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Solving word problems – a crucial step in lower secondary school education

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Abstract. Algebra is considered one of the most important parts of Mathematics teaching and learning, because it lays the foundations of abstract thinking as well as reasoning abilities among the lower secondary school pupils who have just transited from the world of numbers and computations to the area of equalities, signs, symbols and letters. The present article focuses on the fact that how the transition from arithmetic to algebra can be made more smooth. We have concentrated our experiments towards the approach of algebraic reasoning and its utilities in filling the gap between arithmetic and beginning algebra in lower secondary school education. We also underline the importance of another approach in overcoming the challenges in the transition from arithmetic to algebra, to enhance and make algebraic learning more effective, with special considerations to word problem-solving processes. In our opinion, we have to go through three phases in the introducing of algebra in Grade 7 Mathematics education: Regula Falsi method (based only on numerical calculations); functional approach to algebra (which combines the numerical computation with letter-symbolic manipulation); and writing equations to word problems. The conclusions of the present article would be helpful to Mathematics teachers for applying themselves to develop the pupils' interest in word problem-solving processes during algebra teaching classroom activities.

Key words and phrases: Regula Falsi method, functional approach, word problems, arithmetic methods, algebraic thinking.

MSC Subject Classification: 97B10, 97C30, 97C50, 97D10, 97D40.

1. Introduction – a theoretical framework

The main difficulties experienced by lower secondary school pupils learning algebra are well documented in the relevant literature (see Booth, 1984;

Collis, 1974; Kieran, 1981; Filloy & Rojano, 1984; Herscovics & Linchevski, 1991; Arzarello, 1991; Stacey & MacGregor, 1999).

Stacey and MacGregor (1995) have investigated the students' performance in using algebra to solve word problems. They found that a large number of pupils (age 14-15) had not attempted algebraic methods in spite of the instruction to write an equation for each problem and solve it. These pupils' solution methods, usually successful, included mainly informal arithmetic, guess-check-improve procedures, and complex reasoning. For the pupils who tried to use algebra, the main obstacles to success were the incorrect use of algebraic syntax, and the failure to integrate the given information as an equation (Stacey & MacGregor, 1995). Arzarello et al. (1993) have shown that writing equations for word problems is a complex process, where the choice of variables and the understanding of main relations in the problems are the most crucial steps. In our opinion, one of the biggest difficulties in teaching algebra is that most of the secondary school pupils are unable to write an equation to express the structure of the problem situation, some of them do not even understand that an equation is written to represent a problem situation. Until they achieve a certain level of fluency, pupils regard algebra as an extra difficulty or unnecessary task imposed by teachers for no obvious purpose and not a useful tool for making problem solving simple. They also often think that algebra is rule-based and attempt to apply rules by memorizing them, without proper understanding of the structural aspects of the problem situation (Fülöp, 2015). According to Wollman, many students fail to correctly represent word problems as algebraic equations, even when the problems involve relatively simple relationships.

The errors occur not because of computational weaknesses, but due to difficulties in structuring the problem mathematically. Pupils' math instruction may contribute to an impulse to calculate, performing operations on values in a problem without consideration of the relations between the values, which may contribute to the difficulty of word problems for pupils. One example for effective organization of instruction to facilitate learning based on pupils' needs is the use of scaffolding activities that encourage students to perceive the connection between algebra and numbers and operations (Wollman, 1983). Edwards identifies several "big ideas" that are fundamental for middle school students learning algebra: notation, variable, function, properties of numbers, equivalence and equality, generalisation and presentation. He emphasizes that explicitly developing ideas in school helps pupils build a coherent understanding of algebra, rather than viewing it as a collection of disconnected procedures. He encourages teachers to use real-world contexts, multiple representations, and active problem-solving strategies to reinforce these concepts (Edwards, 2000). Cortes et al. (1990) state that if pupils have learned to formulate algebraic equations for solving problems, teachers need to discourage the search for non-algebraic solutions. We cannot totally support this point of view, as algebra is only a tool for solving word problems, and the final goal is to solve the word problem itself, not solely to write equations.

Nathan et al., in their investigation of algebra word-problem comprehension, have underlined that pupils may understand a problem in everyday terms but they are unable to represent its formal aspects as required for an algebraic solution. These researchers suggest that the pupils' cognitive abilities in the representation of the problem determine what information is available for reasoning (see Nathan et al., 1992).

