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Teaching Mathematics and Computer Science

Exploring the basic concepts of Calculus through a case study on motion in gravitational space

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Abstract. In universities, the Calculus course presents significant challenges year after year. In this article, we will demonstrate how to use methods of Realistic Mathematics Education (RME) to introduce the concepts of limits, differentiation, and integration based on high school kinematics and dynamics knowledge. All mathematical concepts are coherently built upon experiences, experiments, and fundamental dynamics knowledge related to motion in a gravitational field. With the help of worksheets created using GeoGebra or Microsoft Excel, students can conduct digital experiments and later independently visualize and relate abstract concepts to practical applications, thereby facilitating their understanding.

Key words and phrases: teaching calculus, kinematics, dynamics, physics, interdisciplinarity, RME, visualization, dynamic illustration, experimental approach, GeoGebra.

MSC Subject Classification: 97D40, 97I40, 97M50.

Introduction

University Calculus courses consistently face challenges, evidenced by low pass rates and students often memorizing the material without understanding it (Kántor, 2021; Radnóti & Nagy, 2014; Barbetta & Skaruppa, 1995). Consequently, the course fails to develop students' problem-solving and modeling abilities, which are crucial objectives. In natural sciences and engineering, it is essential that students have to be able to apply mathematical knowledge to real-world scenarios. An interdisciplinary approach can significantly enhance calculus education and develop complex problem-solving skills (Rátz, 1909). Building on this, researchers in the early 1990s described the methods named Realistic Mathematics Education (RME) to help students develop these skills (Da, 2023).

RME builds on students' experiences and observations, using them as starting points for introducing mathematical concepts. Freudenthal (1991) advocated for mathematics education as a guided re-invention process, where students, guided by the teacher, move from concrete observations to abstract concepts (Khanh, 2015). This approach allows students to experience the research-like process through which mathematical concepts are formed. Consequently, this method deepens understanding and fosters curiosity and motivation (Da, 2022).

The widespread use of computing tools, including smartphones, over the past decades enables students to collect data from precise physical experiments and use it to deduce the laws governing projectile motion. This integrated approach provides an effective way to discover mathematical concepts, transitioning from everyday experiences to abstract notions. It significantly enhance the effectiveness of education and experience-based learning. This extends the possibilities of the RME method and revolutionizes the teaching of calculus concepts.

Publications in the last decades consistently show that integrating digital technologies into mathematics education yields positive results for students, particularly in developing abstraction and problem-solving skills (Forczek & Karsai, 2000; Karsai et al., 2006; Santos-Trigo, 2019; Tamam & Dasari, 2021). Despite this, there is still a notable lack of utilization of computer tools in theoretical subjects. Therefore, it is essential to further develop and integrate these tools into the curriculum to enhance students' understanding and application of abstract mathematical concepts.

We have developed a comprehensive, interdisciplinary educational approach based on the Realistic Mathematics Education (RME) method to introduce the concepts of limits, differentiation, and integration. Each mathematical concept is coherently built on experiences, experiments, and fundamental dynamics knowledge related to motion in a gravitational field. Using worksheets created with GeoGebra or Microsoft Excel, students can conduct digital experiments and independently visualize and connect abstract concepts with practical applications, thereby facilitating their understanding (Vale & Barbosa, 2017; Zengin, 2012; Tall, 1991; Forczek & Karsai, 2000; Karsai, 2003).

The focus on motion in a gravitational field was chosen partly because these phenomena are easily observable and also because the historical development of calculus theory concentrated on understanding kinematics. Isaac Newton's scientific endeavors emphasized the close relationship between mathematics and physics, particularly through the introduction of abstract mathematical concepts like calculus while examining various types of motion. Newton's precise definition of instantaneous velocity laid the groundwork for the derivative, a concept that became ubiquitous in our daily lives but remained enigmatic for centuries. Additionally, integral calculus emerged from applying these theories to solve equations of motion. Empirical facts related to physical phenomena significantly enhance the understanding of fundamental mathematical concepts, particularly real analysis. It is important to preserve Newton's legacy in high school and university education by fostering a closer connection between these disciplines (Tall & Vinner, 1981).

Finally, we note that the content described in the paper can be directly implemented into a curriculum. Additionally, we can start from other phenomena in a similar manner, such as studying harmonic oscillation or the phenomenon of cooling.

Quadratic equation of motion, instantaneous velocity (derivative of functions)

First, let us delve into the example of free-fall without air resistance. Through experimental exploration, we initially unveil the quadratic equation of motion. Subsequently, by examining the average velocity, we introduce the concept of instantaneous velocity, i.e., rate of change, paving the way for the abstract realm of differential Calculus.

Similarly, as done in high schools, record the movement of a free-falling ping-pong ball and analyses it with the Tracker software. From the measured data, we can see experimentally that the position is proportional to the square of the elapsed time: $y(t) = kt^2$, where $k < 0$ is the proportionality factor. Let us also plot $y(t)$ as a function of the square of the time (Figure 1).

By taking the trendline, one can see that the lower graph fits closely a line, confirming the relationship $y \sim t^2$. By plotting the equation of the line, it can be seen that the slope, and hence the proportionality factor is $k \approx -5$, so $y(t) \approx -5t^2$. Repeating this experiment with other bodies (for which the air resistance is negligible), we would obtain similar result.

