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Strategies used in solving proportion problems among 7th-grade students

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Abstract. In the 2023/2024 school year, 146 seventh-grade Hungarian students (aged 12-13) participated in our classroom experiment on solving proportion problems. At the beginning and the end of the teaching phase, both the experimental and the control groups solved a test. Regarding the answers of the students, in the pre- and post-test mostly consisting of word problems, we examined the success of solving the problems, as well as the solution strategies. For this, we used the strategies of proportional thinking that already exist in the literature of mathematical didactics. We intended to answer the following questions: To what extent and in which ways do the different types of problems and texts influence the solution strategies chosen by the students? How successfully do seventh-grade students solve proportion problems?

Key words and phrases: proportion problems, solution strategies, teaching methods, Hungarian mathematics education.

MSC Subject Classification: 97D50, 97F80.

Introduction

Proportional thinking and solving tasks related to proportionality play an important role not only in the school curriculum but also in everyday life. Thanks to this, the framework curricula based on the 2020 Hungarian National Core Curriculum place a lot of emphasis on topics related to proportionality and percentage calculation already in grades 5 and 6 (Education Office, n.d.). By the end of the teaching phase, students must be able to recognize direct and inverse proportionality in concrete, everyday situations.

It should be noted that this goal is not only important because of students' performance in mathematics classes. They encounter tasks related to proportionality in grades 7-8 during natural science subjects (physics, chemistry, biology, geography), too. Models used in scientific measurements and laws are often based on the comparison of quantities directly or inversely proportional. In these tasks (as in the problems of everyday life), it is particularly important to find the appropriate mathematical model, and to apply the characteristic properties between proportional quantities when solving the problem (Nagy et al., 2015).

Theoretical background

Levels of proportional reasoning

Several studies deal with the evolvement of proportional thinking and its age-related characteristics. One of the reasons for this is that its development does not take place during a single subject unit but must be shaped over many years. In 1993/1994, Benő Csapó and his colleagues studied the success of solving a simple proportion problem among grade 3-11 students. Based on the logistic curve fit on the measurement results, the evolution of proportional thinking seems to be taking too long, which calls into question the strength of the school effect. Based on the curve, at the beginning of secondary school, about half of the students, and at the end of secondary school, only two-thirds of them were able to solve the proportion problem (Molnár & Csapó, 2003).

To be able to examine a student's success and strategies in solving proportion problems (such as linear or inverse proportionality problems), it is not enough to know which stage of the development of proportional thinking the given student is at. Before the student reaches the level of formal (quantitative) proportional reasoning, four basic conditions must be met:

(1) When solving a problem, the student needs to recognise and understand the difference between absolute (additive) and relative (multiplicative) changes.

(2) Closely related to this is that the solver of the task must be able to distinguish the situations in which the use of ratios is reasonable from those in which it is not convenient. (For example, it makes no sense to use a ratio to solve the following problem: A 2-year-old child is 90 cm tall. How tall will this child be at the age of 4?)

(3) In addition to these, during problem-solving it is important to be aware of what is constant and what changes, as well as how it changes in the observed process, during which the change of quantities is examined.

(4) When solving problems with a more complex structure, it is essential to designate a unit for comparing quantities which can be used to structure the quantities in the task so that they become comparable. For example, look at the following problem: *In a shopping centre,* 2 *kilograms of apples cost* 700 *HUF. How much are* 6 *kilograms of apples?* One possible way to solve the problem is to use the price of 2 kilograms of apples as a unit, and then the answer to the question will be three times the price of 2 kilograms of apples. However, other different unit choices may be appropriate (such as the price of a kilo of apples) (Langrall & Swafford, 2000).

The quality of mastering these prerequisites greatly influences the level and the timing of the student reaching an understanding of proportionality problems. Many educators and researchers have attempted to explore ways that can support the development of proportional reasoning and understanding of the topic's concepts.

According to the results of a Pakistani study among fifth-grade students, using real-life examples during the learning process helps to deepen the concepts of proportionality. Furthermore, students who had received visual help for the explanations when learning the topic achieved better results in the post-test at the end of the experiment (Saleem et al., 2021).

An experiment conducted by David Ben-Chaim and his colleagues (1998) among seventh-grade students showed that students studying according to reform curricula, supporting problem-solving activities together, performed much better when solving proportion exercises. Cooperative task-solving positively affected not only the performance but also students' logical reasoning, and thus the development of proportional thinking (Nurhayati & Kusumah, 2020). The support of joint problem-solving seemed important also because students' active participation in the class can also positively affect learning motivation, which largely determines their development (Varga et al., 2007).

Strategies in solving proportionality problems

For the improvement of the proportional thinking of students, it is important to map the current situation of their level. Examining the methods of solution used by them can help to achieve this goal. Many pieces of research deal with task-solving strategies used by different age groups, the frequency of appearance and the success of these strategies. By knowing the distribution of correct and incorrect strategies, we can get a more detailed picture of the students' current level of understanding, their blockages, and the limitations of their proportional reasoning. We would like to highlight two surveys of this topic.

Linda Fisher's research (1988) examined how twenty secondary school mathematics teachers solve problems based on direct and inverse proportionality. In the study, the success of solving each exercise was observed, as well as the strategies used by the participants. In this classification, nine categories (four incorrect and four correct solution strategies) were set aside:

F1) *No answer.* In this case, the student does not take any meaningful steps towards a solution, or even starts solving the problem.