According to Van Amerom (2003), the rupture between arithmetic and algebra is related with the type of approach taken during problem solving, since arithmetic problems can be solved directly through direct answers, while algebraic problems are necessarily translated into a formal language and representation and then solved.

The transition from concrete and numerical foundations of arithmetic to the abstract and symbolic reasoning of algebra has been conceptualised as a "didactic cut" (see Filloy & Rojano, 1989), and as a "cognitive gap" (see Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996). Filloy and Rojano divided linear equations into two different kinds: arithmetic (or pre-algebraic) equations, in which the unknown occurs only on one side of the equation, and which hence can be solved by undoing the arithmetic operations (working backwards); and non-arithmetic equations, in which the unknown occurs on both sides of the equation necessitating a solution process that involves operating on the unknown. By making this differentiation, they claimed that there exists a growing level of difficulty in passing from solving arithmetic equations to non-arithmetic ones, which they called a "didactic cut". According to this point of view, if pupils want to adopt an algebraic way of thinking, they have to break away from certain arithmetical conventions and need to learn to deal with algebraic methods and symbolism on a higher abstraction level.

According to Sfard (1991), the abstract mathematical notions can be interpreted in two fundamentally different ways: they are structurally objects, and operationally, processes. In this way, the structural conception is static, instantaneous, and integrative, while the operational conception is dynamic, sequential, and detailed. For example, when pupils tend to solve an equation substituting various values for the variable until the correct value is found refers to the procedural aspect of algebra, and reflects a typical arithmetic way of thinking. On the other hand, when pupils use the balance principle to solve an equation, they are operating on the structural aspects of algebra, and the equal sign gains another meaning, that of equivalence. We can distinguish two forms of the pupils' thinking process: a procedural (or operational) way of thinking that reflects the main aspects of arithmetic, and a structural way of thinking that is inherent to algebra. In a process-oriented view, pupils initially encounter mathematical concepts, they often perceive them as processes, such as calculating or manipulating numbers. For instance, addition is first understood as the process of combining numbers. Meanwhile, the object-oriented view means that pupils, with a higher level of experience and abstraction, begin to see mathematical concepts as objects. For example, they recognize numbers not just as results of counting but as entities with properties and relationships. Structural thinking in word problem solving involves recognizing and utilizing the underlying mathematical structures and relationships within a problem to devise effective solutions. This approach emphasizes understanding the problem's framework rather than relying solely on surface-level cues. Sfard introduces the term reification to describe the cognitive shift from perceiving a concept as a process to understanding it as an object. In the context of mathematical word problem-solving processes, the transition from operational thinking to structural thinking can be understood as the development of a deeper understanding of how problems are formulated, interpreted, and solved. This shift helps pupils move from focusing on step-by-step procedures to grasping the underlying mathematical structures and relationships embedded in the problem. In our opinion, when pupils possess operational knowledge, they are able to solve word problems whose algebraic model is an equation, in which the unknown occurs only on one side of the equation. In this case, they can write and solve equations in a typical arithmetic way of thinking, working backwards. The transition to a structural way of thinking regarding algebra occurs when they are able to solve word problems, whose algebraic models are equations, in which the unknown occurs on both sides. Shortly, the transition from arithmetic to algebra means the shift from word problems, whose algebraic model is an equation $A \cdot x + B = C$, to those where the algebraic model is an equation $A \cdot x + B = C \cdot x + D$. In the stage of algebraic thinking, pupils develop a habit of searching out relationships among quantities across contextualised situations, and learn to solve an equation by attending to its underlying structure. It requires a global representation of the problem from the start of the procedure, and the external symbolic representation is in the form of an equation. As the equation is written, the algebraic calculations often proceed independently of this representation of the situation.

Many researchers widen the definition of algebra and regard it as being more than a system of symbols. Instead, they concentrate on algebraic thinking, which can be interpreted as an approach to quantitative situations that emphasises the general relational aspects with tools that are not necessarily letter-symbolic, but which can be used as cognitive support for introducing school algebra (Johanning, 2004). As beginner algebra learners progress from arithmetic thinking to algebraic thinking, they need to consider the numerical relations of a situation, discuss it explicitly in simple everyday language, and eventually learn to represent them with letters (Herscovics & Linchevski, 1994). This involves a move from knowledge required to solve arithmetic equations (operating on or with numbers) to knowledge required to solve algebraic equations (operating on or with the unknown or variable), and entails a set of changings onto a pre-existing model of arithmetic.

