially CAS we can investigate and explore more thoroughly the nature of algorithms, e.g. the order of the convergence. With the help of CAS we have a real chance to discuss and visualize concepts like the attracting region of fixed and periodic points in a discovery manner. Without exaggeration we can say that a description of the whole complexity of dynamical systems (such as chaotic behaviour) has only become possible with the use of computers.

2. Didactical principles

First of all let us summarize those basic didactical principles and methods which have led us to create a CAS-aided course of dynamical systems.

2.1. Multiple representation

For a mathematical way of thinking and communication it is necessary to represent the elements of mathematical structures as well as the structures themselves. The representations can be divided into two types.

For communication we need external representations, presented as physical objects, pictures, spoken language. (Lesh, Post and Behr, [1].)

When thinking internal representation is used. This representation particularly concerns mental images corresponding to internal formulations we construct from reality. According to cognitive science there is a relationship between internal and external representations. This indicates that the effectiveness of teaching and the comprehension of mathematical concepts depend greatly on the appropriate representation. On the basis of examining the mathematical learning process we can say that the internal representations are greatly determined by the mathematical ones.

The effectiveness of mathematical knowledge and in a narrow sense the comprehension itself can only be approached from the point of view of the organisation of the knowledge elements.

Hiebert and Carpenter [2] define comprehension in mathematics as follows:

A mathematical idea, procedure or fact can only be understood if its mental representation is part of the network of representations. The degree of comprehension is determinated by the number and strength of connections. A mathematical idea, procedure or fact is understood thoroughly if it is linked to existing networks with strong and numerous connections. Mathematical structures can be represented in various modes. Here we distinguish and use four types of representations: numeric, symbolic, graphic and descriptive. (See Figure 1.)

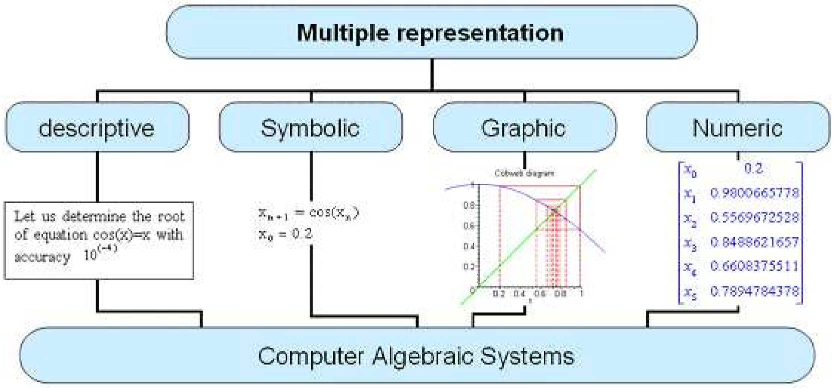

Figure 1. Multiple representation in CAS-environment

According to the US NCTM – 2000 Standard each mathematical topic should be represented by the above four types. In our opinion the use of more than one representation – but not always all the four, or sometimes one type repeatedly-helps to provide a better picture of a mathematical concept or idea. Kaput [3] says: Complex ideas are seldom represented adequately by using a single form of representation. Each representation reveals a different aspect of an idea while concealing other aspects. The ability to link different representations helps us reveal the different facets of a complex idea explicitly and dynamically.

The literature shows that students are able to work with different types of representations, and even the use of the CAS can assure a greater effectiveness of concept building and of discovery learning. But as Schneider and Peschek [4] emphasize: If the easy availability of various CAS-representations and the changing of the various forms of representation is to be used in a sensible manner, then further efforts at interpretation, that can not be underestimated, need to be employed.

Porzio [5] found that students are better able to see or make a connection between different representations when one or more of the representations is emphasized in the instructional approach that they experienced and when instructional approaches include having students solve problems specifically designed to explore the connection(s) between representations.

2.2. Modularization

Using CAS we have the possibility to use predefined functions or procedures, and we can establish such structures ourselves too. In using these structures – so-called modules – one needs to know their effects and the "interfaces" to the outside very well, so that one can apply them correctly, but at the same time, it is not necessary to know their internal structure. The main function of modules is to reduce the complexity of problems, and doing so to release the load on the mind. When working with modules, we always realize the so-called outsourcing method. In mathematics it is a constantly applied method. For example when we would like to find the fixed points of a function f we need to solve the equation $f(x) = x$. When we are working with CAS we can solve this equation with the Maple module 'fsolve'. Here we don't need to know how the applied approximating algorithm works when finding roots of this equation. This outsourcing of operative (also symbolical) knowledge to the machine frees time for thinking and reduces the complexity.

2.3. Constructivistic view of learning

As it appears to us, the constructivistic view of learning is the most supporting paradigm in CAS-aided teaching. The constructivistic view involves two main principles (Glasersfeld, [6]):

1. Knowledge is actively constructed by the learner, not passively received from the environment.
2. Coming to know is a process of adaptation based on and constantly modified by the learner's experience of the world.

Actually, learning is only successful if it is realised that a fundamental metamorphosis should develop in the mind, together with conceptual change. If the new knowledge is to fit organically into the already possessed knowledge it is necessary to produce the conditions for this conceptual change. Distrust of the interpretations and theoretical systems possessed must be created at each step. The students must be confronted with the contradictions between their existing ideas and the new knowledge element. Proceeding with constructivism in mind, we realize K. Popper's theory of knowledge, the so-called "falsification". Namely, we should attempt to disprove rather than verify our hypotheses. That means we should disprove the false hypothesis to get gradually nearer to reality. The APOS theory – developed by Dubinsky (see in [7]) – is based on the constructivistic principle. This theory begins with actions and moves through processes to encapsulated objects. These are then integrated into schemas – consisting of actions, processes and objects – which can themselves be encapsulated as objects. (See Figure 2.)

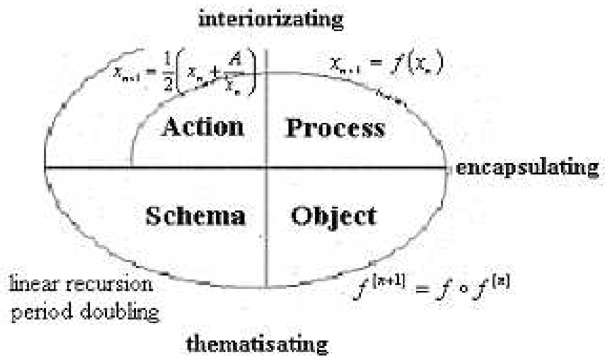

Figure 2. Learning process on the basis of APOS theory

3. Conceptual and modelling system of dynamical systems' study materials

From this point on we deal with our development. We are preparing course material concerned with discrete dynamical systems based on the previously discussed learning methods. We apply the Maple CAS, the local network and also the internet.

We can elaborate the subtle conceptual mesh and the system of the complicated relations of the discrete dynamical systems in many ways for beginners. In this paper we deal with some stages of the way which the user can walk through after accomplishing the introductory calculus course.

The knowledge of the fundamentals of differential calculus enables us to understand the process of finding the roots of equations by Newton's method. However, with the appearance of CAS it is necessary to enlarge the motivation base when we teach these methods. Namely CAS permits, in most of the cases, finding the roots by using the built-in procedures together with the graphic visualisation even without knowing the inner properties or the detailed circumstances of using the procedures. Changing and renewing the motivation can be provoked by two things. On the one hand, Newton's iteration may become chaotic when the conditions are not fulfilled, for example the function is not differentiable at the zero (see 4.4). On the other hand, the concept of iteration is a necessary requirement for the conceptual establishment of discrete dynamical systems.