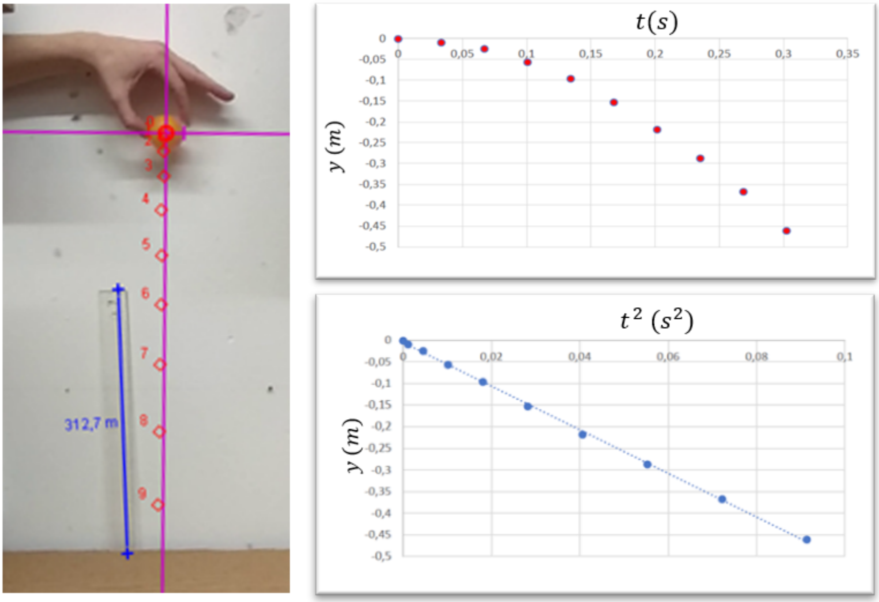


Figure 1. Examination of freefall

On the other hand, this result can be theoretically derived by using the high school definition of instantaneous velocity (treated in higher classes). Textbooks typically illustrate instantaneous velocity by calculating the average velocity over smaller intervals, often using the analogy of a car's speedometer (Csajági et al., 2021). Understanding this concept leads to a challenging abstraction, i.e., the notion of differential quotient.

Consider the motion on a sufficiently small time interval $(t, t+\Delta t)$, and take average velocity:

$$v_m = \frac{y(t+\Delta t) - y(t)}{t+\Delta t - t}.$$

In a geometric context, this is exactly the slope of the line passing through the points $(t, y(t))$ and $(t+\Delta t, y(t+\Delta t))$. Then, try to take $\Delta t > 0$ as small as possible. What happens? As the endpoints are arbitrarily close, the line "becomes" tangent, and the slope then can be considered as the instantaneous velocity at time t. This process is well visible on a dynamic figure (GeoGebra [2]), and in Figure 2.

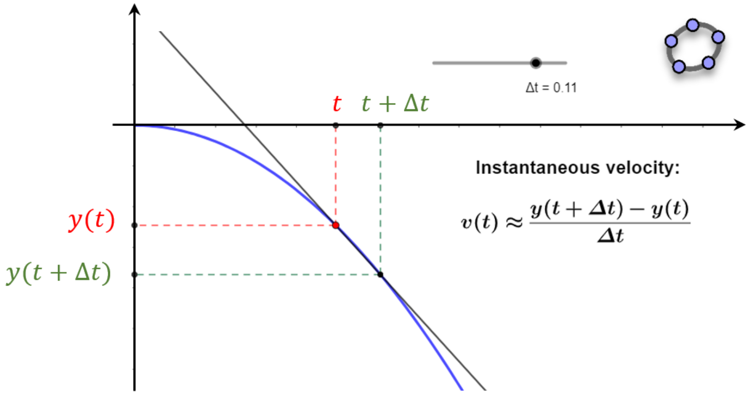


Figure 2. Introduction of instantaneous velocity

Utilizing the concept of instantaneous velocity provided in this way, we can calculate the coefficient k value with even high school-level knowledge:

$$v(t) \approx \frac{y(t+\Delta t)-y(t)}{\Delta t} = \frac{-k(t+\Delta t)^2+kt^2}{\Delta t} = \frac{-kt^2-2kt\Delta t-k\Delta t^2+kt^2}{\Delta t}$$
$$\approx -2kt-k\Delta t \approx -2kt.$$

Since Δt is arbitrarily small, we can have that $v(t) = -2kt$. Analogously, the rate of change of velocity can be introduced. This is called acceleration and is denoted by $a(t)$:

$$a(t) \approx \frac{v(t+\Delta t)-v(t)}{\Delta t} = \frac{-2k(t+\Delta t)+kt}{\Delta t} = -2k.$$

In the case of free fall, $a(t)$ is called gravitational acceleration. In Hungary, it is approximately $g \approx 9.81\frac{m}{s^2}$. We note that the value of g varies across different locations on Earth, typically ranging from about $9.78\frac{m}{s^2}$ at the equator to $9.83\frac{m}{s^2}$ at the poles (Budó & Pócza, 1978). For the sake of simplification, we will use $g \approx 10\frac{m}{s^2}$ in the remaining examples. Hence, $k = -\frac{1}{2}g$.

Our result is that the position of the falling body is described by the function $y(t) = -\frac{1}{2}gt^2$ if we start the motion from height 0. The quadratic equation of motion has been proved. This procedure can be made for more general motions, using the finite limit at finite location. We illustrate is with a simple example.

Example 1. *Consider the motion of a vertically upward thrown point-like body whose position is described by the function* $y(t) = 1+5t-5t^2\,(m)$*. Give the velocity of the body at time* $t_0 = 1\,s$*.*

The dynamic graph available at the link (Figure 3) allows us to examine the magnitude of the average velocity for times close to, less than or greater than 1 s.