According to Rézio (2014), it is necessary to attenuate the difference between arithmetic and algebraic problems, solving these as arithmetic-algebraic problems; that is, numbers are not only understood as entities but also as relationships between those numbers and between numbers and operations. This perspective brings out the importance of the smooth transition from arithmetic to algebra teaching.

Yerushalmy (2000) supports that the concept of function should be present in the school curriculum in the teaching and learning of algebra from the very start. The concept of function is one of the central ideas of applied mathematics, so many researchers suggest that the teaching of algebra needs to be based on a functional approach (Leitzel, 1989; Thorpe, 1989; Kieran, 1997). In our opinion, the functional approach is a very good opportunity to give pupils a way to answer problems that are modelled by an equation with the unknown on both sides. The familiarity of the situational context consists in that pupils can create tables and try out the values of the variables they have chosen themselves. In this way, the pupils have at their disposal a suitable referential field in which the variables and unknowns, denoted by symbols, acquire meaning.

2. The research design

In the Hungarian curriculum for the Grade 7, equations and inequalities precede the functions, and in the students' textbooks they are found in two different chapters. At the beginning, students are introduced to the algebraic expressions, where the addition and subtraction of simple letter expressions, calculation of the substitution value, multiplication of one- and two-member algebraic expressions by a number, and extraction of a common factor from two or more members of an algebraic expression play the main role. This is followed by solving equations working backwards and using the balance principle. These educational processes provide the opportunity to learn operations with algebraic expressions and equation solving methods. However, it is not possible to develop the appropriate structural thinking that is a prerequisite knowledge for solving word problems involving equations with unknowns on both sides. After these learning processes, according to the curricular requirements, pupils have to solve word problems from maths, other subjects and everyday life or on economic and financial topics. Functional theory is introduced much later, in a different chapter, and the material taught is not in line with the learning outcomes expected in the teaching of algebraic expressions.

2.1. The background of the research

The present research is part of a series of studies we carried out among lower secondary school pupils on the transition process from arithmetic to algebra. Our aim is to find methods that strengthen structural thinking and make the transition to algebra more smooth. Grade 6 pupils learn about arithmetic methods and the properties of basic mathematical operations. Their thinking is almost exclusively procedural, limited to word problems whose algebraic model is an equation of the form $A \cdot x + B = C$, and working backwards is one of the most frequent problem-solving methods. Since manipulating with concrete numbers is one of the main features of procedural thinking, we have been studying the possibilities of introducing the Regula Falsi method into Mathematics teaching practice starting from the 2014-15 school year. Our experience and conclusions on the implementation of this method have been reported in several studies (see Fülöp, 2016; Fülöp, 2020). Our main conclusion was that the Regula Falsi method which we applied gave the chance for pupils to solve problems with quite different approaches and strategies. This teaching approach encourages pupils to manipulate the situation context of the problem giving specific values to unknown quantities, and to check the variation of the error. Pupils could seek and recognize the relation between co-variable quantities, and thereafter write this relation in a letter-symbolic representation. From our teaching experience, after the consolidation of algebraic skills, pupils set up equations and the false position method becomes embarrassing for them in opposition to the straightforward methods of algebra. So, the Regula Falsi method is useful in the phase of early algebra, but later pupils may ignore it, and then they will use purely algebraic methods.

Another way of introducing algebra, that has been the subject of our research, was the functional approach to algebra. For three school years, we studied ways to make the algebraic approach to word problems more effective, in a way that the teaching of functions precedes, in chronological order, the introduction of algebra. We carried out our experiments in the Reformed High School of Gödöllő, and the results we have obtained were published in several studies (see Fülöp, 2023). This method gave the chance for pupils to solve problems with different approaches and strategies. This teaching approach encourages the pupils to manipulate the situation context of the problem giving specific values to unknown quantities, to make tables, to analyse the relations between two or more quantities. Pupils also appreciated the opportunity to solve problems with different approaches and strategies using various representations of the function such as the graph, the table or the letter-symbolic form of the function and not only by the formulation and solution of an equation. The pupils could seek and recognize the relation between co-variable quantities, and thereafter write this relation in a letter-symbolic representation. The symbols gained meaning and importance for the pupils, after they treated the situation context of the problem with concrete numbers. In this way, the transition from procedural thinking to structural thinking, which predates the transition from arithmetic to algebra, is more smooth.