Introducing the concepts for the subject of dynamical systems demands special elaborateness. Namely, it is such a new (since the 1980s) and forcefully growing mathematical topic for which it is typical that the basic concepts are interrelated with each other in a complicated manner. The course contains a lot of new concepts and constructions (e.g.: discrete dynamical system, orbit of iteration, periodic points, attraction, repulsion, bifurcation, period doubling and characteristics of the chaos, etc.) (Benkő–Klincsik, [8])

It is well-known that a discrete dynamical system is an (X, f) pair, where X is a nonempty set and f is a function mapping from X into X, i.e. $f : X \to X$. The 'dynamic' word in this definition emphasises the behaviour of the orbit: $(x_0, f(x_0), f(f(x_0)), f(f(x_0)), \dots)$ in the set X. This behaviour can be for example periodic, aperiodic, stable, unstable, everywhere dense in X, chaotic etc.

Because of the complexity of the subject it is necessary to present the proposed ways of thinking together with their visually arranged model. The hyperlink possibility of the programs (Maple, HTML) guarantees quick recalling of the concepts, and then permits the possibility of progress for the user on a personal path. The student can either continue a previously reviewed topic, because in this indirect way he/she has acquired the necessary concepts to go further, or if they find a new interesting problem or concept they can go further to understand and learn in that direction.

Perspicuity of the conceptual system and the contructions is increased by the constant display or availability of

- The column visualizing the concepts and interrelations
- The row containing the well-known examples
- The map showing the mesh of the references

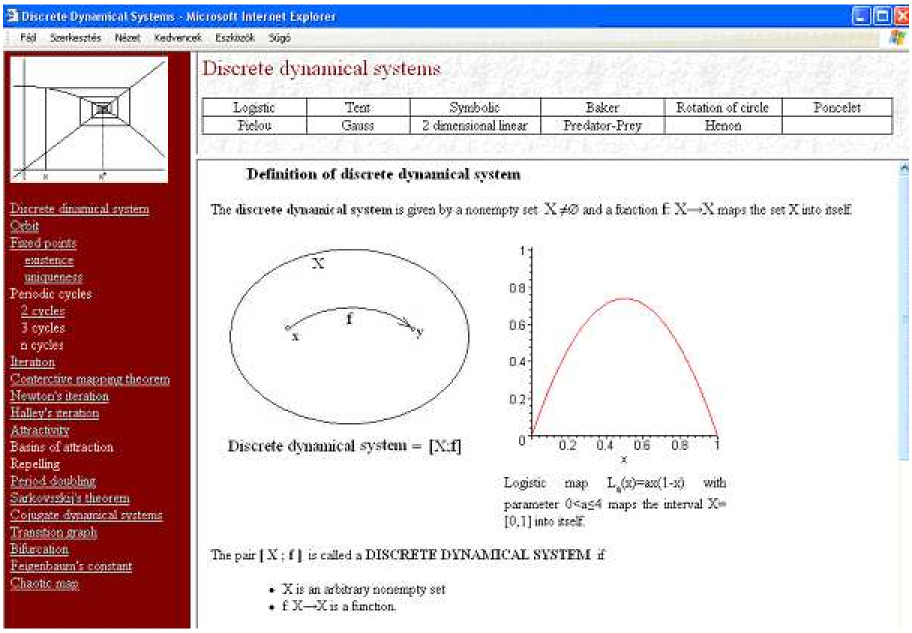

Figure 3. The structural frame of the learning material

4. On the way towards the dynamical systems

4.1. Constructive approach to iterations

Contraction mapping theorem is a milestone in the discussion of iterations. If there is a nonnegative number $\lambda < 1$ such that the inequality $|f(x) - f(y)| \leq \lambda|x - y|$ holds for all pairs of x, y from the domain of function f then f is called a contraction mapping or shortly contractive.

Contractive mapping theorem ([9]). *Let the function f be contractive and map from the subset C of the real numbers onto itself. Then f possesses exactly one fixed point $p \in C$ and moreover for any initial number x_0 from C the iteration sequence is converges to p.*

In order to the explore the evolvement of the theorem first of all we investigate special iterations, the linear ones in the form $x_{n+1} = ax_n + b$. We write a

CAS procedure performing both graphical and numerical representations of the iterations. We execute the iterations with different slopes. We divine that if the slope of the line is between -1 and $+1$, the sequence is convergent, and otherwise the iterative sequence is divergent. Additionally we can see that the dynamical behaviour of the sequences is independent from the parameter value b. (See Figure 4.)

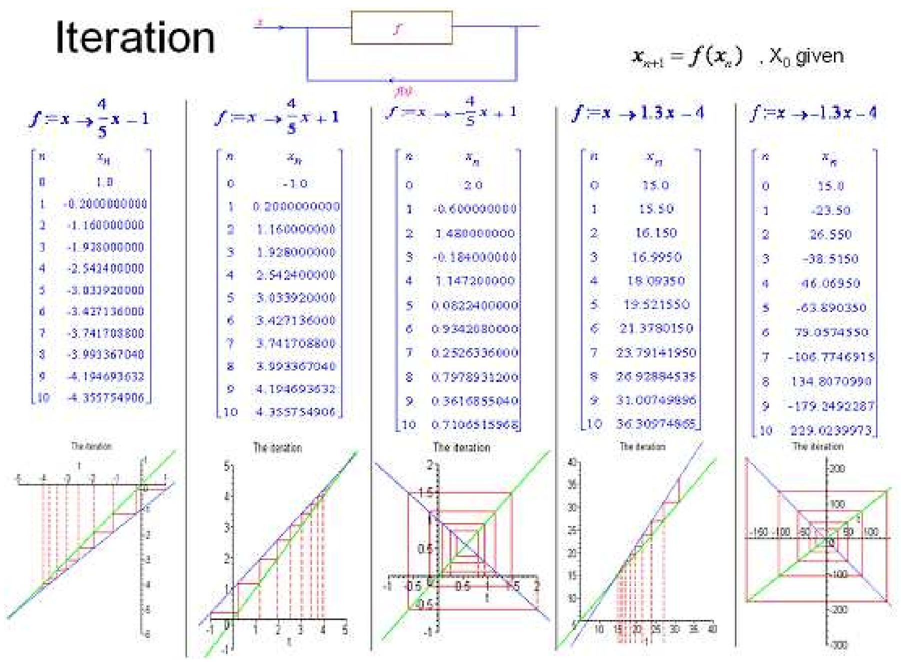

Figure 4. Dynamical investigation of the iteration sequences generated by linear functions

The proof of this conjecture is accomplished by the joint application of the symbolical and analytical representations. After this the notion of fixed point can be introduced. It is obvious that we want to search for such conditions which guarantee the convergence of iterative sequences with arbitrary differentiable functions. Whether it is enough to assume the generalization of the condition obtained in case of the linear iterations, i.e. besides the fulfilment of the $|\frac{d}{dx}f(x)| < 1$ inequality do we have to impose additional conditions? We can work by the constructive learning aspect. Investigating it in pursuance of Popper's falsification theory we are eliminating the unsuitable class of functions, and we are looking for simple conditions, guaranteeing the existence of a fixed point in the remaining set of functions.