F2) *Intuitive.* This includes guesswork and illogical calculations. These are often just random operations on numbers in the problem text.

F3) *Additive.* In this case, the problem solver incorrectly focuses on the differences between the quantities instead of examining the multiplicative relationship between them.

F4) *Proportion attempt.* The solver recognises the proportionality but fails to express it correctly. There may be an incorrectly written proportionality equation or a calculation with an incorrect proportionality type.

F5) *Incorrect other.* This category includes any incorrect attempt to solve a problem that does not fit into any of the previous categories.

F6) *Proportion formula.* With this correct strategy, the equality of two ratios appears in the case of direct proportionality, and the equality of two multiples appears in the case of inverse proportionality.

F7) *Proportional reasoning.* This category includes correct solutions that contain a correct proportionality argument but do not show the proportionality formula in the explanation.

F8) *Algebra.* This category includes correct solutions that lead to the correct result using an algebraic equation but do not include the proportion formula. This usually refers to cases where the solver does not write down the proportion formula, but works with a rearranged form of it from the beginning.

F9) *Correct other.* This category includes all correct solutions that do not fit into any of the previous categories.

Based on the results, it was established that the examined teachers prioritise the formal solution (proportion formula) over the less formal solution schemes preferred by the students. This is also surprising, since these alternative strategies are more natural for students and can help them understand generalizations better than the use of formulas (Fisher, 1988).

In a study conducted in the 2009/2010 school year, the strategies used by 278 Turkish sixth-grade students when solving proportionality problems were examined (Avcu & Avcu, 2010). The results showed that most students, instead of the strategies supporting conceptual understanding, more often used the strategies developing procedural skills preferred by the textbooks. In contrast to the classification described above, the emphasis here is on the right strategies. For each strategy, the proper way of solving the following problem is presented: 2 *kilograms of apples in a shopping centre cost* 700 *HUF. How much do* 6 *kilograms of apples cost?* (It should be mentioned that the strategies are not ordered by the level of complexity.)

A1) *No answer.* In this case, the student does not take any meaningful steps towards a solution, or even starts solving the problem.

A2) *Incorrect.* This category includes all incorrect attempts to solve the problem.

A3) *Cross product algorithm strategy.* In this strategy, after writing down the proportion formula, the solver will use that the equivalence of the ratios can also be written down as the equivalence of the cross products.

Solving the problem: $\frac{700}{2} = \frac{x}{6} \iff x = \frac{(700 \cdot 6)}{2} = 2100,$

6 *kilograms of apples cost* 2100 *HUF.*

A4) *Unit rate strategy.* Using this strategy, the solver determines one unit of a quantity, and then use it to calculate the result by multiplication with the other quantity.

Solving the problem: *If* 2 *kilograms cost* 700 *HUF,* 1 *kilogram costs* 350 *HUF. So,* 6 *kilograms of apples cost* $6 \cdot 350 = 2100$ *HUF.*

A5) *Factor of change strategy.* The solution involves determining the multiplication factor of change of a quantity, and then multiplying the other quantity by this number.

Solving the problem: $\frac{6}{b} = 3 \longrightarrow 3 \cdot 700 = 2100,$

6 *kilograms of apples cost* 2100 *HUF.*

A6) *Equivalent fractions strategy.* Using this strategy, the solver multiplies the fraction created by defining the pair of rates by a fraction whose numerator and denominator are equal (i.e., the fraction is 1). In this way, he/she tries to achieve the third quantity given in the numerator or denominator.

Solving the problem: $\frac{2}{700} = \frac{6}{x} \longrightarrow \frac{2}{700} = \frac{2}{700} \cdot \frac{3}{3} = \frac{6}{2100} \longrightarrow x = 2100,$

6 *kilograms of apples cost* 2100 *HUF.*

A7) *Equivalence class strategy.* In this strategy, the solver treats the pair of rates as a fraction and, by expanding the fraction containing the known numbers, tries to construct a fraction with the numerator or denominator of the given number.

Solving the problem: $\frac{2}{700} = \frac{4}{1400} = \frac{6}{2100},$

6 *kilograms of apples cost* 2100 *HUF.*

A8) *Build-up strategy.* Those who use this strategy are usually characterised by additive thinking. By adding smaller quantities together, they can create the desired larger quantity.

Solving the problem: *If* 2 *kilograms cost* 700 *HUF, then* 4 *kilograms cost* 1400 *HUF. Since* $6\,\mathrm{kg} = 2\,\mathrm{kg} + 4\,\mathrm{kg}$, *we get that* 6 *kilograms of apples cost* $700 + 1400 = 2100$ *HUF.*

A9) *Correct other.* This category includes all correct solutions that do not fit into any of the previous categories.

Among the six different solution strategies, the "cross product algorithm" was the most common (Avcu & Avcu, 2010).

Methods

In recent years, no research has been carried out in Hungary that would have examined the strategies used by elementary or secondary school students when solving proportion problems. We set ourselves the goal of conducting a classroom quasi-experiment with the participation of seventh-grade students in order to map the schemes used. Note that in Hungary, regardless of the type of school (elementary school, eight-grade or six-grade grammar school), the numbering of grades is uniform from 1 to 12. The age range for seventh-grade pupils is 12-13 years. The reason for choosing this grade was that it is the first grade to teach not only direct proportionality but also inverse proportionality. We organised a control group study to examine the strategies used during proportional reasoning and student development.