It should be underlined that we implemented the experimental work on these two methods separately, and formulated our conclusions from this point of view. In some studies, the introduction of algebra was preceded by a presentation of the Regula Falsi method, while in other cases, we applied a functional approach to algebra.

2.2. Aim and methology

As a result of our previous research, an idea emerged to implement our attempts to introduce algebra by combining the two aforementioned methods.

In the school year 2023-24, we found an opportunity to put this idea into practice. The experiment was carried out in the Reformed High School of Gödöllő and involved 19 pupils from Grade 7, who were enrolled in a special Mathematics class, most of them high achievers. As these pupils were attending 6 maths lessons a week, we had the opportunity to work on the solution methods of word problems in more detail than in the traditional curriculum. The classroom activities and the testing of knowledge were conducted by the author.

These pupils had completed their first six grades in other different schools, therefore they had quite different levels of knowledge in solving word problems at the beginning of Grade 7. Most of them were familiar with traditional arithmetic methods, but some had already learned how to approach word problems by the use of algebraic methods. The entire research programme was carried out in three phases:

(1) problem solving by the Regula Falsi method;

(2) functional approach in solving word problems;

(3) use of algebraic methods, writing equations to word problems.

The Regula Falsi method is to impose some condition(s) on the unknown quantities of the word problem and compare the real situation (the data of the problem) with the situation created by the hypotheses. By taking the difference into account, simple calculations can be used to easily deduce how much the false assumption differs from the correct solution. This method is neglected in traditional curricula, or in some cases is only described as systematic trial-and-error. The use of this method is based on the manipulation of concrete numbers, and pupils could understand the relationships between the data of the problem and the overall structure of the task. In this way, the Regula Falsi method may reinforce and develop structural thinking.

The functional approach includes teaching first-degree functions, making tables and the concept of variables. The method is based on considering the two sides of the equation modelling the word problem as two different functions. In solving the problem, we have to find the exact value of the variable for which the values of the two functions are equal. When calculating the value of the unknown, pupils could start from a graphical representation of the functions, but for some problems they used the equation-solving skills they had learned in the meantime, based on the balance principle. This approach serves the further deepening of structural thinking, as students can see the whole structure of the task as they construct the table of values and write the rule of the two functions.

In both methods, Regula Falsi and functional approach, we move towards abstraction starting from concrete numbers, reinforcing procedural thinking and laying the foundations for a move towards structural thinking.

In the third phase, we followed the curriculum requirements in all respects. Pupils had to write down the relationship between the data in the word problem (where the unknown is denoted by x), and then the equation needed to solve the task. This requires a higher level of abstraction, but it was a big help that in the previous phases the students had already learned the actual meaning of variable and unknown, namely that x actually stands for certain numbers.

In this paper, we report results for the three items mentioned before, with the greatest emphasis on the transition from operational (arithmetic) thinking to algebraic thinking. In the following, we discuss the following questions:

(1) How do pupils solve word problems exclusively by numerical manipulation? How well-developed is their structural thinking, i.e., how well do they understand the overall structure of the problem when solving it?

(2) How well do pupils understand the concept of variables and the co-variation of unknown quantities?

(3) To what extent can a functional approach to algebra influence the development of algebraic thinking?

(4) What are the roots of specific errors and misunderstandings?

3. Classroom activities and testing results

In order to investigate our conceptions about the aforementioned teaching strategies, we developed a course consisting of 22 lessons of 45 minutes each, focussing on solving word problems. *Both the practice exercises and the tasks on the test papers were chosen to monitor the development of students' thinking as effectively as possible. In this way, we have omitted the very common type tasks.*

3.1. Regula Falsi method – the first step in solving word problems

In the first 4 lessons, we introduced the Regula Falsi method, in a way we had adapted to suit the train of thought of nowadays pupils. These strategies have been described in more detail in a previous paper (see Fülöp, 2020), so here we will just illustrate the essence of the method through the following exercise.

Problem 3.1. *Sarah has 5 times more plums than apples in her basket. If she adds 2 more apples and take out 14 plums, the basket will have 3 times more plums than apples. How many plums and how many apples were originally in the basket?*

This exercise was also solved during the classroom practice activity, in the following, we will show the description of one pupils' work.

First position: Let us consider, for example, initially there are 3 apples and 15 plums in the basket. If we add 2 more apples and take out 14 plums, there will be 5 apples and 1 single plum. However, the basket should contain 15 plums, since the number of plums is three times the number of apples. Therefore, the error of our assumption is equal to 14.