Let us consider the function $f(x) = \sqrt{1+x^2}$. The derivative of f is $f'(x) = \frac{x}{\sqrt{1+x^2}}$ and the absolute value of this is less than 1 for all real x, even if the iteration is not convergent (Figure 5).

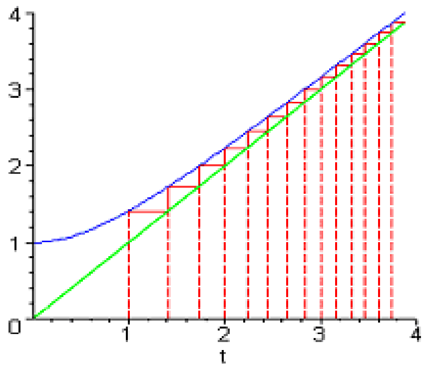

Figure 5. Although the condition $|f'(x)| < 1$ is fulfilled, the iterative sequence is divergent

There is no such interval for this function that the function maps onto itself. Now it is interesting to note, that if we consider a continuous function $f : [a, b] \to [a, b]$, i.e. f maps the interval $[a, b]$ onto itself, then a fixed point exists. Let us consider, indeed, the auxiliary function $g(x) = x - f(x)$! We can show there is a zero of the function g in the interval $[a, b]$, or, equivalently, there is a fixed point of function f. Scilicet, we can obtain from the definition of the function f the inequalities

$$a \le f(a) \le b, \quad \text{and} \quad a \le f(b) \le b.$$

From this follows that $g(a) = a - f(a) \le 0$, and $g(b) = b - f(b) \ge 0$. The continuity of the function g implies that there is a zero of g.

The emergence of this idea is crucial in developing the notion of dynamical system.

4.2. Special iteration: Newton's method and its domain of attraction

The detailed discussion of Newton's iteration is motivated by several things. On one hand, it is a natural and useful application of the differential calculus acquired; on the other hand we can investigate the applicability conditions of the method by using CAS. We note that searching for the root of the equation $f(x) = 0$ by using Newton's method is equivalent with the iteration method generated by the function $N(x) = x - f(x)/f'(x)$. In other words, it means that the set of zeros of a differentiable function $f(x)$ and the set of fixed points of the function $N(x)$ are the same. It is important to choose such an initial value of Newton's iteration, which is adequate to the conditions. Newton's iterations with different initial values lead to the classification of the fixed points or rather to the investigation of the domain of attraction. A fixed point p is called attracting if there is a $\delta > 0$ neighbourhood of p such that starting the iteration anywhere in the interval $(p-\delta, p+\delta)$ the iteration converges to p. If the condition $|N'(p)| < 1$ is fulfilled – assuming the function f is at least twice differentiable – then the fixed point p of the map $N(x)$ becomes attracting. Calculating the derivative of the function $N(x) = x - \frac{f(x)}{f'(x)}$, we obtain the formula $N'(x) = \frac{f(x) \cdot f''(x)}{(f'(x))^2}$. We can get from this that when the function is twice differentiable and $f'(p) \neq 0$, then $N'(p) = 0$ since $f(p) = 0$. Consequently, Newton's iteration is an adequate method for determining the zeros of the functions, satisfying the previously mentioned conditions. That is, if we start the iteration sufficiently close to the zero, then the iteration converges to the zero.

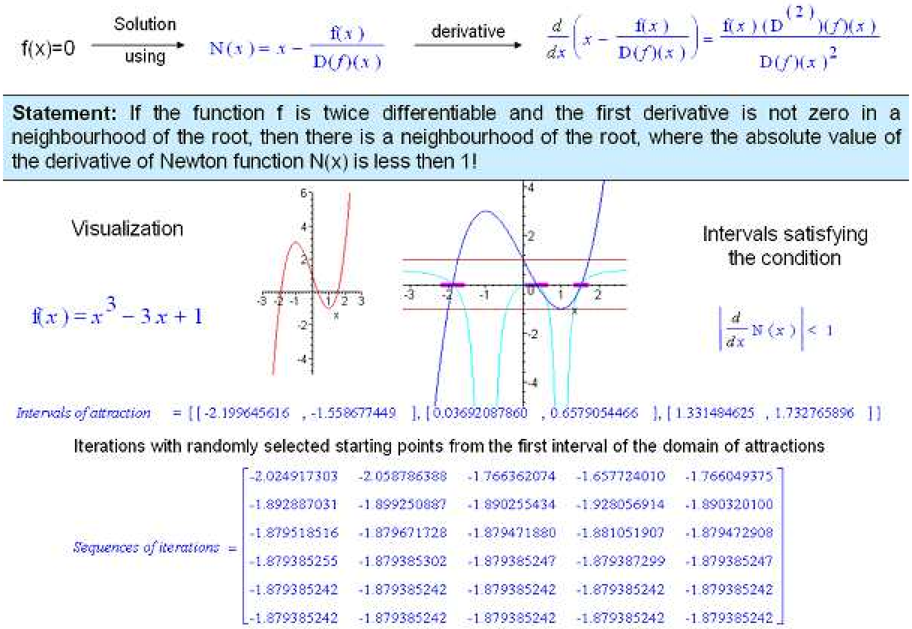

Figure 6. Investigation of the domain of attraction for Newton's iteration

The set of all points from which the procedure starts and the iteration sequence tends to the attracting fixed point p is called the domain of attraction or the basin of the attractive fixed point p. From both practical experiences and theoretical considerations it is important to analyze the domain of attraction. Figure 6 shows the results of the CAS procedure written for the investigation of the domain of attraction. The subsets of the domain of attraction are also obtained in graphical and in numerical form. The procedure is based on checking whether the condition $|N'(x)| < 1$ is fulfilled or not. It gives the domain of attraction for different fixed points and plots them. We test the procedure with randomly selected initial values from the domain of attraction.

4.3. More thorough investigation of the domain of attraction

When we investigate the iteration $x_{n+1} = f(x_n)$, the question emerges, whether the condition $|\frac{d}{dx}f(x)| < 1$ is necessary on the whole basin of attraction or not. Choosing again an experimental way and using different representations we show that the domain of attraction can be extended to such intervals where the above-mentioned condition fails. Let us analyse the iterations for the function $f(x) = x^3$. As it is well-known, f maps the interval $[-1, 1]$ onto itself, i.e. the pair $([-1, 1], x \to x^3)$ is a dynamical system. We have got three fixed points -1, 0 and 1. Now the derivative is $f'(x) = 3x^2$ and the value of the derivative in these fixed points is 3, 0, 3, respectively. Therefore the fixed point $p = 0$ is attracting. And what about the other two fixed points? These are repelling fixed points, i.e. each of them has a neighbourhood such that an iteration starting from any of its points will leave the interval. This fact can easily be verified by the closed form $x_n = x_0^{3^n}$ given for the n-th iteration. It is clear from this formula, that the entire domain of attraction of the fixed point $p = 0$ is the whole open interval $(-1, 1)$. However there are places in this interval where the derivative is greater than 1. This means that the condition $|f'(x)| < 1$ is not necessarily fulfilled in each point of the domain of attraction. However, in this case we have to look for another method for surveying the domain of attraction. Nevertheless, presenting this method is not simple. We would like to demonstrate this with the following example (Alligood, K. T., Sauer, T. D. & Yorke, J. A., [10]).