In the experiment carried out in the 2023/24 school year, 146 students from one county seat and three cities participated. The experimental groups consisted of 89 students (45 girls, and 44 boys), and the control groups consisted of 57 students (25 girls, and 32 boys). Control and experimental groups were randomly selected from classes/groups in participating schools. 36 students from an elementary school, 30 students from an eight-grade grammar school and 80 students from three six-grade grammar schools took part in the classroom experiment. Since seventh-grade students from only four cities took part in the survey, we cannot consider the study to be representative. Nevertheless, the experienced results carry useful information regarding the teaching of the topic and the knowledge of students.

The groups participating in the experiment were connected to the study when discussing the topic containing the knowledge of ratio, proportionality and percentage calculation. Each group spent seven lessons working on the topic. During the given time, the control groups dealt with learning the elements of the subject based on the curriculum and according to the professional decisions of the teacher of the group. The experimental groups, on the other hand, followed a common curriculum. The learning material used by the experimental groups placed great emphasis on the inclusion of real-life problems with a variety of wording (Saleem et al., 2021). Another main pillar of the work of the experimental groups was to focus on working together (Ben-Chaim et al., 1998; Nurhayati & Kusumah, 2020; Varga et al., 2007). During the lessons, students were required to work in methods that provide more scope for working actively and together (such as rally coach, tiered assignments, jigsaw method). In designing the learning materials, we have aimed to ensure that students work with these methods for at least half of the time they spend on the topic. In addition to the lessons, students also had to work at home on predetermined measurement problems and projects which were also reported on during a product presentation class. Projects included, among other things, measuring the flow rate of liquids, as well as research projects focusing on the unit price of goods in shops and on the calculation of bank interest.

Both the experimental and control groups took part in a pre-test before the start of the topic, and a post-test at the end of the topic. The students in the experimental and control groups completed the same pre- and post-test under the same conditions. They could solve the problems on the printed worksheet in any order they liked. The pre-test included seven, and the post-test eight (mostly open-ended) word problems. This means that when formulating the problems, we did not give any answer options, so the students had to find the correct solution on their own. (It should be mentioned that the meaning of the expression "problem" depends very much on the task-solver. In our opinion, solving this type of word problems is not a routine task for this age group.) The pre- and post-tests included the following problems:

Direct proportionality:

- Pre-test – Problem 1: *In a department store, the price of* 6 *bottles of mineral water of the same size altogether is* 720 *HUF. How much do we have to pay in this store if we buy* 9 *bottles of this type of mineral water?*
- Post-test – Problem 1: *In a department store, the price of* 7 *bottles of mineral water of the same size altogether is* 840 *HUF. How much do we have to pay in this store if we buy* 10 *bottles of this type of mineral water?*

Inverse proportionality:

- Pre-test – Problem 2: *Grandma made peach jam. There were* 14 *jars of* 4 *decilitres filled. How many jars would she have needed if she had bought* 7*-decilitre jars?*
- Post-test – Problem 2: *Csongi wants to make lemonade. One tablet of the sweetener used by his mother corresponds to* 6 *grams of sugar, so he usually puts* 6 *tablets in the one-and-a-half litre lemonade. How many tablets should Béla use if he also wants to make one-and-a-half litres and the sweetener he just found at home corresponds to* 9 *grams of sugar per tablet?*

Recognising fractions and percentages using a figure:

- Pre-test – Problem 3: *Under the figures is written how much of the whole is painted. Circle the letters of the figures* (a, b, c, d) *that correspond to the text underneath them.* (See Figure 1.)

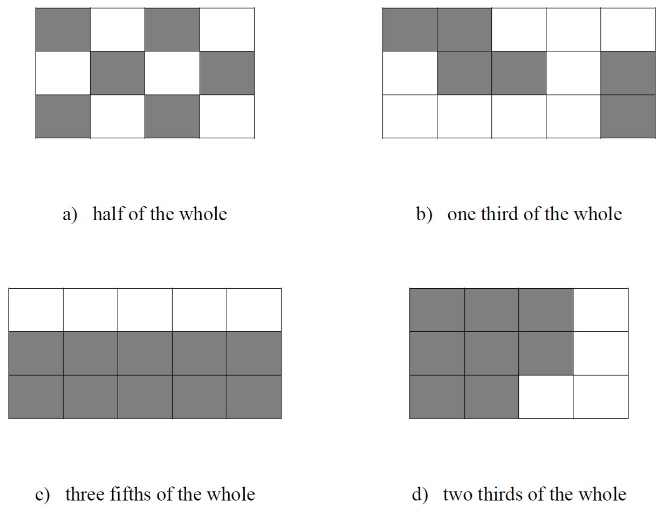

Figure 1. Figure for Problem 3 of the pre-test

- Post-test – Problem 3: *Colour in the given part of the figure.* (See Figure 1.)