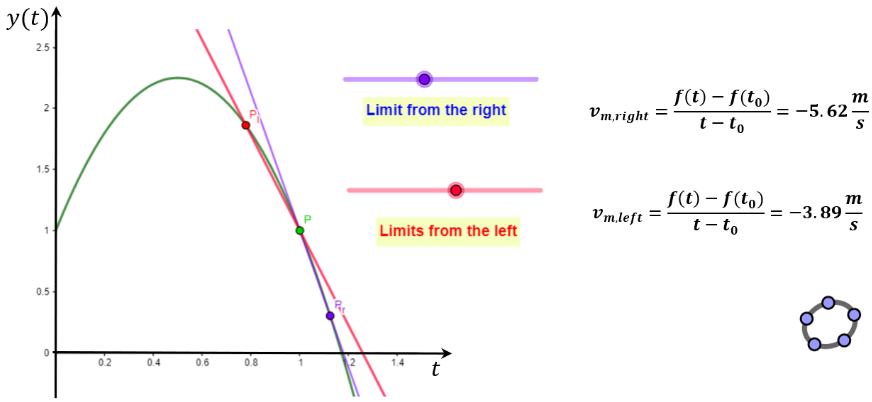


Figure 3. Approximations of instantaneous velocity (GeoGebra [3])

Our empirical results are:

- if $t < t_0$ and $t \approx t_0$, then $\frac{y(t)-y(t_0)}{t-t_0} \approx -5 \Rightarrow \lim\limits_{t \to t_0^-} \dfrac{y(t) - y(t_0)}{t - t_0} = -5$;
- if $t > t_0$ and $t \approx t_0$, then $\frac{y(t)-y(t_0)}{t-t_0} \approx -5 \Rightarrow \lim\limits_{t \to t_0^+} \dfrac{y(t) - y(t_0)}{t - t_0} = -5$.

Whether we choose times close to 1 from the left or from the right, the average velocities are close to $-5\,\frac{m}{s}$ (limit). This brings us to the notion of the differential coefficient of a function f at t_0 in the general case:

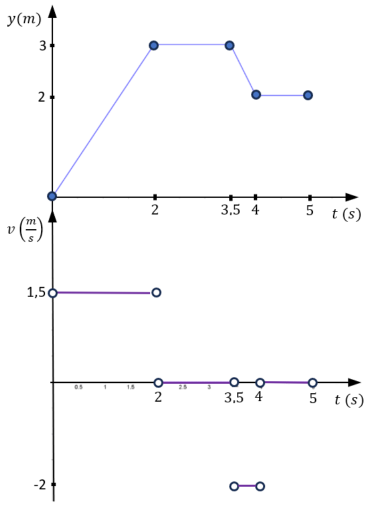
$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0},$$

provided that the limit exists and is finite.

Graphical determination of the velocity function (investigation of function)

The teaching of differential Calculus typically tends to focus mainly on formal calculations during university lectures. Graphical analysis is rarely emphasized in education, despite its significant contribution to understanding.

Consider the instantaneous velocity of a body moving on a straight line over the interval given below (Figure 4, left).



$$\lim_{t\to 2^-} \frac{f(t)-f(t_0)}{t-t_0} = \lim_{t\to 2^-} m_1 = \frac{3}{2}$$

$$\lim_{t\to 2^+} \frac{f(t)-f(t_0)}{t-t_0} = \lim_{t\to 2^+} m_2 = 0$$

Figure 4. Relation between $y(t)$ and $v(t)$

The motion can be decomposed into 4 stages where the body is moving in a uniform motion. At these intervals, the velocity is equal to the slope of the corresponding line segment. The question is what we can say about the velocity at the endpoints of the motion intervals. Let us examine the velocity in $t = 2$ by approximating from the left and from the right (Figure 4, right). We can see that the left and right limits do not coincide, indicating that the limit does not exist. We can also find the same situation at $t = 3.5$ and $t = 4$. Consequently, instantaneous velocity cannot be attributed to the body at these specific points. Note that this example is an abstraction, since instantaneous change of velocity would require an infinitely fast change of kinetic energy, what is physically impossible. In real cases (e.g., accidents) such changes happen under very short time with extremely high value of acceleration.

In practical applications, especially experimental sciences, only graphical description of processes can be obtained (for example, by measurements). Qualitative study of their behavior must be done graphically with no formulas. Therefore, when developing the concept of differential coefficient, it is important to use only the geometric meaning to graphically visualize the derivatives of functions (Karsai et al., 2008, 2006).