Let us investigate the fixed points of the function $f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x$ and their domains of attraction. Similarly to the previous example the fixed points are $p_1 = -1$, $p_2 = 0$ and $p_3 = 1$. Calculating the derivative we find that the fixed points p_1 and p_3 are attractive, and p_2 is repelling. Let us determine the basins of the attracting fixed points -1 and 1. With the investigation of the iteration sequences we have a presentiment and then we can prove that the open interval $(-\sqrt{3}, 0)$ belongs to the domain of attraction of the fixed point $p_1 = -1$, while the interval $(0, \sqrt{3})$ is a subset of the domain of attraction for the fixed point $p_3 = +1$.

A more precise survey of the domains of attraction leads to an extremely interesting result. Namely, let us analyze what the image of the different intervals is under this special map f!

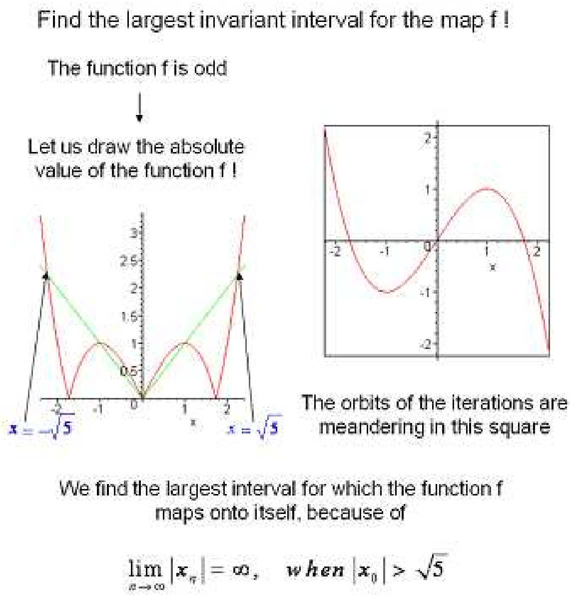

Figure 7. The interval $[\sqrt{-5}, \sqrt{5}]$ is invariant under the map f

On analyzing Figure 7 we can see that the function f maps the interval $[-\sqrt{5}, \sqrt{5}]$ onto itself and it is the largest interval with this property. Therefore we can say that the pair $(|-\sqrt{5}, \sqrt{5}|, f)$ is a dynamical system. It can also be proved that the iteration sequence with the initial values of $|x_0| > \sqrt{5}$ tends to $-\infty$ or $+\infty$, alternatively if we separate the terms of the sequence with odd and with even indices. Hereafter we have to investigate only the iterations starting from the remainder intervals $(-\sqrt{5}, -\sqrt{3})$ and $(\sqrt{3}, \sqrt{5})$ respectively.

$$\text{Orbits} = \begin{bmatrix} 1.71 & 0.06489 & 0.09721 & 0.1453 & 0.2165 & 0.3197 & 0.4632 & 0.6451 & 0.8334 & 0.9607 & 0.9977 & 1.000 & 1.000 \\ 1.72 & 0.03578 & 0.05364 & 0.08038 & 0.1203 & 0.1796 & 0.2665 & 0.3903 & 0.5557 & 0.7478 & 0.9126 & 0.9889 & 0.9998 \\ 1.73 & 0.006142 & 0.009212 & 0.01382 & 0.02073 & 0.03108 & 0.04661 & 0.06987 & 0.1046 & 0.1564 & 0.2326 & 0.3427 & 0.4939 \\ 1.74 & -0.02401 & -0.03601 & -0.05399 & -0.08091 & -0.1211 & -0.1808 & -0.2682 & -0.3926 & -0.5587 & -0.7509 & -0.9146 & -0.9894 \\ 1.75 & -0.05469 & -0.08195 & -0.1226 & -0.1831 & -0.2715 & -0.3973 & -0.5645 & -0.7568 & -0.9185 & -0.9903 & -0.9999 & -1.000 \\ 1.76 & -0.08589 & -0.1285 & -0.1917 & -0.2840 & -0.4146 & -0.5863 & -0.7787 & -0.9319 & -0.9932 & -0.9999 & -1.000 & -1.000 \end{bmatrix}$$

Figure 8. The dynamical behaviour of the iteration changes at the zero of the function

Let us consider the iteration sequences together with Figure 8. It is distinctly visible that there is a turning point in the dynamical behaviour of the iteration sequences when the initial values walk through at the $z_0 = \sqrt{3} \approx 1.732$ zero of the function f. Scilicet, the iteration starting from the left hand side neighbourhood of z_0 tends to the fixed point 1, but for greater initial values (at least on a piece of the interval) the sequence tends to the fixed point (-1). The same phenomenon can be observed at the other zero $(-z_0)$ of the function f. It seems that we have described the phenomena entirely. But if we investigate further then it turns out that similar turning points follow each other sequentially. The next turning point, moving from the origin in positive direction on the x-axis, will be at z_1, where the function takes the value $-z_0 = -\sqrt{3}$, i.e. $f(z_1) = -z_0$. Since the function f is odd, the next turning point in the negative direction will be at the point $(-z_1)$, where $f(-z_1) = z_0$. This process can be continued ad infinitum, scilicet the $(n+1)$-th turning point z_{n+1} in positive direction can be obtained from the n-th one z_n by the iteration formula $f(z_{n+1}) = -z_n$. We obtain a monotone increasing sequence of the turning points $z_0 = \sqrt{3} < z_1 < z_2 < \cdots < z_n < z_{n+1} < \cdots < \sqrt{5}$ on the x-axis. The turning points in a negative direction constitute a monotone decreasing sequence and can be obtained by reflecting the sequence $(z_n, n = 0, 1, 2, \dots)$ to the origin. Figure 9 shows the obtained infinite system of disjoint intervals becoming smaller and smaller, and belonging to the basins of the attracting fixed points $p_1 = 1$ and $p_3 = 1$ alternately. It is difficult to demonstrate in CAS environment the reality of the infinite system of the turning points, where the nature of the attraction is changing. One reason is that the lengths of the intervals of the domains of attraction become smaller and smaller. Therefore when we take a step with an initial value using the iteration then we don't know exactly the number of turning points we walk through. The second reason is that we can illustrate only a finite number of the turning points using the CAS. Therefore, for an exploration of this phenomenon we have to rely on a common application of the graphical, numerical and descriptive representations.

As we have seen this whole process is a complete realization of Popper's falsification method.