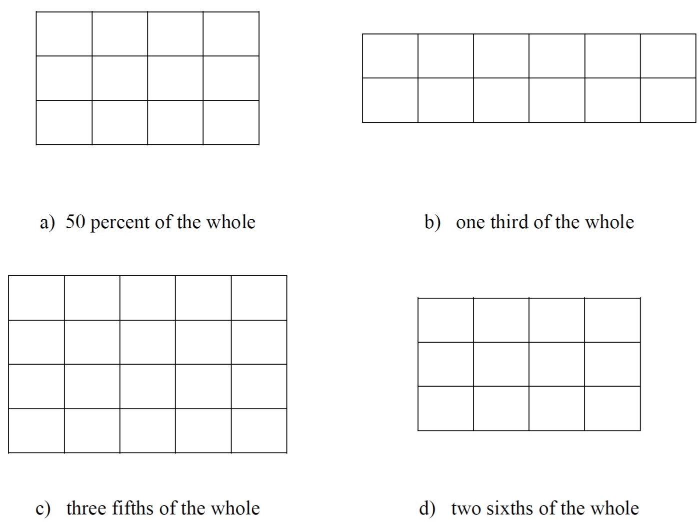

Figure 2. Figure for Problem 3 of the post-test

Complex proportionality:

- Pre-test – Problem 4: *On an average day,* 3 *production lines operate in the chocolate factory. Thus,* 2700 *bars of chocolate are produced in* 4 *days. How many chocolate bars are produced in* 9 *days if* 5 *production lines are in operation due to the Christmas holidays? (The performance of the production lines is equal.)*
- Post-test – Problem 4: *Grandma Ági asks her grandchildren to dig up her vegetable garden every year. Last year, her* 4 *grandchildren finished in* 3 *days, working* 4 *hours a day. In how many days will the work be completed this year if only* 2 *of her grandchildren can help but they work for her* 6 *hours a day? (All her grandchildren work just as skillfully and just as quickly.)*

Proportional division:

- Pre-test – Problem 5: *Csaba and Attila jointly borrowed a grinding machine for a week, which Csaba used for* 5 *days and Attila for* 2*. They agreed to split the rental fee between them in proportion to the number of days they used the machine. How many HUF should Attila pay if the rental fee was* 6650 *HUF?*
- Post-test – Problem 5: *Bronze is an alloy of copper. The most commonly used alloy is the so-called tin bronze, which is made using copper and tin. How much copper and tin are needed to make* 20 *kg of bronze if the ratio of tin to copper in bronze is* $1:7$*?*

Conversion from percentages to fractions:

- Pre-test – Problem 6: *A beginner cycling team went around Lake Balaton in the summer. They covered* 25% *of the total distance (about* 200 *kilometres) on the first day,* 35% *on the second day, and* 20% *on the third day. What fraction of the total distance remained for the fourth day?*
- Post-test – Problem 7: *Danuta received a lot of Hungarian Christmas candies ('szaloncukor') for Christmas and she wants to eat them before New Year's Eve. On the 27th of December she ate* 15% *of the amount she received, on the 28th* 35%*, and on the 29th* 40%*. On the 30th she ate the rest of the candies. What proportion of the total amount of candy is left by the 30th?*

Conversion from fractions to percentages:

- Pre-test – Problem 7: *Four children ate from a large cake. Peti ate* $\frac{1}{5}$ *part, Judit* $\frac{1}{10}$ *part, Anna* $\frac{1}{4}$ *part, Zoli* $\frac{2}{5}$ *part of it. What percentage of the cake is left?*

- Post-test – Problem 8: *Dóri baked a cake for her friends for the holidays. Patricia ate* 2/5 *of the cake, Vivi* 35%*. Daniella arrived later, so she only had what Patricia and Vivi left. What percentage of the cake did Daniella eat?*

Price changes:

- Post-test – Problem 6: *Between April and August 2023, the average price of petrol increased by* 5% *from* 600 *HUF per litre, and then decreased by* 5% *between August and November. What was the average price of a litre of petrol in November?*

When changing the context and numbers of the problems, we considered whether the students had encountered the problem type before the topic was discussed. In view of this, for example, inverse proportionality problems may have been given more emphasis in the post-test than in the pre-test.

When evaluating the completed worksheets, we assessed the students' success using a three-point scale for each problem ('left empty', 'solved incorrectly', 'solved correctly'). In addition, we used two classification systems already existing in the literature to classify the solution strategies used in proportion problems (Fisher, 1988; Avcu & Avcu, 2010). Whenever it was possible, we classified the method using both systems to get a more accurate picture of the strategy used by the student. In the case of incorrect solutions (if they could be discovered and recognised), we also indicated the strategy used (according to the Avcu classification). In this study, we would like to present the methods observed when solving the proportion problems (Problems 1, 2, 4, and 5) found at the beginning of the worksheets and their changes between the two tests. (Nevertheless, we thought it important to present the whole survey so that the reader can put the highlighted problems in context.)

Research questions

Considering the theoretical background presented earlier, our research aimed to answer the following research questions:

- How successfully do students solve proportionality problems before and after the teaching unit?
- Is there a noticeable difference between the success rates of solving direct and inverse proportionality problems at each test?

- What strategies do students most often use when solving proportionality problems?
- Do the strategies used by seventh-grade students change over the duration of the teaching unit?

Results

Direct proportionality problems

Regarding direct proportionality, we used the same problem in the pre- and post-test, only the numerical data were changed. Table 1 shows the accuracy of the solutions of the experimental and the control groups for the two tests. A high proportion of the students solved the problems in both tests correctly. A significant part of the students (96% of the experimental group, and 86% of the control group) solved the problem without any mistakes during both tests. Surprisingly, regarding this problem type, 2% of the experimental group and 9% of the control group result deteriorated compared to their performance at the beginning of the topic. This means three different cases: (1) after the correct solution in the pre-test, students solved the problem incorrectly in the post-test, (2) they solved the problem in the pre-test correctly but did not write anything in the post-test, (3) they solved the problem in the pre-test incorrectly and did not write anything in the post-test.