Second position: Now, we assume that there are 4 apples and 20 plums in the basket. If we add 2 more apples and take out 14 plums, there will be 6 apples and 6 plums. But the number of plums must be equal to 18, so the error is equal to 12.

We can see that if we increase the initial number of apples by one (so the number of plums will increase by 5), then the error will decrease by two. We can conclude that the error will disappear (will be equal to 0) if we increase the initial number of apples by $14 : 2 = 7$. So there were originally 10 apples and 50 plums in the basket.

In four lessons, we solved word problems, and during these activities all the rules about the application of the Regula Falsi method were explained to the class. The main emphasis was on the fact that this method differs from simple guess-and-check in that it requires at most two attempts to deduce the correct solution from the evolution of the error of the assumptions. During the lessons, we solved, in roughly equal proportions, arithmetic and algebraic problems. It is necessary to mention that we talk about an arithmetic problem if the word problem can be solved by an equation of type $A \cdot x + B = C$ (where the unknown occurs only on one side of the equation), while algebraic problems can only be solved by an equation of type $A \cdot x + B = C \cdot x + D$ (i.e., the unknown is on both sides of the equation). For the next two lessons, students worked entirely alone and had to complete a worksheet in each lesson. They were asked to work exclusively with the Regula Falsi method. The distribution of right and wrong answers is shown in Tables 1 and 2.

The first worksheet contained three arithmetic (Exercises 1, 2 and 4) and two algebraic problems (Exercises 3 and 5), while the second worksheet contained only algebraic problems. *As we can see in Table 1 and Table 2, the proportion of right*

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Right answer	17	11	11	11	10
Wrong answer	2	7	8	6	8
No response	0	1	0	2	1

Table 1. Regula Falsi – Worksheet 1

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Right answer	12	8	15	7	11
Wrong answer	6	10	4	10	7
No response	1	1	0	2	1

Table 2. Regula Falsi – Worksheet 2

answers was 68% for arithmetic problems, and 55% for algebraic problems, respectively. This denotes that students' operational thinking overtakes their structural thinking, as they have more difficulty dealing with problems where the unknown quantity occurs on both sides of the equation in the algebraic model, even when the solution process involves only numerical manipulation.

We are not able to publish all the exercises, but we would like to mention one of them. Students were least effective on Exercise 4, Worksheet 2. This task, however, is a word problem that is difficult to handle even with algebraic tools, and its text is as follows:.

Problem 3.2. *From a warehouse are sold 10 tonnes of coal. Then the amount of coal entering the warehouse is equal to the amount left after the sale. Now 30 tonnes of coal are sold again. Three times as much coal is then delivered to the warehouse as was left after the second sale. Another 360 tonnes are sold, leaving the same amount of coal in the warehouse as originally. How many tonnes of coal were originally in the warehouse?*

As we can see in the foregoing example, the exercises in the test paper were chosen especially to analyse how the pupils can deal with high-level exercises by numerical manipulation. Tables 1 and 2 show that in the aggregate only 59% of the solutions contained right answers. This provides information on the pupils' structural thinking, as the Regula Falsi method requires a global overview of the problem situation, followed by a relatively simple numerical manipulation. The percentage of 59% denotes that in many cases pupils do not understand the entire structure *of the problem situation, especially the relation between co-varying quantities. We also have to mention that only a few wrong answers were due to computational errors.*

3.2. The functional approach – the foundation of structural thinking

For the next 6 lessons, students were introduced to the concept of functions. Since the chapter on functions comes later in the curriculum, we only introduced the concept of first-degree linear functions, through making value tables and plotting graphs. The main aim of the activities was to use functional knowledge to write equations where the unknown occurs on both sides of the equation. When pupils become able to alternate between different representations of the concept, the functional approach may be the most adequate method to introduce the algebraic symbolism, and to fill the literal symbols with meaning.

The methods used during classroom activities are illustrated through the following exercise.