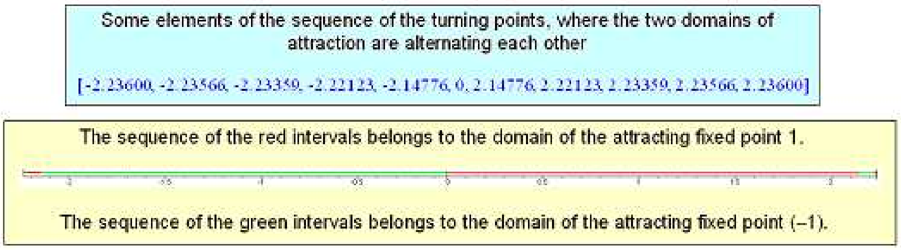

Figure 9. Infinite system of disjoint intervals as the domains of attraction

We can get closer to the explanation of the asymptotic behaviour of this dynamical system if we notice that the points $C_2 = \{-\sqrt{5}, \sqrt{5}\}$ constitute a cycle with period two, i.e. $f(-\sqrt{5}\,) = \sqrt{5}$, $f(\sqrt{5}\,) = -\sqrt{5}$. This cycle is repelling the orbits starting near to the neighbourhood of the points C_2, because in the points $-\sqrt{5}$ and $\sqrt{5}$ the derivative of the second iteration $f^{[2]}$ is greater than 1:

$$\begin{aligned}\left[f^{[2]}(x)\right]'\big|_{x=\sqrt{5}} &= \left[f(f(x))\right]'\big|_{x=\sqrt{5}} = f'(f(\sqrt{5}\,)) \cdot f'(\sqrt{5}\,) = f'(-\sqrt{5}\,) \cdot f'(\sqrt{5}\,) \\ &= \left[f(f(x))\right]'\big|_{x=-\sqrt{5}} = 36.\end{aligned}$$

The existence of an infinite sequence of intervals belonging to the domains of attraction of different fixed points is occurring because

- there are two attracting fixed points: -1, 1;
- the function f changes its sign at the zeros $-\sqrt{3}$, $\sqrt{3}$;
- the orbits move away from the two-cycle $C_2 = \{-\sqrt{5}, \sqrt{5}\}$;
- there are points x where the equation $f^{[n]}(x) = 0$ holds for all $n = 1, 2, 3, \ldots$

Now we characterise the global behaviour of the sequences $x_{n+1} = f(x_n)$, where the function is $f(x) = \frac{3x - x^3}{2}$. But we can demonstrate easily that this behaviour is changing dramatically when we multiply the function f by (-1), i.e. consider the dynamic which is induced by the function $g(x) = -f(x) = \frac{x^3 - 3x}{2}$. The fixed points of the map g are $p_1 = -\sqrt{5}$, $p_2 = 0$, $p_3 = \sqrt{5}$ and each of them is repelling, because of $g'(-\sqrt{5}\,) = g'(\sqrt{5}\,) = g'(\sqrt{5}\,) = 6$, $g'(0) = -\frac{3}{2}$. But we have a new cycle $K_2 = \{-1, 1\}$ with the length of period two

$$g(-1) = 1, \quad g(1) = -1,$$

which is now attractive:

$$\big[g^{[2]}(x)\big]'|_{x=-1} = \big[g(g(x))\big]'|_{x=-1} = g'(g(-1)) \cdot g'(-1)$$
$$= g'(1) \cdot g'(-1) = \big[g(g(x))\big]'|_{x=1} = 0.$$

The terms of the iteration sequences $x_{n+1} = g(x_n)$ with initial value $x_0 \in (-1, 1)$ tend periodically to the points of the cycle $K_2 = \{-1, 1\}$, closer and closer to 1 and after (-1) and so on (Figure 10).

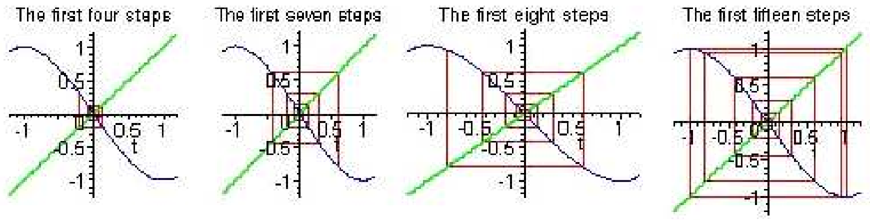

Figure 10. The cobweb diagrams of the function g from the initial value $x_0 = -0.04$

We write this situation in the form

$$\lim_{n\to\infty} d\big(x_n, \{-1, 1\}\big) = 0,$$

where $d(x_n, \{-1, 1\})$ denotes the distance between the n-th iterated value x_n and the set $\{-1, 1\}$. This situation remains true in the whole interval $(-\sqrt{5}, \sqrt{5}\,)$ except for the zeros of the iterated functions

$$g(x),\ g^{[2]}(x) = g(g(x)),\ g^{[3]}(x) = g(g(g(x))), \ldots,\ g^{[n+1]}(x) = g(g^{[n]}(x)).$$

These zeros are the same as we have seen above for the function f, and starting from these zeros the iteration is terminated at the fixed point $p_2 = 0$ by finitely many steps of iteration.

4.4. When Newton's iteration becomes chaotic

If we omit one of the conditions for the function f, e.g. if the function is not differentiable at its zero, then the behaviour of Newton's iteration sequence may be *unpredictable, i.e. chaotic.* It is well known that the iteration sequence for the logistic map $L_a(x) = ax(1-x)$ with parameter $a = 4$ is chaotic on a Cantor-type subset of the interval $[0, 1]$ (see in 5.2). Now we are looking for a function $f(x)$, for which Newton's iteration sequence is exactly the same as in the logistic map $L_4(x)$ using a simple iteration sequence. For this we have to solve the differential equation $L_4(x) = N_f(x)$, i.e.

$$4x(1-x) = x - \frac{f(x)}{f'(x)}.$$

The solution of this differential equation is the function $f(x) = \sqrt[3]{\frac{4x-3}{x}}$, which satisfies the initial condition $f(1) = 1$. If we perform Newton's iteration with this function, then it shows chaotic behaviour because it is the same sequence as the iteration with the logistic map L_4.

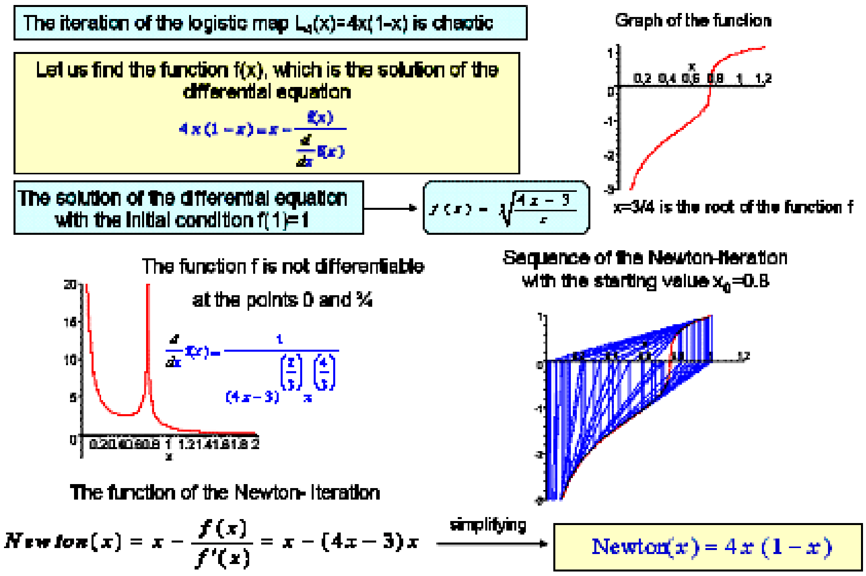

Figure 11. When the Newton-iteration becomes chaotic

We can explain this unpredictability property of the Newton-iteration sequence by the fact that the function f is not differentiable at the zero $x = \frac{3}{4}$.