Example 2. *On a windy day in autumn, observe the height* $(y(t))$ *of a light rubber ball shown in the graph below (Figure 5). Use the digital graph at* GeoGebra [4] *to illustrate the velocity-time graph of the body. Formulate conjectures about the relationship between a function and its derivative!*

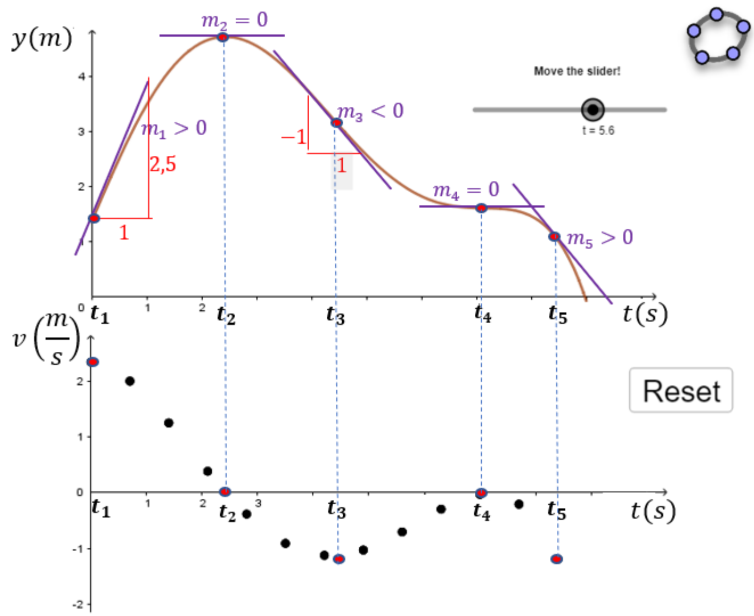


Figure 5. The relation between $y(t)$ and $v(t)$

It can be seen that the slope of the tangent is 0 at two time points (t_2 and t_4), in which case the instantaneous velocity is also 0. Among the intermediate times, it is worth considering those at which the velocity will have an extreme value (t_3, t_4). These are the inflection points of $y(t)$.

The quadratic equation of motion based on dynamics (antiderivative)

In the previous section, using kinematic analysis, we derived the quadratic equation of motion and introduced the concept of the differential coefficient.

In this section, we will build upon this by examining the laws of dynamics, starting with Newton's second law. For a single point-particle with fixed mass (m), it says that $F = ma$, where F is the resultant of forces acting on the body and a is the acceleration. In the simplest case is like in our examples considered, $a(t)$ depends only on the time. From this, we need to determine the instantaneous velocity, for which $v'(t)= a(t)$, and the position of the body with $s'(t)= v(t)$. A more general and abstract consideration of this problem can lead to the concepts of antiderivative and indefinite integral. To illustrate this, let us consider a practical example:

Example 3. *A point-like body of mass m is thrown obliquely from a height of* $2\,m$ *with an initial velocity of* $10\,\frac{m}{s}$ *and an angle of* $30°$ *with the horizontal. Give the relation describing the position of the body if drag is not taken into account.*

Velocity-time functions. The initial parameters are:

$$v_{x_0} = 10 \cdot \cos(30) = \sqrt{3} \cdot 5, \qquad v_{y_0} = 10 \cdot \sin(30) = 5.$$

According to Newton's second law:

$$v_x'(t) = a_x(t) = 0, \qquad v_y'(t) = a_y(t) = -g\,.$$

Analyse the graphical report. For the horizontal component, we know that the derivative of the velocity function at each time is 0 , so the slope of a tangent drawn at any point on the graph of the velocity function is 0. Let us draw line segments with this slope (Figure 6, left).

Among the infinitely many possible constant functions, choose the one that satisfies the initial value, i.e., the one starting from $\sqrt{3} \cdot 5\,\frac{m}{s}$. A similar procedure is followed for the vertical component. Let us draw straight lines with slope $-10\,\frac{m}{s^2}$. Out of infinitely many possible linear functions, let us select the one that satisfies the initial condition, i.e., the one starting from $5\,\frac{m}{s}$ (Figure 5, right). Symbolically:

$$v_x(t) = v_{x_0} = \sqrt{3} \cdot 5, \qquad v_y(t) = -gt + v_{y_0} = -10t + 5.$$

Similarly, we can determine position-time functions.

Position-time functions. The initial parameters are: $x_0 = 0$, $y_0 = 2$. Knowing $x'(t) = v_x(t)$ and $y'(t) = v_y(t)$, the tangent fields of both the horizontal and vertical coordinates can be easily plotted as above (Figure 7). Symbolically:

$$x(t) = v_{x_0}t + x_0 = \sqrt{3} \cdot 5t, \qquad y(t) = -\frac{g}{2}t^2 + v_{y_0}t + y_0 = -5t^2 + 5t + 2.$$

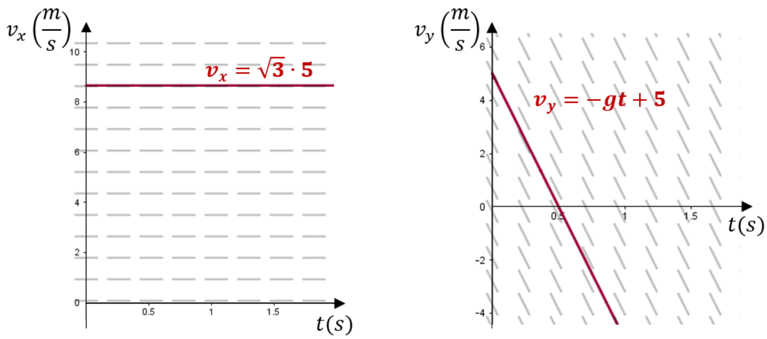


Figure 6. Tangent line field representation for velocity-time functions

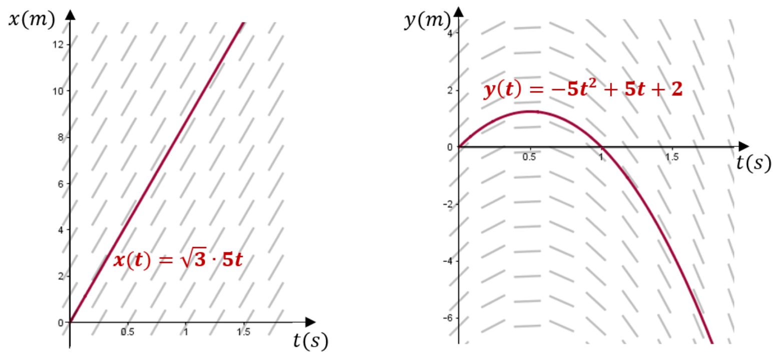


Figure 7. Tangent line field representation for position-time functions

We can see that the calculations are consistent with the graphical solution. Although formal treatment is usual in Calculus education, emphasizing the graphical meanings significantly helps better understanding of the theoretical concepts.