Problem 3.3. *Andrew has 8 euros more than twice John's money. Paul has 40 euros more than five times John's money. Make a table of the possible values of the three boys' money! Write the money of the boys separately as a function of x, where x is John's money! For what value of x will Paul have three times as much money as the two boys together? Write an equation for the problem and solve it!*

John's money	1	2	3	4	$f(x) = x$
Andrew's money	10	12	14	16	$g(x) = 2 \cdot x + 8$
Paul's money	45	50	55	60	$h(x) = 5 \cdot x + 40$

Table 3. The value-table of the functions

Firstly, we made a table, where the relationships between the data of the problem are shown solely in a numerical context, analogously to the guess-and-check or to the Regula Falsi method. In this process, the manipulation with numbers gives a possibility of thinking on the structure of the problem as a whole. Moving from column to column, pupils can gain a recursive view whose algebraic model is $f(x + 1) = f(x) + a$. But finally, pupils have to write the explicit rules of the functions, from the numerical data of the table (see the last column of Table 3). In this way, they translate the relationships between the data of the problem in the language of symbolic algebra. Writing the rule of several functions, they could gain an explicit view and could easily alternate between the two representations (recursive and explicit) of the function concept. Therefore, after the proper development of the structure sense, the process of reification appears almost instantaneous when they have to write the equation. In this way, to solve the problem means to find the value of the variable, denoted by x, for which the value of the functions $h(x)$ must be three times the value of $f(x)+g(x)$ (and this is the unknown in the equation). Therefore the terms 'variable' and 'unknown' take new meaning, and the equal sign indicates equivalence, where both sides of the equation (the values of the functions) are equal.

In conclusion, the functional approach has three crucial steps: making a value table, writing the rule of the functions, and writing the equation by equating the corresponding functions. In this approach, it is appropriate to provide students with specific instructions for the steps of the problem solving, such as *make the value table*, *write the rule of the function*, *write the equation*, etc.

In our classroom experiment, the majority of students had no problem following the instructions in the task. It was very easy for them to create the table of values, and even writing down the rule of the functions was not a difficult challenge. Several pupils noticed the recursive relationships in the rows of the table, but they easily realised that this could not help them to write the rule of the function. When writing the equation, it was relatively easy for them to identify what to write on the two different sides of the equation, based on the relationships between the function rules and values. We have to mention that, in planning the exercises, we chose data for which it was difficult to carry out a graphical solution. This was done because the final objective was to write equations through the application of functional knowledge, rather than to find any solution to the problem.

Following the practice lessons, the students solved a four-task worksheet in a 45-minute lesson. The formulation of the exercises included instructions on the main steps of solving the problem by a functional approach. For example, let us see Exercise 1 from the worksheet:

Problem 3.4. *The number of cows on a farm is three times the number of horses, and the number of sheep is 36 more than twice the number of horses. Make a table of the possible number of animals in the problem! Write them as a function of x, where x is the number of horses! How many horses, cows and sheep are there on the farm if we know that the number of sheep is twice the number of cows? Write an equation for the problem and solve it!*

As we can see, the pupils have to carry out the instructions, i.e., making tables, writing the rules of the function and writing the equation. Moreover, the placement of information in the text also suggested which information should be used to write the equation (i.e., we know that the number of sheep is twice the number of cows).

The distribution of right and wrong answers is shown in the following table:

	Exercise 1	Exercise 2	Exercise 3	Exercise 4
Right answer	15	15	13	14
Wrong answer	4	4	6	4
No response	0	0	0	1

Table 4. Worksheet – functional approach

As we can see, 75% of the answers were correct, this may represent an improvement in students' structural thinking. This is even more emphasised by the fact that here the pupils were working not only with numerical values but also with letter symbols. However, it should also be noted that the wording of the exercises included additional helpful instructions.

3.3. Writing equations – the final step

After teaching the functional approach, 9 lessons were devoted to practise solving word problems by the use of the methods according to the traditional curriculum. In solving the problems, the main focus was on strengthening the next solution steps: from the data in the word problem, find the unknown quantity, which henceforth is denoted by x; write down the relationships between unknown data using x; write an equation and solve it.

The students were given the word problems in a formulation that lacked the instructions which were present in the functional approach. For example, let us consider the reformulated form of Problem 3.4 in the following way:

Problem 3.5. *The number of cows on a farm is three times the number of horses, and the number of sheep is 36 more than twice the number of horses. How many horses, cows and sheep are there on the farm if we know that the number of sheep is twice the number of cows?*

In this case, the pupils have to choose (and denote by x) the unknown quantity that can be used to express the other unknowns in the easiest possible way.

For example, we can see that in this problem maybe the number of horses represents this unknown quantity. And pupils have to find, on their own, how to use the three main pieces of information existing in the exercise. They have to decide which two pieces of information are used to write down the relationships between the data, and obviously, the third piece of information is left to write the equation. As such, they cannot solve the task correctly unless they have a global view of the overall structure of the task, i.e., it is necessary to have a certain degree of development of structural thinking. Pupils with procedural thinking read and interpret the problem word by word or sentence by sentence, which is the main reason for errors in finding connections between data and writing the equation. This is why it is important to use a functional approach in the foregoing, as following the instructions in the task, linked to manipulation of numbers and letters, strengthen the structural thinking of the pupils.