5. Our experiences in teaching math with help of CAS

We have used the Maple computer algebraic system in math education at the Pollack Mihály Engineering Faculty of Pécs University since the beginning of the academic year 1997–1998. The curriculum of the calculus and that of linear algebra are treated by the application of CAS. We use the system to train students not only at the lectures and the exercises, but also with assessments (tests, examinations). The Maple contributes to making the training activity more effective within three main areas:

- High accuracy numerical calculations (there may be represented numbers with approximately 500 digits)
- Symbolic calculations (manipulating expressions, simplification, derivation, integration, solving differential equations in symbolic way and so on)
- Graphical representation (2D and 3D graphics, animations, presentation of direction and vector fields)

At the beginning of our teaching period we mostly teach the built-in procedures of the system to students. The high accuracy, fast calculator function and the great variety of graphical visualisations have got a great importance in this phase. It was soon becoming clear that there are great possibilities in the area of symbolic calculations.

The Maple system is possessed, as are other CAS, of modular structure and open architecture. This means that the user can have access generally to the source codes of the built-in system procedures and can modify them; can broaden them by new procedures and packages.

We now summarise our teaching experiences concerning algorithms, procedures and models in the following two points:

5.1. The principle of the stepwise refinement of the algorithm

It is expedient to follow the principle of the *stepwise refinement* when the algorithms are represented by using CAS. Now let us consider this principle attentively and what it means in the case of iterations. In the first step the *calculation* of the iteration $x_{n+1} = f(x_n)$ is performable when the starting value x_0 and the function f are known at least up to a fixed number of terms. In the second step we organize the written program for a unique module, the so called *procedure* in order to be able to call it with different parameters *many times.* Further refinements concern the activity mode of the procedures: we built the *stopping conditions* into the algorithm. For example the stopping conditions in the case of the Newton-iteration can be $|f(x_n)| < \varepsilon$ or $|x_{n+1} - x_n| < \varepsilon$ beside the limitation of the number of steps.

How can we develop and refine the written procedure further? For this purpose we use multiple representations. We can use our procedure to produce a *cobweb diagram.* The procedure can also be developed so as to enable us to provide the graphical and numerical representations of the *orbit.*

5.2. Investigation of the representability of equivalence models

In our course we investigate some different – but equivalent – models showing chaotic behaviour, which demonstrate the common properties (periodicity, stability, instability, chaos) of discrete dynamical systems. Each model is a variation on the same theme, chaos. In this manner we introduce the concept of the chaos versatilely to the students. When we define and compare the models with each other then we have possibilities for representation without using CAS (descriptive, symbolic, numeric representations) and also using CAS (graphical, symbolic, numeric representations). However, we emphasize the most profitable representation for each construction.

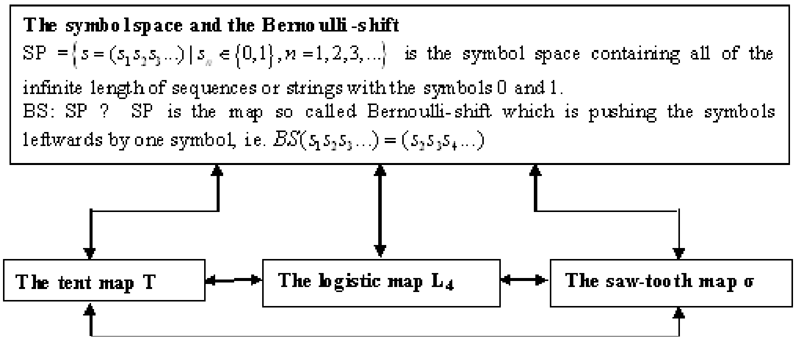

Figure 12. Equivalent models, where the symbol space is the essenrial one

Figure 12 shows four well-known chaotic discrete dynamical systems or models where the equivalences are indicated by the double arrows between the boxes. We give several such equivalence relations – which are called conjugacies in the theory of dynamical systems – and we hope that the students will discover the others.

We consider the symbol space SP with the Bernoulli-shift map BS as the essential dynamical system out of these four because so we can exhibit the *chaotic properties* more easily than in the other cases. These properties are among others the existence of an *everywhere dense orbit* and the *sensitive dependence on the initial values.* Because the elements of the symbol space are the infinite sequences with 0 or 1, the descriptive representation and approach will better fit this model than the CAS realization which is not so adequate in this case. However, the three other dynamical systems are suitable for representation of their properties using CAS.

In the following five points are collected our investigations into the models and the equivalence relations between them, and furthermore about using the appropriate representations for these constructions.

INVESTIGATION 1. *The Bernoulli-shift BS describes well the phenomenon of chaos in the symbol space SP but it cannot be represented well by using CAS.*

To be able to describe the properties of the chaos we have to measure the distance between the elements of the symbol space SP. Therefore we define the distance between two arbitrary strings $s = (s_1 s_2 s_3, \dots s_n \dots)$ and $t = (t_1 t_2 t_3, \dots t_n \dots)$ of the symbol space SP by the sum of the following convergent series:

$$d(s,t) = \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{2^n}.$$

The distance defined in this way will be not greater than 1 for an arbitrary pair of s, t. Now we construct the following string in the space SP:

$$s^* = (0|1|00|01|10|11|000|001|010|011|100|101|110|111|0000|\dots),$$

where we put into a list and separated by vertical lines the substrings whose lengths are equal to $1, 2, 3, \dots$ respectively. Thus the first two digits of s^* are 0 and 1 because these are the strings with length 1; the next substrings 00, 01, 10 and 11 of s^* are the strings with lengths 2 and this procedure is continued with the enumeration of the 3 length substrings and so on. We can show on the basis of this construction that using the Bernoulli-shift BS the orbit of s^* will get arbitrarily close to each string $t = t_1 t_2 t_3 \cdots \in SP$. For a verification using the map BS of this *everywhere dense property* of the orbit s^* we have to choose natural numbers n and m so large, that applying the Bernoulli-shift map BS the n-th iterated value of s^* pushes out the digits of s^* so that the first m digits of $BS^{[n]}(s^*)$ and of t will be the same. We can demonstrate this property by using CAS only for those t containing a finite number of digits 1, but using the descriptive representation we have a chance to characterise all of the cases. So we have demonstrated using the map BS that the orbit of s^* is an everywhere dense subset of the space SP.