Approximate analysis of throwing based on dynamics (definite integral)

Let us illustrate the inclined deflection without air resistance with Microsoft Excel. Using the following approximation based on Newton's laws of dynamics, we can visualize the motion of the body as if we were taking a stroboscopic picture.

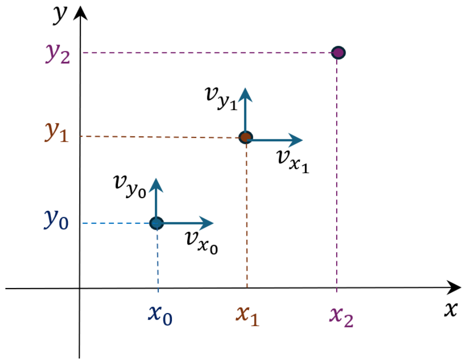


Figure 8. Projectile motion approximate

Given a point-like body at a point in the plane (x_0, y_0), which is thrown with velocity components v_{x_0} and v_{y_0}, where v_x and v_y are nonzero, the initial position is arbitrary (Figure 8). If Δt is chosen to be sufficiently small, we can estimate the motion by a uniform motion between the two points, where the distance traveled in Δt is given by $v \cdot \Delta t$. From these, the position of the body after Δt can be given:

$$x(t_0 + \Delta t) \approx x_1 = x_0 + v_{x_0}\Delta t, \qquad y(t_0 + \Delta t) \approx y_1 = y_0 + v_{y_0}\Delta t.$$

The next position of the body, Δt time later, is approximated in a similar way:

$$x(t_0 + 2\Delta t) \approx x_2 = x_1 + v_{x_1}\Delta t, \qquad y(t_0 + 2\Delta t) \approx y_2 = y_1 + v_{y_1}\Delta t.$$

It is apparent that to obtain this, it is necessary to know the velocity components at (x_1, y_1). Write down the forces acting on the body in the x and y directions, and then use Newton's second law to give the accelerations:

$$F_x = 0 = m \cdot a_x \Rightarrow a_x = 0, \qquad F_y = m \cdot a_y \Rightarrow a_y = -g.$$

It is clear that the velocity will be constant in the x direction $v_{x_1} = v_{x_0}$. Acceleration in the y direction: $a_y = -g$; so the change in velocity over time Δt:

$$\Delta v_y = a_y \cdot \Delta t = -g \cdot \Delta t, \qquad v_{y_1} = v_{y_0} + a_y \cdot \Delta t = v_{y_0} - g \cdot \Delta t.$$

In this way, the coordinates (x_2, y_2) can be determined, and the position of the body can be recursively calculated in a similar way after an arbitrary positive integer Δt:

$$x_n = x_{n-1} + v_{x_0} \cdot \Delta t, \qquad y_n = y_{n-1} + v_{y_{n-1}} \cdot \Delta t.$$

We can calculate and plot the approximate motion by iteration (Excel teaching aids [1]).

Example 4. *Using the above approximation, we can also investigate the projectile motion with drag* ($F_x = -kv_x$; $F_y = -kv_y - mg$) (Excel teaching aids [2]), *or even in winds* (Excel teaching aids [3]).

Examine the geometric meaning of the terms representing changes in coordinates:

$$x_n - x_{n-1} = v_{x_0} \cdot \Delta t, \qquad y_n - y_{n-1} = v_{y_{n-1}} \cdot \Delta t.$$

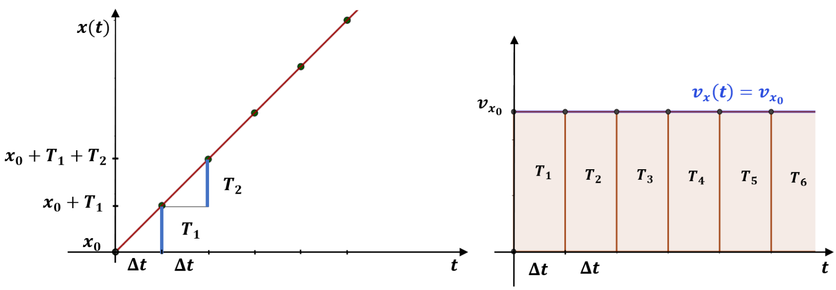


Figure 9. Projectile motion horizontal component, if $\Delta = 0.1$

Concerning the horizontal coordinate (Figure 9), it is obvious that the displacement term $v_{x_0} \cdot \Delta t$ coincides with the area of the rectangle T_1.

After time $2 \cdot \Delta t$, the algorithm above is $x_2 = x_1 + v_{x_0} \cdot \Delta t$. The member representing the displacement with respect to x_1 is equal to the area T_2, so the displacement with respect to the initial location is $T_1 + T_2$.