Pre-test	**Post-test**	**Control**	**Experi-mental**
Incorrect	Empty	1	0
Incorrect	Incorrect	0	1
Incorrect	Correct	3	1
Correct	Empty	0	1
Correct	Incorrect	4	1
Correct	Correct	49	85
Total		57	89

Table 1. The categorised answers given in numbers for the problems related to direct proportionality

When examining the solution strategies of the students in the case of this problem by Fisher's classification (Fisher, 1988), it is important to highlight that neither in the pre- nor the post-test was there a solution in which someone made an additive error, also, there was not a student who used the proportion formula during the correct solution (Table 2). Most of the correct problem solvers used proportional reasoning, and the vital part of the participants did this during both tests (90% of the experimental group, 84% of the control group).

Pre-test	**Post-test**	**Control**	**Experi-mental**
Intuitive (incorrect)	No answer (empty)	1	0
Proportion attempt (incorrect)	Proportional reasoning (correct)	2	0
Incorrect other (incorrect)	Incorrect other (incorrect)	0	1
Incorrect other (incorrect)	Proportional reasoning (correct)	1	1
Intuitive (correct)	Proportional reasoning (correct)	0	1
Proportional reasoning (correct)	No answer (empty)	0	1
Proportional reasoning (correct)	Intuitive (incorrect)	1	0
Proportional reasoning (correct)	Proportion attempt (incorrect)	2	0
Proportional reasoning (correct)	Incorrect other (incorrect)	1	1
Proportional reasoning (correct)	Intuitive (correct)	1	0
Proportional reasoning (correct)	Proportional reasoning (correct)	48	80
Proportional reasoning (correct)	Algebra (correct)	0	1
Proportional reasoning (correct)	Correct other (correct)	0	1
Correct other (correct)	Proportional reasoning (correct)	0	2
Total		57	89

Table 2. Number of students using different strategies (Fisher's classification) in the problems related to direct proportionality of the pre- and post-test

The Avcu classification (Avcu & Avcu, 2010) gives a more detailed insight into the methods of correct solutions. The most popular category was the "unit rate strategy", which was used by 93% of the experimental group and 96% of the control group on at least one test (successfully or unsuccessfully). This is a much higher rate than what Ceneida Fernández and his colleagues experienced during their research among 12-13-year-old students (Fernández et al., 2008).

Table 3 shows the distribution of the users of the method and the success of the application.

Pre-test	Post-test	Control	Experi-mental
Other incorrect	No answer	1	0
Other incorrect	Unit rate strategy (correct)	1	0
Other incorrect	Build-up strategy (correct)	1	0
Unit rate strategy (incorrect)	Unit rate strategy (incorrect)	0	1
Unit rate strategy (incorrect)	Unit rate strategy (correct)	1	1
Cross product algorithm strategy (correct)	Unit rate strategy (correct)	0	2
Unit rate strategy (correct)	No answer	0	1
Unit rate strategy (correct)	Other incorrect	2	0
Unit rate strategy (correct)	Unit rate strategy (incorrect)	1	0
Unit rate strategy (correct)	Factor of change strategy (incorrect)	1	0
Unit rate strategy (correct)	Cross product algorithm strategy (correct)	0	1
Unit rate strategy (correct)	Unit rate strategy (correct)	40	58
Unit rate strategy (correct)	Factor of change strategy (correct)	1	1
Unit rate strategy (correct)	Correct other	1	0
Factor of change strategy (correct)	Unit rate strategy (correct)	2	3
Factor of change strategy (correct)	Factor of change strategy (correct)	0	3
Build-up strategy (correct)	Unit rate strategy (incorrect)	0	1
Build-up strategy (correct)	Cross product algorithm strategy (correct)	0	1
Build-up strategy (correct)	Unit rate strategy (correct)	5	14
Build-up strategy (correct)	Build-up strategy (correct)	0	1
Correct other	Unit rate strategy (correct)	0	1
Total		57	89

Table 3. Number of students using different strategies (Avcu classification) in the problems related to direct proportionality of the pre- and post-test

We also investigated whether students' strategies were consistent or changed across tests. Most of the students having solved both problems correctly did not change their strategy when writing the post-test compared to the pre-test (73% of the experimental group, 82% of the control group). 24% of the students of the experimental group having answered correctly both times switched from the strategy used in the pre-test to the “unit rate strategy”, while only 2% used it in the pre-test and then switched to another strategy. In the case of the control group, these rates were 14% and 4%. As for the most typical strategy change, the “build-up strategy” used in the pre-test was replaced by the “unit rate strategy” (16% for the experimental group, 10% for the control group). This may indicate the development of proportional thinking and the strengthening of the multiplicative approach instead of the additive one.

Inverse proportionality problems

The problems containing inverse proportionality were given different contexts in the cases of the two tests. Compared to the problem related to direct proportionality, the rate of successful solutions was lower for this type. In the pre-test, 87% of the experimental group and 79% of the control group were able to solve the problem, and at the post-test, 91% of the experimental group and 81% of the control group were able to solve it. 82% of the experimental group and 65% of the control group successfully solved this type in both tests (Table 4).

Pre-test	**Post-test**	**Control**	**Experi-mental**
Empty	Empty	1	2
Empty	Incorrect	0	1
Empty	Correct	3	2
Incorrect	Incorrect	2	1
Incorrect	Correct	6	6
Correct	Empty	3	1
Correct	Incorrect	5	3
Correct	Correct	37	73
Total		57	89

Table 4. The categorised answers given in numbers for the problems related to direct proportionality

The strategy used by most students “bypassed” proportional reasoning and focused on the equality of products (to calculate the total amount of jam and the total amount of sugar). This strategy was used for at least one of the tests by 90% of the experimental group and 95% of the control group (Figure 3).