By using the descriptive representation we can also prove the validity of another property of the chaos for the discrete dynamical system (SP, BS). This is the so called *sensitive dependence on the initial values.* We have to show the existence of a real number $0 < \delta < 1$, such that for every $\varepsilon > 0$ and for every string $s = s_1s_2s_3\ldots \in SP$ there is a string $t = t_1t_2t_3\ldots \in SP$ and a natural number n, so that the distance of t and s is smaller than ε, i.e. $d(s,t) < \varepsilon$, but the distance between the n-th iterated value of s and t under BS greater than δ, i.e. $d\big(BS^{[n]}(s), BS^{[n]}(t)\big) > \delta$. We can express this property in other words: if we start the iteration from two arbitrarily close initial values then the distance between the elements of the two iterations can get greater than a given value δ at some step. This is a simple consequence of the shifting property of the map BS. Let the number $0 < \delta < 0.5$ be fixed, while the value $0 < \varepsilon < 1$ and the string $s = s_1s_2s_3\ldots \in SP$ are given arbitrarily. Choose the natural number n so that the inequality $2^{-n} < \varepsilon$ holds and define the string $t = s_1s_2s_3\ldots s_nt_{n+1}t_{n+2}\ldots \in SP$ so that the first n digits of t and s are equal, but the $(n+1)$-th digits are different: $t_{n+1} \neq s_{n+1}$. Then from the definition of the distance we get

$$d(s,t) = \sum_{k=n+1}^{\infty} \frac{|s_k - t_k|}{2^k} \leq 2^{-n} < \varepsilon$$

but

$$d\big(BS^{[n]}(s), BS^{[n]}(t)\big) = \sum_{k=1}^{\infty} \frac{|s_{n+k} - t_{n+k}|}{2^k} \geq \frac{1}{2} > \delta.$$

Investigation 2. *The equivalence between the maps T and L_4 can be represented well by using CAS.*

When the students have become familiar with the properties of the chaos using the previous digital model then they are able to investigate the chaotic behavior of other models and to discover their connections, too. Let us continue the investigation with the comparison of the iterations $x_{n+1} = L_4(x_n)$ and $y_{n+1} = T(y_n)$, where the function $L_4(x) = 4x(1-x)$ is the so called logistic map with parameter 4 (Figure 13/a), and the function $T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x \leq 1 \end{cases}$ is the tent map (Figure 13/b). So we see the importance of the different models and the necessity of discovering similarity relations between the models.

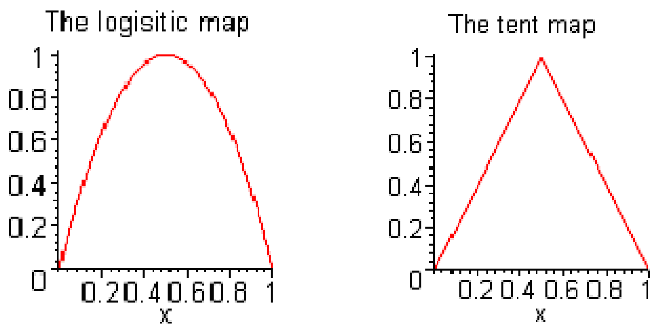

Figure 13. a) Graph of the logistic map parameter 4.
b) Graph of the tent map

Obviously the shape of these two graphs is similar if we compare their monotonic and extreme value properties. The similarities between these two maps can also be extended to the iteration sequences if we compare the cobweb diagrams of the iterations (Figure 14).

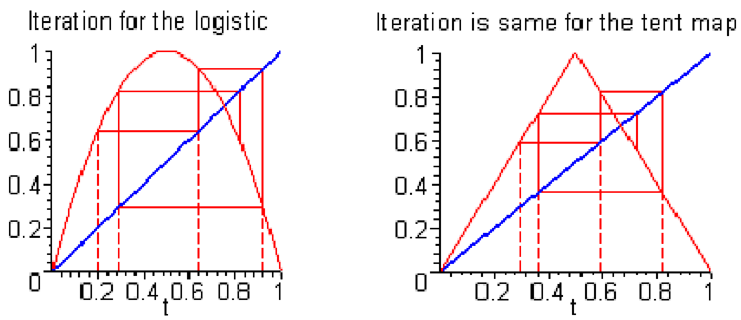

Figure 14. The iteration sequences are also similar

We can easily verify that the one-to-one map $h(x) = \sin^2(\frac{\pi}{2}x)$ on the interval $[0, 1]$ establishes these similarities between the tent and the logistic maps (Figure 15). If we compute the composite function $L_4 \circ h$, then we get

$$\begin{aligned}(L_4 \circ h)(x) = L_4(h(x)) &= 4 \cdot \sin^2\left(\frac{\pi}{2}x\right) \cdot \left(1 - \sin^2\left(\frac{\pi}{2}x\right)\right) \\ &= \left[2 \cdot \sin\left(\frac{\pi}{2}x\right) \cdot \cos\left(\frac{\pi}{2}x\right)\right]^2 = \sin^2(\pi x).\end{aligned}$$

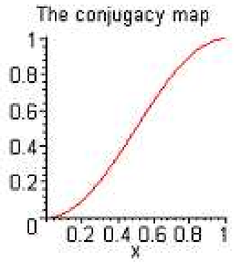

Figure 15. The conjugacy map between the tent and the logistic map

Now we obtain the same function for the composition $h \circ T$:

$$(h \circ T)(x) = h(T(x)) = \left\{ \begin{array}{ll} \sin^2\left(\frac{\pi}{2} \cdot (2x)\right), & \text{if } \ 0 \le x \le \frac{1}{2} \\ \sin^2\left(\frac{\pi}{2} 2(1-x)\right), & \text{if } \ \frac{1}{2} \le x \le 1 \end{array} \right\} = \sin^2(\pi x).$$

The equality $L_4 \circ h = h \circ T$ means conjugacy between the maps L_4 and T. Since the inverse function h^{-1} of h exists on the interval $[0,1]$, the relation

$$L_4 = h \circ T \circ h^{-1}$$

holds. Thus the similarity of the iteration sequences is a consequence of the relations

$$L_4^{[n]} = h \circ T^{[n]} \circ h^{-1} \quad (n = 1, 2, 3, \dots),$$

where $T_4^{[n]}$ and $L_4^{[n]}$ is the composition of T and of L_4 n-times by itself, respectively. We saw that the equivalence property of the tent and the logistic model can easily be visualized and calculated symbolically, analytically and numerically by use of CAS.

Investigation 3. *The equivalency of the logistic and the saw-tooth map can be represented well by the CAS.*

Let us consider the so called saw-tooth map $\sigma(x) = \left\{ \begin{array}{ll} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1, & \text{if } \frac{1}{2} \le x \le 1 \end{array} \right\} =$ $\operatorname{frac}(2x)$ on the interval $[0,1]$, where the symbol $\operatorname{frac}(2x)$ denotes the fractional part of the real number $2x$ (Figure 16).

This function σ has a discontinuity property at the value $x = 1/2$. We can compare the behavior of the logistic iteration $x_{n+1} = L_4(x_n)$ with the behavior of the saw-tooth iteration $z_{n+1} = \sigma(z_n)$ where the map L_4 is continuous and the

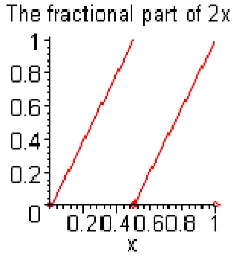

Figure 16. The saw-tooth map on the interval $[0, 1]$

map σ is discontinuous. We can carry out of this comparison by considering the function

$$g(z) = \sin^2(\pi z)$$

on the interval $[0, 1]$ (Figure 17).

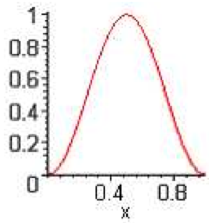

Figure 17. The semi-conjugacy map between the logistic and the saw-tooth map

We can show that the saw-tooth iteration $z_{n+1} = \sigma(z_n)$ and the logistic iteration $x_{n+1} = L_4(x_n)$ are connected by the relation $x_{n+1} = g(z_{n+1}) = \sin^2(\pi z_{n+1})$ for all natural numbers $n = 0, 1, 2, 3, \ldots$. This relation can be obtained from the equality

$$\begin{aligned}(g \circ \sigma)(z) &= \sin^2(\pi \,\mathrm{frac}\,(2z)) = \sin^2(2\pi z) = 4 \cdot \sin^2(\pi z) \cdot (1 - \sin^2(\pi z)) \\ &= L_4(g(z)) = (L_4 \circ g)(z)\end{aligned}$$

which holds for all z in the interval $[0, 1]$ because the function g has period one, i.e. $g(z+1) = g(z)$. This relation is only a semi-conjugacy, because the function g is not a one-to-one map. This similarity relation between the logistic and the saw-tooth map enables us to work with a map realizing a more effective representation.