After the practice lessons, the students worked independently on tasks for the next 3 lessons. One of the worksheets contained 9 arithmetic problems, i.e., exercises whose algebraic model is an equation of type $A \cdot x + B = C$. The other two worksheets contained so-called algebraic problems (five problems on each sheet), which require an equation of the type $A \cdot x + B = C \cdot x + D$. Pupils were told that they can use any method they have learned.

The pupils' efficiency regarding the problem solving on the first worksheet is illustrated in the following table:

	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8	Ex. 9
Right answer	17	19	2	15	16	11	12	14	16
Wrong answer	2	0	15	3	3	6	5	4	2
No response	0	0	2	1	0	2	2	1	1

Table 5. Arithmetic problems

All of the pupils worked with algebraic equation, only one pupil used the Regula Falsi method to solve one single problem. The sources of the errors were the following: 9% of the total solutions contained erroneously written equations, 7% contained errors in writing the relationships between data using algebraic expressions, and 6% contained computational errors. This is a very good ratio considering that Grade 7 pupils, according to relevant literature, still have difficulty solving text problems using algebra. However, it should also be mentioned that procedural thinking is sufficient for solving word problems with $A \cdot x + B = C$ equations, so the main challenge is to translate the arithmetic thinking into algebraic expressions and equations. Exercise 3 proved to be the most difficult one, and only 2 pupils could give the right answer. The text of this exercise is the following:

Problem 3.6. *Ann paid for a 2410 HUF book with only 20 HUF and 50 HUF coins. The number of 20 HUF coins was 5 more than the number of 50 HUF coins. How many 20 HUF coins and how many 50 HUF coins did she use, respectively?*

Most students wrote the equation $x+x+5 = 2410$, mainly due to the fact that students are not always able to correctly interpret the meaning behind algebraic expressions, so they did not understand that the value of the coins is equal to $20\cdot(x+5)$ and $50\cdot x$, respectively. On the other hand, we have to mention this type of exercise is much easier to solve using the Regula Falsi method. For example, when we tested the pupils' knowledge on the Regula Falsi method, they had to solve a very similar exercise, Exercise 1 from Table 1, which is the following:

Problem 3.7. *Paul paid for a 9600 HUF toy with only 20 HUF and 100 HUF coins. He used a total of 200 coins. How many 20 HUF coins and how many 100 HUF coins did she use, respectively?*

As we can see in Table 1, 17 pupils gave the right answer by the use of the Regula Falsi method. Although in the case of word problems of type $A\cdot x+B = C$ students find it easier to write algebraic equations, there are also some problems where manipulation with numerical data is more useful.

The pupils' results from the second and third worksheets are shown in Tables 6 and 7, respectively. As we have mentioned, in the case of algebraic word problems (where the unknown occurs on both sides of the equation), pupils have to possess a structural thinking to write the equation. As the tables show, the students had no problems with this type of exercises, the repartition of right answers was 77%.

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Right answer	16	16	14	15	16
Wrong answer	3	3	5	4	3
No response	0	0	0	0	0

Table 6. Algebraic problems – Worksheet I

However, it is worth noting that for this type of problem, students do not always use the tools of algebra, namely writing equations to word problems, but

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5
Right answer	17	15	16	13	12
Wrong answer	2	4	3	6	7
No response	0	0	0	0	0

Table 7. Algebraic problems – Worksheet II

they also use prior knowledge, such as the Regula Falsi method or the functional approach. The distribution of methods used by students is shown in Table 8.

	Algebra	Regula Falsi	Functional approach	Total
Right answer	65%	8%	6%	79%
Wrong answer	15%	3%	3%	21%

Table 8. Repartition of the pupils' solution method

As can be seen from the table, in 20% of the solutions the students continued to use the Regula Falsi method or the functional approach. On the other hand, they also used the tools of algebra very properly for difficult word problems, such as Exercise 3 in Worksheet I, which is shown in the following:

Problem 3.8. *There are white and red marbles in a box. The number of white marbles is six times the number of red marbles. If 43 red marbles are put in and 86 white marbles are taken out, the number of white marbles will be twice the number of red marbles. How many red marbles and how many white marbles are in the box, separately?*

In the case of the above exercise, only two students tried to solve the problem by a functional approach (one gave right answer, the other did not). The other pupils wrote equations, and only 4 of them made mistakes.