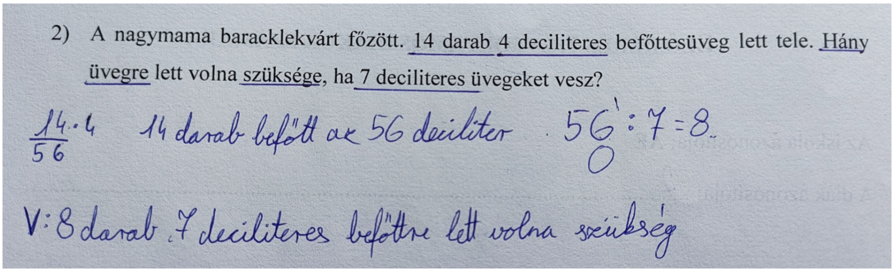

Figure 3. A correct student solution to Problem 2 of the pre-test. (Problem: *Grandma made peach jam. There were* 14 *jars of* 4 *decilitres filled. How many jars would she have needed if she had bought* 7*-decilitre jars?* Solution: *The* 14 *jars are* 56 *decilitres.* 8 *jars of* 7 *decilitres would have been needed.*)

Table 5 shows that by the end of the teaching unit, the proportion of students following this type of strategy had increased among the students of the control group, while there was no significant change in the case of the experimental group. Among the students who calculated in this way only in the pre-test, there were two options for the scheme used in the post-test. These students of the control group (except one student) could not solve the inverse proportionality problem of the post-test correctly, on the other hand, two-thirds of these students in the experimental group reached the correct result in the similar problem of the post-test using proportional reasoning or an algebraic solution.

When studying the conversion between the schemes used for solving the pre- and post-test problems, the rate of conversion was the same for both groups (19% of both the experimental and control groups changed). However, it is important to note that the way the strategies changed was not similar for the two groups. While in the experimental group, 43% of the students showing a schema change used proportionality reasoning during the post-test instead of calculating all the quantities, in the control group, the proportion of the change in the opposite direction was 71%. From this, we can deduce the assumption that the recognition of the nature of the proportion problem in the case of the control group students was more influenced by the text of the problem than in the case of the experimental group.

Pre-test	Post-test	Control	Experi-mental
No answer (empty)	No answer (empty)	1	2
No answer (empty)	Proportion attempt (incorrect)	0	1
No answer (empty)	Correct other (correct)	3	2
Intuitive (incorrect)	Proportion attempt (incorrect)	0	1
Intuitive (incorrect)	Correct other (correct)	3	3
Proportion attempt (incorrect)	Intuitive (incorrect)	1	0
Proportion attempt (incorrect)	Proportion attempt (incorrect)	1	0
Proportion attempt (incorrect)	Algebra (correct)	0	1
Proportion attempt (incorrect)	Correct other (correct)	3	2
Intuitive (correct)	Proportional reasoning (correct)	0	1
Intuitive (correct)	Correct other (correct)	1	1
Proportional reasoning (correct)	Proportional reasoning (correct)	0	2
Proportional reasoning (correct)	Algebra (correct)	0	1
Proportional reasoning (correct)	Correct other (correct)	5	4
Correct other (correct)	No answer (empty)	3	1
Correct other (correct)	Intuitive (incorrect)	3	1
Correct other (correct)	Proportion attempt (incorrect)	2	1
Correct other (correct)	Incorrect other (incorrect)	0	1
Correct other (correct)	Proportional reasoning (correct)	1	6
Correct other (correct)	Algebra (correct)	0	1
Correct other (correct)	Correct other (correct)	30	57
Total		57	89

Table 5. Number of students using different strategies (Fisher's classification) in the problems related to inverse proportionality of the pre- and post-test

It is also worth mentioning that in the case of the inverse proportion problem, nearly 9% of the students recognised the presence of proportionality during the solution of the problem but expressed it incorrectly. In most cases, this was manifested by students calculating as if the problem were about directly proportional quantities instead of inverse proportional ones. So, for example, they calculated as if the quotient of the corresponding quantities were constant, rather than the product of them. A form of this error is also called the "illusion of linearity". In this case, students mistakenly believe that the linear model is applicable in the given situation (De Bock et al., 2002). The research of De Bock et al. (2015) among young adults also highlighted that students tend to confuse inverse proportional models with direct proportional models, and they highlighted that the mistake also depends on the representation of the model used.

Complex proportionality problems

This problem was especially obvious when solving the problem related to complex proportionality. While in the problem in the pre-test, any two of the three quantities had a direct proportional relationship with each other, in the similar problem in the post-test, the relationship between the quantities was inversely proportional.

In the pre-test, 72% of the experimental group and 49% of the control group solved the problem correctly (Table 6). On the other hand, in the post-test, the rate of successful solutions was 42% for the experimental group and 46% for the control group. In the case of both groups, by the increase in the difficulty of the problem, there was a decline, but while in the experimental group, this is large and spectacular, with the control group it is not a drastic change in magnitude.

In the case of this problem, proportional reasoning was also the most widely used. In the post-test, with the appearance of inverse proportionality, the number of proportion attempts increased for both groups (Table 7). We can conclude from this that, although students recognise the relationship, the practice is not yet at the level of reproduction. Students are more comfortable with reasoning strategies at this age.