Investigation 4. *The equivalence between the logistic or the tent model and the symbolic model can be represented well by using CAS.*

The syllabus continues with a comparison between the Bernoulli-shift with the tent map and the Bernoulli-shift and the logistic map. We can characterize these two equivalences in a similar way, by the so called itinerary calculus. We register 0 or 1 in a diary about the orbit starting from the initial value $x_0 \in [0,1]$ when the n-th element of the iteration is in the left half or the right half of the interval $[0,1]$, respectively. This registration procedure establishes order in the chaos. Symbolically we now define two functions g_1 and g_2 mapping from the interval $[0,1]$ to the symbol space SP. Our function in the case of the logistic map is

$$g_1(x_0) = s_1s_2s_3s_4\dots \text{ where } s_n = \begin{cases} 0, & \text{if } 0 \le L_4^{[n]}(x_0) \le \dfrac{1}{2} \\ 1, & \text{if } \dfrac{1}{2} < L_4^{[n]}(x_0) \le 1 \end{cases} \quad (n = 1,2,3,4,\dots),$$

and in the case of the tent map

$$g_2(x_0) = s_1s_2s_3s_4\dots \text{ where } s_n = \begin{cases} 1, & \text{if } 0 \le T^{[n]}(x_0) \le \dfrac{1}{2} \\ 1, & \text{if } \dfrac{1}{2} < T^{[n]}(x_0) \le 1 \end{cases} \quad (n = 1,2,3,4,\dots).$$

We can more easily give the systems of intervals mapping the n-th iteration $T^{[n]}$ to the whole interval $[0,1]$ for the tent map than for the logistic map. But it is not an easy task to demonstrate to the students that the functions g_1 and g_2 map the interval $[0,1]$ onto the whole symbol space SP. We can verify the last statement in the case of the logistic map by drawing the orbits for rational initial values. These figures consist of criss-cross lines which are intuitively the chaos. But the orbits in the case of the tent map for rational initial values are terminated in a periodic cycle.

The conjugacy relationships

$$BS \circ g_1 = g_1 \circ L_4, \quad BS \circ g_2 = g_2 \circ T,$$

hold with the functions g_1 and g_2 and can be deduced easily from the definitions. Let us consider the proof of the relation for example in the case of the logistic map. We have $(BS \circ g_1)(x_0) = BS(g_1(x_0)) = BS(s_1s_2s_3\dots) = s_2s_3s_4\dots$ and $(g_1 \circ L_4)(x_0) = g_1(L_4(x_0)) = s_2s_3s_4\dots$, because of $L_4\big(L_4^{[n]}(x_0)\big) = L_4^{[n+1]}(x_0)$. In the case of the tent map we can reason similarly.

Investigation 5. *The equivalence between the saw-tooth and the symbolic models can be represented cumbersomely by using CAS.*

We can show the equivalency between the dynamical systems $([0,1],\sigma)$ and (SP, BS) using the binary representation or expansion of the real number $z \in [0,1]$

$$z = \frac{s_1}{2} + \frac{s_2}{2^2} + \frac{s_3}{2^3} + \dots,$$

where s_n is the n-th binary digit of the number z which may be 0 or 1 $(n = 1,2,3,\dots)$. So we establish a correspondence between an arbitrary element z of the interval $[0,1]$ and the element $s = s_1 s_2 s_3 \dots$ of the symbol space SP, which we write as $g_3(z) = s_1 s_2 s_3 \dots$. The basic relationship between these two dynamical systems follows from the fact that the binary expression of the number $\sigma(z)$ will be the same as the binary expression of the number z, after omitting the first digit of the number z:

$$\begin{aligned}\sigma(z) &= \operatorname{frac}(2z) = \operatorname{frac}\left(2 \cdot \left(\frac{s_1}{2} + \frac{s_2}{2^2} + \frac{s_3}{2^3} + \dots\right)\right)\\ &= \operatorname{frac}\left(s_1 + \frac{s_2}{2} + \frac{s_3}{2^2} + \frac{s_4}{2^3} + \dots\right) = \frac{s_2}{2} + \frac{s_3}{2^2} + \frac{s_4}{2^3} + \dots\end{aligned}$$

So we have got the conjugacy relationship $g_3(\sigma(z)) = BS(g_3(z)) = BS(s_1 s_2 s_3 \dots)$ $= s_2 s_3 s_4 \dots$ On the basis of this equivalence we can investigate the chaotic behavior of the saw-tooth map on the interval $[0,1]$, but, basically, this equivalence cannot be well represented by using CAS.

References

[1] R. Lesh, T. Post and M. Behr, Representation and translations among representation in mathematics teaching and problem solving, *Problems of representation in the teaching and learning of mathematics*, (C. Janvier, ed.), Lawrence Erlbaum Associates, NJ, 1987, 33–40.

[2] J. Hiebert and T. P. Carpenter, Learning and teaching with understanding, *Handbook of research on mathematics teaching and learning*, (D. A. Grouws, ed.), Macmillan, New York, 1992, 65–97.

[3] J. J. Kaput, Technology and mathematics education, *Handbook of research on mathematics teaching and learning*, (D. A. Grouws, ed.), Macmillan, New York, 1992, 515–556.

[4] E. Schneider and W. Peschek, Computer Algebra Systems (CAS) and Mathematical Communication, *The International Journal of Computer Algebra in Mathematics Education* **9**, no. 3 (2002), 229–242.

[5] D. T. Porzio, The effect of differing technological approaches to calculus on students' use and understanding of multiple representations when solving problems, *Dissertation Abstracts International* **55**(10) (1994), 3128A, (University Mikrofilms No. AAI 9505274).

[6] E. von Glasersfeld, An introduction to radical constructivism, *The invented reality*, (P. Watzlawick, ed.), W. W. Naughton & Co., London, 1987.

[7] E. Dubinsky (with J. Cottrill, D. Nichols, K. Schwingendorf, K. Thomas and D. Vidakovic), Understanding the Limit Concept: Beginning with a Coordinated Process Schema, *Journal of Mathematical Behavior* **15**, 2 (1996), 167–192.

[8] J. Benkő and M. Klincsik, *Discrete dynamical systems*, Maple V programs on floppy disc, University Press, Pécs, 2000 (in Hungarian).

[9] David Kincaid and Ward Cheney, *Numerical Analysis*, Belmont, 1991.

[10] K. T. Alligood, T. D. Sauer and J. A. Yorke, Chaos, *An introduction to the dynamical systems*, Springer, 1997.

CSABA SÁRVÁRI
UNIVERSITY OF PÉCS
7624 PÉCS, BOSZORKÁNY ÚT 2.
HUNGARY

MIHÁLY KLINCSIK
UNIVERSITY OF PÉCS
7624 PÉCS, BOSZORKÁNY ÚT 2.
HUNGARY

(Received July, 2003)