It can be clearly seen that the area under the graph of the $v_x(t)$ function gives the displacement in the horizontal direction.

Similarly for the vertical component (Figure 10), the displacement during Δt is given by T_1, i.e., $y_1 = y_0 + T_1$. The position after $2\Delta t$ time, $y_2 = y_1 + v_{y_1} \cdot \Delta t =$

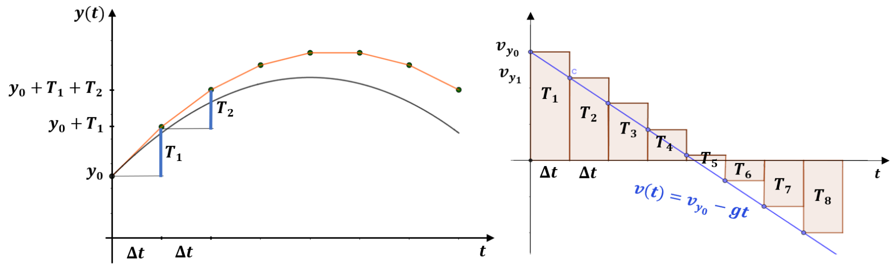


Figure 10. Projectile motion vertical component, if $\Delta = 0.1$

$y_0 + T_1 + T_2$. After $6\Delta t$ time, v_{y_5} is negative, hence the displacement with respect to y_5 is $-T_6$. So the displacement with the initial position can be calculated by the signed sum of the areas:

$$y_6 = y_0 + T_1 + T_2 + T_3 + T_4 + T_5 - T_6.$$

Based on these observations and decreasing Δt, we can formulate the conjecture that the signed area under the v_y graph estimates the true position function. In the GeoGebra utility below, observe the estimated and actual motion trajectories as the time unit Δt is chosen to be smaller and smaller (GeoGebra [5]).

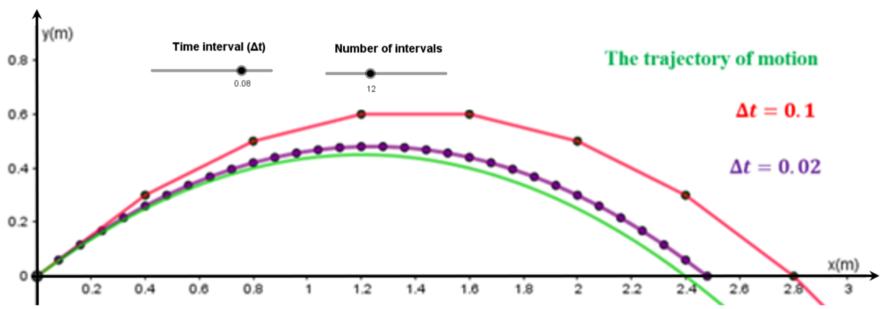


Figure 11. The trajectory of motion

If we choose $\Delta t = 0.02$ time unit instead of $\Delta t = 0,1$, the observed time of the body motion is proportionally reduced, but the sum of the areas of the rectangles is getting closer and closer to the area under the graph of the velocity function, so the errors from the approximation will be smaller. Such a sequence of procedures is shown in (Figure 11).

To refine the procedure, we can employ the concept of limits. Therefore, as $\Delta t \to 0$, the signed area under the velocity function's graph, according to our conjecture, approximates the position of the object over the interval. With this practical example, we can introduce the notion of definite integral and the Newton–Leibniz theorem:

$$y(t_2) - y(t_1) = \int_{t_1}^{t_2} v(t)\ dt = \int_{t_1}^{t_2} y'(t)\ dt.$$

The specific terms and concepts are not addressed within the scope of this paper. Instead, we will explore the meaning of the graphic in the GeoGebra [6] teaching aid help below.

Analysis of throwing under drag (limit, differential equations)

Let us explore the motion of a parachute jump, see Figure 12, Skydiving analysis (Next Generation Science, 2023). We can notice that a person jumping out of an aircraft accelerates less and less as he falls, and after a certain time fall, velocity becomes nearly constant. After opening the parachute, the fall velocity decreases significantly and becomes nearly constant (much lower than before) again. We will experience the same phenomenon, for example, in the case of a thrown basketball as well (Azhikannickal, 2019).

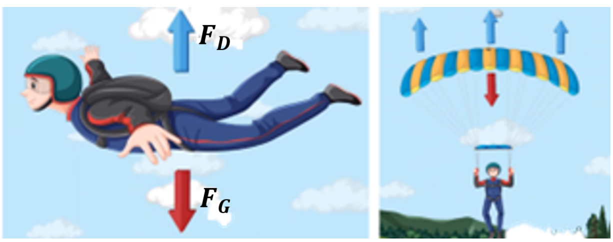


Figure 12. Skydiving analysis

There are many questions arising about this motion:

(1) What factors affect this kind of motion?

(2) What is the fall velocity of a person with or without using a parachute?

(3) What function describes the position and velocity of the body during the jump?

Limiting velocity based on the laws of Dynamics

The first question can be answered easily. The parachutist is subject to the force of gravity ($F_g = mg$), and to the drag in the opposite direction to the motion, which can be empirically described by the formula (NASA, n.d., Forces on Falling Object):

$$F_D = \frac{1}{2} k \cdot A \cdot \rho \cdot v^2,$$

where k, A, ρ, v denote the drag coefficient, area, density of the air, velocity, respectively. It can be observed that after a while the fall speed is approximately constant. In the limit case, the two forces are equal, i.e,

$$\frac{1}{2} k \cdot A \cdot \rho \cdot v^2 = mg.$$

For the second question, let us examine the problem in a simpler case.