We have to mention that this three-step method (Regula Falsi – Functional approach – Writing equations) has its own useful role to develop and strengthen structural thinking that is inherent to algebra. The increase in the proportion of right answers (Regula Falsi 59%, Functional approach 75%, Writing equations to arithmetic problems 73%, Writing equations to algebraic problems 79%) denotes a rising tendency in pupils' problem-solving thinking. While using the Regula Falsi method, the 59% proportion of correct answers reflects the fact that students have only partial structural knowledge necessary to fully understand the structure of the problem before being taught the algebra chapter.

In the functional approach, the proportion of correct answers increased significantly, and pupils were able to understand the concept of variables, the operations on variables and the co-variation of unknown quantities in a meaningful way. This kind of approach contributes significantly to the development of structural thinking (which is inherent to algebraic thinking) as demonstrated by the effective use of algebraic methods in the later educational stages. However, it should be noted that there occurred some common type errors in the teaching of algebra. The most frequent errors occurred in the manipulation of letter symbols. For example, we have to mention that the great number of errors in solving Exercise 3 from the test paper regarding arithmetic problems (see Table 5) affected seriously the rate of right answers (without this exercise this rating is 79%). In addition to the incorrect representation of the letter symbols, the difficulty of concatenation also appeared in this problem situation. The case of this exercise proves that there are still errors in the manipulation of letter symbols, especially for more complex relationships between unknown quantities. The reversal error has also occurred in a few cases for both arithmetic and algebraic problems.

4. Summary and conclusions

In the present research, we tried to design the strategies for solving word problems in such a way that, through the development of structural thinking, we could reach a level where pupils can write equations for non-stereotypical word problems. Pupils possessing an arithmetic thinking are able to manipulate numerical data, they know the properties of operations, and they see the equals sign as a symbol behind which the result of a series of operations must be written. The arithmetic thinking (called as procedural or operational), focusing mainly on operations, in some cases can be a major obstacle to the introduction of algebra. Algebraic reasoning with the help of some basic principles like Doing–Undoing (Reversibility), abstracting from computation, building rules to represent the function, structural understanding, coordination of numeric and spatial structures unable the learners to solve the problems of algebra (Driscolle, 1999). The major aim of early algebra is to educate elementary pupils such that to cultivate and develop habits of mind that focus on the deeper, underlying structure of Mathematics. These “habits of mind” include two essential features: (1) generalizing, or identifying, expressing and justifying mathematical structure, properties, and relationships, and (2) reasoning and actions based on promoting generalizations. To develop these central features, algebraic reasoning is necessary to link the gaps between arithmetic and algebra.

In our three-step educational process, we took into account the above-mentioned points of view. Using the Regula Falsi method, students focus their thinking not only on arithmetic operations, but also on manipulating the numerical data, to see the relationships between the unknown quantities in the problem, and to have a global view on the overall structure of the problem. Thus, the use of Regula Falsi is the first step in the development of structural thinking through concrete arithmetic calculations. In our study, after teaching the method, about 59% of the students gave correct answers when solving the word problems. For us, this meant that students were not always aware of how to interpret the relationships between the unknown data in the task, and also the co-variations between data. In the functional approach, by making tables about unknown quantities, students are able see how the unknown quantities change together and learn how to express them in algebraic expressions. So, the functional approach promotes a step-by-step transition from numerical calculations to algebraic expressions. At this stage, a significant improvement in students' structural thinking was evident in the classroom activities, and this was also reflected in the survey through the 75% rate of the right answers. In the last step, the students learned how to write down equations to solve different word problems, in line with the actual curriculum requirements. At this stage, the students had not met any particular problem writing equations, even when they had to solve problems of a type that had not yet been solved before (the only exception is Problem 3.6). The rate of the right answers in writing equations to arithmetic problems (problems with algebraic model $A \cdot x + B = C$) was 73%, which confirms that the majority of students have the structural thinking necessary to write equations in order to solve various word problems. Furthermore, the 79% efficiency of solving typical algebraic problems highlights the advantages of the three-step method

As our research was limited to a small group, we cannot draw general conclusions about the application of the method. However, based on our own positive experience, we highly recommend it to our colleagues in lower secondary school education.

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