Pre-test	Post-test	Control	Experi-mental
Empty	Empty	1	2
Empty	Incorrect	5	1
Empty	Correct	0	1
Incorrect	Empty	4	1
Incorrect	Incorrect	9	14
Incorrect	Correct	10	6
Correct	Empty	0	2
Correct	Incorrect	12	32
Correct	Correct	16	30
Total		57	89

Table 6. The categorised answers given in numbers for the problems related to complex proportionality

Pre-test	Post-test	Control	Experi-mental
No answer (empty)	No answer (empty)	1	2
No answer (empty)	Intuitive (incorrect)	2	0
No answer (empty)	Proportion attempt (incorrect)	3	1
No answer (empty)	Proportional reasoning (correct)	0	1
Intuitive (incorrect)	No answer (empty)	1	0
Intuitive (incorrect)	Intuitive (incorrect)	1	0
Intuitive (incorrect)	Proportion attempt (incorrect)	3	1
Intuitive (incorrect)	Proportional reasoning (correct)	3	1
Additive (incorrect)	No answer (empty)	1	0

Additive (incorrect)	Intuitive (incorrect)	0	1
Additive (incorrect)	Proportion attempt (incorrect)	0	1
Additive (incorrect)	Proportional reasoning (correct)	1	1
Proportion attempt (incorrect)	No answer (empty)	1	1
Proportion attempt (incorrect)	Intuitive (incorrect)	2	2
Proportion attempt (incorrect)	Additive (incorrect)	1	1
Proportion attempt (incorrect)	Proportion attempt (incorrect)	0	3
Proportion attempt (incorrect)	Incorrect other (incorrect)	1	0
Proportion attempt (incorrect)	Proportional reasoning (correct)	5	1
Incorrect other (incorrect)	No answer (empty)	1	0
Incorrect other (incorrect)	Intuitive (incorrect)	1	0
Incorrect other (incorrect)	Proportion attempt (incorrect)	0	4
Incorrect other (incorrect)	Incorrect other (incorrect)	0	1
Incorrect other (incorrect)	Proportional reasoning (correct)	1	3
Intuitive (correct)	Proportional reasoning (correct)	1	0
Proportional reasoning (correct)	No answer (empty)	0	2
Proportional reasoning (correct)	Intuitive (incorrect)	0	6
Proportional reasoning (correct)	Additive (incorrect)	2	1
Proportional reasoning (correct)	Proportion attempt (incorrect)	9	23
Proportional reasoning (correct)	Incorrect other (incorrect)	1	2
Proportional reasoning (correct)	Intuitive (correct)	1	0
Proportional reasoning (correct)	Proportional reasoning (correct)	14	30
Total		57	89

Table 7. Number of students using different strategies (Fisher's classification) in the problems related to complex proportionality of the pre- and post-test

When solving complex problems of this type, the strategy most often used is to fix one quantity and examine the relationship between the other two quantities. However, the appearance of inverse proportionality makes it difficult to assess the relationships between the quantities (Arican, 2019). When applying this strategy ("factor of change strategy"), a special case is when the student intends to get to a unit in the case of two quantities ("unit rate strategy"). However, not all users of the strategy solved the problems correctly (Table 8). In the pre-test, with one or two exceptions, the correct solutions were obtained by applying these strategies, while this is no longer true in the post-test (especially in the case of the experimental group).

Pre-test	**Post-test**	**Control**	**Experi-mental**
No answer	No answer	1	2
No answer	Other incorrect	2	0
No answer	Factor of change strategy (incorrect)	3	1
No answer	Factor of change strategy (correct)	0	1
Other incorrect	No answer	1	0
Other incorrect	Other incorrect	2	0
Other incorrect	Factor of change strategy (incorrect)	2	2
Other incorrect	Factor of change strategy (correct)	2	1
Other incorrect	Correct other	1	0
Unit rate strategy (incorrect)	No answer	1	0
Unit rate strategy (incorrect)	Other incorrect	0	1
Unit rate strategy (incorrect)	Unit rate strategy (incorrect)	1	0
Unit rate strategy (incorrect)	Factor of change strategy (incorrect)	0	3
Unit rate strategy (incorrect)	Unit rate strategy (correct)	1	1
Unit rate strategy (incorrect)	Factor of change strategy (correct)	1	2
Factor of change strategy (incorrect)	No answer	2	1
Factor of change strategy (incorrect)	Other incorrect	4	4
Factor of change strategy (incorrect)	Factor of change strategy (incorrect)	0	4

Factor of change strategy (incorrect)	Unit rate strategy (correct)	2	0
Factor of change strategy (incorrect)	Factor of change strategy (correct)	1	2
Factor of change strategy (incorrect)	Correct other	2	0
Factor of change strategy (incorrect)	Correct other	2	0
Unit rate strategy (correct)	No answer	0	2
Unit rate strategy (correct)	Other incorrect	1	12
Unit rate strategy (correct)	Unit rate strategy (incorrect)	0	1
Unit rate strategy (correct)	Factor of change strategy (incorrect)	8	17
Unit rate strategy (correct)	Unit rate strategy (correct)	5	3
Unit rate strategy (correct)	Factor of change strategy (correct)	4	13
Unit rate strategy (correct)	Correct other	2	12
Factor of change strategy (correct)	Other incorrect	1	0
Factor of change strategy (correct)	Factor of change strategy (incorrect)	2	1
Factor of change strategy (correct)	Unit rate strategy (correct)	3	0
Factor of change strategy (correct)	Factor of change strategy (correct)	1	1
Build-up strategy (correct)	Unit rate strategy (incorrect)	0	1
Build-up strategy (correct)	Factor of change strategy (correct)	0	1
Correct other	Unit rate strategy (correct)	1	0
Total		57	89