Example 5. *What is the falling speed of a 1-cm-diameter ice ball?*

The parameters are $\rho_{\text{air}} = 1.3\,\frac{kg}{m^3}$, $k = 0.45$, $A = 25 \cdot 10^{-6}\pi\, m^2$, $\rho_{\text{ice}} = 920\,\frac{kg}{m^3}$, $V_{\text{ice}} = \frac{4}{3} \cdot 5^3 \cdot 10^{-9}\pi\, m^3$ (Budó & Pócza, 1978), and

$$v = \sqrt{\frac{8\rho_{\text{ice}} r g}{3k\rho_{\text{air}}}} \approx 14.48\,\frac{m}{s} \approx 52.13\,\frac{km}{h}.$$

Reconsidering the original skydiving exercise in light of the above, as the speed of a falling person can be measured during the fall (for example, with a smartphone), it becomes possible to determine the drag coefficient of the human body and the parachute. A similar exercise:

Example 6. *What is the speed at which a 10-cm-diameter spherical iron ball will hit the seabed when thrown into the Mariana Trench? We can assume that the resistance of the medium acts on the body as a square function of the velocity,* $\rho_{\text{iron}} = 7860\,\frac{kg}{m^3}$, $\rho_{\text{sea}} = 1020\,\frac{kg}{m^3}$.

However, the relations between position-time and velocity-time are not yet clear even for this simplified problem. We discuss this in the following subchapter.

Differential equations – Limiting of a function

Let us look at the last question within the Calculus domain. Give the time evolution of an object subjected to a drag force proportional to its velocity, if the drag coefficient is denoted by k and the mass of the body by m. Write Newton's second law for the center of mass:

$$m \cdot a(t) = -mg - k \cdot v(t) \quad \Rightarrow \quad a(t) = -g - \frac{k}{m} \cdot v(t).$$

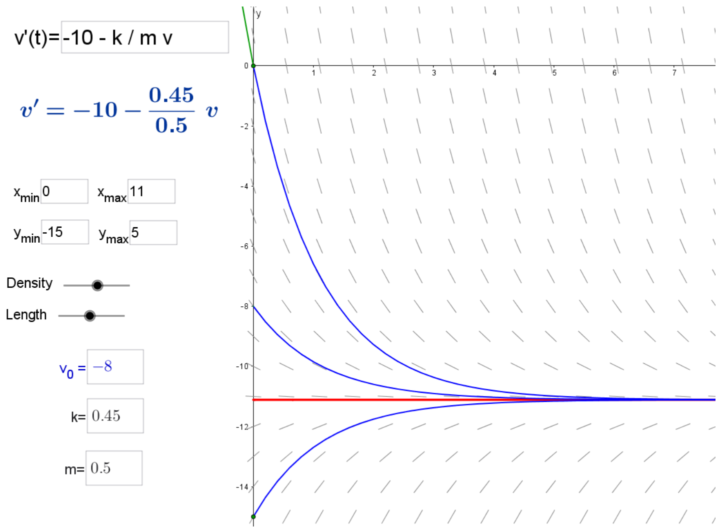


Figure 13. Skydiving analysis (GeoGebra [8])

We express the connection using derivatives. We formulated an equation containing the function and its derivative (differential equations):

$$v'(t) = -g - \frac{k}{m} \cdot v(t).$$

As we saw in the previous chapter, we can draw the tangent field for the differential equation to plot the graph of the velocity function depending on the initial velocity v_0 (Figure 13). With GeoGebra [8], you can also set the initial velocity, the drag coefficient and the mass of the body. We can see that the mathematical model reproduces our dynamic tests, since the velocity is approximated to a constant value. The constant velocity can be calculated by solving the equation $a(t) = v'(t) = 0$:

$$v_{\text{const}} = -g \cdot \frac{m}{k}.$$

When altering the air resistance coefficient and mass values within the GeoGebra aids, there is a noticeable impact on the terminal velocity value.

Solving the initial value problem is a topic covered in university courses (or in high-level classes in high schools). The methodology for addressing such differential equations is detailed in the digital textbook designed for students in life sciences (Karsai, 2021). Here, we obtain the solution with the help of WolframAlpha. The initial values should be $v(0) = v_0$ and $y(0) = h_0$. Then, we obtain

$$v(t) = e^{\frac{-kt}{m}} \cdot \left(v_0 + \frac{gm}{k}\right) - \frac{gm}{k}, \tag{1}$$

$$y(t) = \frac{-m \cdot e^{-\frac{kt}{m}}(gm + kv_0) + gm(m - kt) + h_0k^2 + kmv_0}{k^2}. \tag{2}$$

For simplicity, let us examine the case where the initial velocity $v_0 = 0\,\frac{m}{s}$:

$$v(t) = \frac{mg}{k}(e^{-\frac{kt}{m}} - 1), \qquad y(t) = \frac{m \cdot g(m - kt - m \cdot e^{-\frac{kt}{m}})}{k^2} + h_0.$$

Example 7. *Let us examine the motion of a falling ice ball with the following* GeoGebra teaching aids [1] *(see Figure 14).*

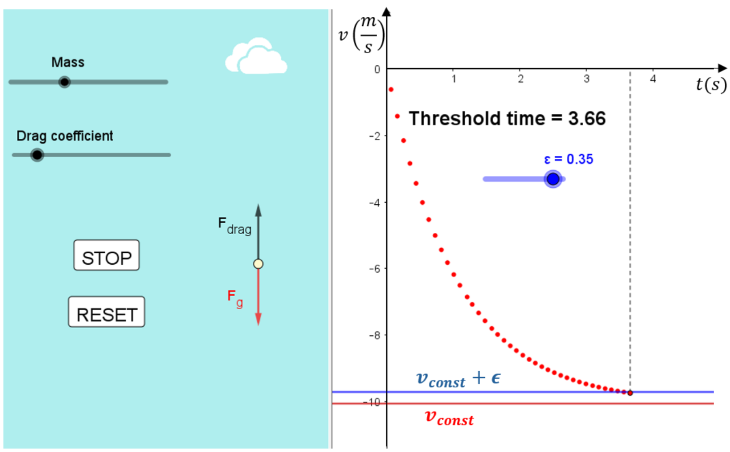


Figure 14. The ice ball $v(t)$, *GeoGebra* [1] function

When we start the animation, the program indicates the instantaneous velocity of the object, and on the right side, you can observe the graph of the velocity function. The animation stops when the instantaneous velocity value is closer to the constant velocity value we calculated above than the set value ϵ. Change the value of ϵ in GeoGebra to a smaller value and observe how long it takes the ice ball to reach that velocity in each case. We find that for any arbitrary drag coefficient (k), velocity becomes arbitrarily close to v_{const} as time is large enough. So the velocity approaches a limit as time approaches infinity. This allows us to illustrate the difficult definition of the limit through a practical example. Using the velocity function gives a similar result:

$$a(t) = v'(t) = -g \cdot e^{-\frac{kt}{m}}.$$

It is obvious that $\lim_{t\to\infty} a(t) = 0$, so the observations made in the previous subchapter were correct, the velocity will be nearly constant after a sufficiently long time. The parameters k and m give the rate of convergence. For high air resistance, the convergence to 0 is faster, for high mass, it is slower. Finally, we introduce two intriguing practical examples.

Example 8. *Let the mass of the body be $m = 0.5\,kg$ and the drag coefficient $k = 0.4\,\frac{kg}{s}$. After how much time can the motion of the body be considered to be nearly uniform?*

With the given parameters, the velocity function and its limit are:

$$v_{\text{const}} = -\frac{-gm}{k} = -\frac{5\,m}{0.4\,s} = -12.5\,\frac{m}{s}, \qquad v(t) = \frac{5}{0.4}(e^{-\frac{0.4t}{0.5}} - 1).$$

Consider motion almost uniform if $|v(t) - v_{\text{const}}| < 0.001\frac{m}{s}$. Our task is to determine the threshold value of t after which it happens:

$$\left|12.5(e^{-\frac{4t}{5}} - 1) - (-12.5)\right| = 12.5e^{-\frac{4t}{5}} < 0.001,$$

$$t > -\frac{5}{4}\ln\left(\frac{0.001}{12.5}\right) \approx 11.792.$$

Example 9. *We throw a ball upwards from the ground with an initial velocity of $v_0 = 15\,\frac{m}{s}$. We know that the mass is $0.2\,kg$ and the coefficient of drag is $k = 0.2\,\frac{kg}{s}$.*

(a) *What is the maximum height of the ball?*

(b) *What is the velocity at which the ball impacts the ground?*

Substituting the parameters ($v_0 = 15\,\frac{m}{s}$, $g = 10\,\frac{m}{s^2}$, $h_0 = 0\,m$, $m = 0.2\,kg$) into the general formula, Equation (2), we obtain:

$$y(t) = -10t - 25e^{-t} + 25.$$

The solution to problem (a) is easily obtained by solving the equation $y'(t) = 0$. The body reaches its highest point at about $t \approx 0.92\,s$. The height is $5.837\,m$.

To determine the time of landing, we need to solve: $y(t) = 0$. Obviously, $t = 0$ is the solution to this equation. But the solution to the problem is the other root of the equation. We cannot solve the equation algebraically, numerical approximation is needed to find it. Using numerical methods, the body in the last example will hit the ground after $2.232\,s$. The body hits the ground at the velocity:

$$v(2.232) = -7.317\,\frac{m}{s} = -26.34\,\frac{km}{h}.$$

In high school, it is generally assumed that all the equations being discussed are solvable. On the other hand, equations that arise in nature cannot usually be solved symbolically. Numerical methods are required, but these are not taken into account in high school, although they, for example, Newton's iteration, can be easily understood by graphical interpretations. It is illustrated in the dynamic diagram at GeoGebra [7], and classroom introduction is detailed in the note (Karsai, 2021).

Summary

This article demonstrates how Realistic Mathematics Education methods can be enriched with digital tools to effectively teach the concepts of limits, differentiation, and integration through the study of motion in a gravitational field. Our interdisciplinary approach combines physical experiments with mathematical theories, thereby deepening students' understanding and encouraging creativity and innovation in addressing real-world problems. We selected motion in a gravitational field as a model example because these phenomena are easily observable and effectively illustrate the historical development of Calculus. This approach is particularly valuable for non-mathematics majors, as it enables them to experientially discover mathematical concepts, thereby developing the modeling skills essential in the modern job market. By linking abstract mathematical concepts with practical applications, we enhance the quality of mathematics and physics education at both the high school and university levels, showcasing the benefits of experiential learning in developing problem-solving abilities.

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