Table 8. Number of students using different strategies (Avcu classification) in the problems related to complex proportionality of the pre- and post-test

The other correct solutions of the problem found in the post-test were created without exception by following a single scheme: the students led back the problem to solving a simpler type by grouping two quantities. For example, the total number of working hours was calculated from the number of working hours per day and the number of days worked, and then this was compared to the accurate number of grandchildren. With this merger, a "detour" path similar to the one already seen in the inverse proportion problem was created. The decline observed in the experimental group is strongly contributed to by the fact that the appearance of inverse proportionality in the problem increased the appearance of "proportion attempt" in the students' solutions.

The variety of relationships between the quantities included in the complex proportion problem may have caused the phenomenon that, compared to the problems discussed so far, an exceptionally large number of changes can be observed when examining the solution of these problems of the pre- and post-tests. In the case of those solving both problems correctly (experimental: 30 students, control: 16 students), 87% of the experimental group and 63% of the control group changed their previously used strategy; and concerning the experimental group, half of the students in question changed from "unit rate strategy" to "factor of change strategy", and half of them to the method using grouping.

Proportional division problems

The difference in the success data for the pair of proportional division problems of the two tests is similarly shocking (Table 9). The number of students skipping the problem increased significantly compared to what had been experienced in the pre-test. While only 3% of all students left the problem related to proportional division blank in the pre-test, in the case of the post-test this proportion was 15%. A particularly spectacular difference can be seen in the case of the experimental group.

Like what was seen in direct and inverse proportion problems, only one strategy dominated during the correct solutions of the problems related to proportional division. In the calculations of most students, the determination of how many equal parts the whole must be divided into can be found, and then the quantities in question were determined by multiplying the unit. In the case of the control group, there was no strategy change at all in the solutions of the students who performed well in both the pre- and post-tests, and in the case of the experimental group, there was no significant change of strategy (12%), either.

According to a domestic research paper conducted on a similar topic, this age group finds it difficult to represent quantities and proportions (Bereczki, 2023). Our experience supports this claim. When solving the task related to proportional division, the students of the control group did not use visual aids either in the pre- or the post-tests. Even in the case of the experimental group, only a few students made diagrams: 3 in the pre-test, 7 in the post-test (all but one student solved the task correctly). Although this number increased in this group, the proportion of those who used visual representation during the solution process remained below 10%.

Pre-test	**Post-test**	**Control**	**Experi-mental**
Empty	Empty	1	1
Empty	Incorrect	1	0
Empty	Correct	2	0
Incorrect	Empty	4	6
Incorrect	Incorrect	3	0
Incorrect	Correct	2	3
Correct	Empty	1	9
Correct	Incorrect	6	10
Correct	Correct	37	60
Total		57	89

Table 9. The categorised answers given in numbers for the problems related to proportional division

Summary

In our research, we examined the success of 146 seventh-grade students in solving proportional problems and the strategies they used. The consistency and the change of the schemes used by the students were established with the help of a pre- and a post-test.

The proportion of correct solutions to the problems related to direct proportionality was high (over 90%) in both tests. For the inverse proportionality problems, 84% of students answered correctly in the pre-test and 87% in the post-test. Students were significantly less successful in solving complex proportionality problems: 63% of students gave the correct solution in the pre-test and 43% in the post-test. Proportional division problems were solved correctly by 84% and 71% of students in each tests.

The success rates in solving problems also indicate that students have more difficulty solving problems related to inverse proportionality than problems related to direct proportionality. This is also reflected in the experience that the nature of the relationship between quantities in complex proportionality strongly influenced both the proportion of correct solutions to the problem and the chosen strategy.

When solving the direct proportionality problem, students most often used the "unit rate strategy", while in the inverse proportionality problem, the "avoidance" of proportional reasoning could be observed among half of the students. In this type of problem, nearly 9% of students incorrectly used direct proportional reasoning. When solving the pair of problems related to proportional division, the vast majority of students in both groups used only one strategy.

When comparing the strategies used in the pre- and post-tests, it is noticeable that in the case of the pair of problems on direct proportionality, the changing of the scheme was characteristic of about a fifth of the students. For the inverse proportionality problems, 69% of the experimental group and 56% of the control group persevered when writing the post-test with the strategy used in the similar problem of the pre-test. In the case of the complex proportion problem, there was an exceptionally high percentage of strategy change. In contrast, more than 90% of those correctly solving proportional division problems, both in the pre- and the post-test, used the same strategy chosen during the initial test of the topic.

It was found that for problems related to inverse proportionality, proportional division and complex proportionality, the strategies described by the classifications used were not applied. Therefore, it would be worthwhile to establish a new classification system that is more adapted to the local conditions.

It can be stated that in the case of simple problems related to direct and inverse proportionality, the proportion of students working correctly in the experimental group increased further compared to the proportion experienced in the control group. In addition, in their case, a change in the applied strategies could be observed several times, which in most cases may indicate a change in proportional reasoning.

We did not find that a collaborative and project-based approach was more successful in improving students' performance. Teachers and pupils in the experimental groups also highlighted the novelty of the methods, which probably did not produce the expected results due to the short time available. A new, revised research is needed to investigate this issue.

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