

Psychology – an inherent part of mathematics education

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Abstract. On the chronology of individual stations of psychology and their effect on mathematics education designed as working document for use in teacher training.

The article is structured as a literature survey which covers the numerous movements of psychology towards mathematics education. The current role of psychology in mathematics education documented by different statements and models of mathematics education should provide a basis for the subsequent investigations. A longitudinal analysis pausing at essential marks takes centre of the continuative considerations. The observed space of time in the chapter covers a wide range. It starts with the separation of psychology from philosophy as a self-contained discipline in the middle of the 19th and ends with the beginning of the 21st century. Each stop states the names of the originators and the branches of psychology they founded. These stops are accompanied by short descriptions of each single research objective on the one hand, and their contributions to mathematics education on the other hand. For this purpose, context-relevant publications in mathematics education are integrated and analysed. The evaluation of the influence of concepts of psychology on teaching technology in mathematics is addressed repeatedly and of great importance. The layout of this paper is designed for the use as a template for a unit in teacher-training courses. The conclusion of the article where the author refers to experiences when teaching elements of psychology in mathematics education courses at several universities in Austria is intended for a proof on behalf of the requested use.

Key words and phrases: branches of psychology, mathematics education, longitudinal analysis.

MSC Subject Classification: 01A70, 01-XX, 97-03, 97D80.

Prologue and configuration

The prologue is opened by one of the main ideas of the International Group for the Psychology of Mathematics Education (PME) announced in their Research Report (2022):

[...] to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

A similar line take John R. Anderson, Lynne M. Reder and Herbert A. Simon from the Department of Psychology, Carnegie Mellon University, Pittsburgh, quoting Richard Lesh and Susan J. Lamon (1992) in the Abstract of the paper “Applications and misapplications of cognitive psychology to mathematics education” (2000):

[...] Behavioral psychology (based on factual and procedural rules) has given way to cognitive psychology (based on models for making sense of real-life experiences), and technology-based tools have radically expanded the kinds of situations in which mathematics is useful, while simultaneously increasing the kinds of mathematics that are useful and the kinds of people who use mathematics on a daily basis. In response to these trends, professional and governmental organizations have reached an unprecedented, theoretically sound, and future-oriented new consensus about the foundations of mathematics in an age of information. (Lesh & Lamon, 1992, p. 18–19)

Andreas Obersteiner, Kristina Reiss and Aiso Heinze regard Psychology and Mathematics Education in their publication “Psychological theories in mathematics education” (2018) as closely related fields, too. The authors keep hold of the circumstance that although researchers in both disciplines do not always speak the same language, psychologists and mathematics educators share a common focus on the process of concept formation in several topics of mathematics.

Finally, we focus on the relation between the two disciplines from a mathematical point of view. Karl Josef Fuchs and Hans-Stefan Siller embed Psychology as a building block of the requirements of theory in their ‘Theory-Practice Model of Mathematics Education’ (Fuchs & Siller, 2009) (Figure 1).

Already in the 70s of the last century Erich Wittmann presented his early model of mathematics education integrating the *reference science psychology* (Wittmann, 1974, p. 2) (Figure 2).

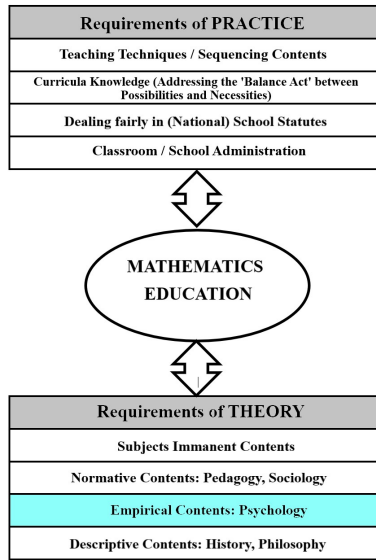


Figure 1. Theory-Practice Model of Fuchs and Siller

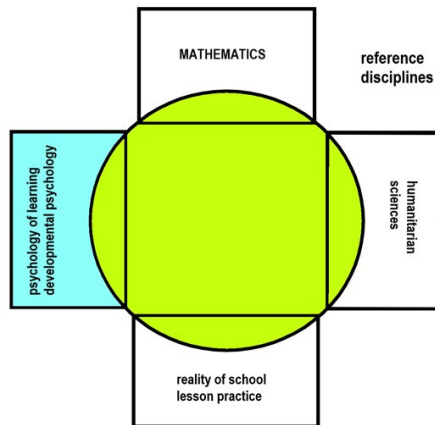


Figure 2. Theory-Practice Model of Wittmann

The main part of the discussion is designed in a longitudinal view. The integration process of the two disciplines should be communicated to the reader

by pausing at essential marks in history. Each mark incorporates originators of a branch of psychology which is named and characterised at this point. Additionally, the contributions of this branch of psychology to Mathematics Education will be described considerably.

Longitudinal analysis of essential marks and relations

We start our journey through the time with the separation of psychology from philosophy as a self-contained science in the middle of the 19th century. The postulation of a psychology as an experimental, empirical discipline had a not-inconsiderable role in this separation (Sachs-Hombach, 2005).

In the beginning of the 20th century, the German psychologists Max Wertheimer, Kurt Koffka and Wolfgang Köhler founded a school of thought, the so-called *Gestalt Psychology*. The purpose of this branch of psychology was to describe human sensations by order principles like *closeness*, *similarity*, *continuity*, *common destiny* or *the law of conciseness*.

In the beginning of the 21st century, Edith Luchins, Professor of Mathematics, gave the following answer to a student who asked for information about how Gestalt theory relates to mathematics:

[...] The book *Productive Thinking* of Max Wertheimer (1945) has many references to teaching math., e.g., to teaching the formula for the area of a parallelogram in a manner that is blind to the structure of the figure (contrastructural) or that takes the structure into account (prostructural); different methods of finding the sum of the interior or exterior angles of a polygon; and different methods of finding the sum of arithmetic series. A thesis is that methods are needed that help the learner to grasp the structure whether it be of a geometric figure or an arithmetic series, etc. (Luchins, 2004)

Currently, Ismael Norulhuda picked up Gestalt principles in her publication “Teaching mathematics with gestalt psychology” (2020). Different principles such as *closure*, *proximity* and *continuation* are used in the paper. Subsequently, the *principle of continuation* out of the list of the presented principles is shown (Norulhuda, 2020):

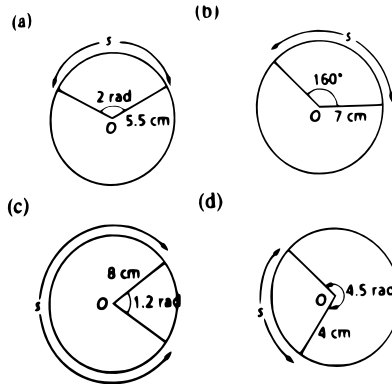


Figure 3. Principle of continuation

From the first half of the 20th century, the US American psychologist John Broadus Watson, founder of the *Behaviourism*, must be mentioned. The Behaviourism, a topic of *General Psychology (GP)*, is a branch of research of psychology which tends to observable, checkable empirical data of human behaviour: factors of cognition as well as those of subjective experience. Introspection does not play any role.

The Behaviourism in the impression of Watson found an Enhancement by the US American psychologist Burrhus Frederic Skinner in the middle of the 20th century. The enhancement consists of adding the feedback to the *stimulus response scheme*.

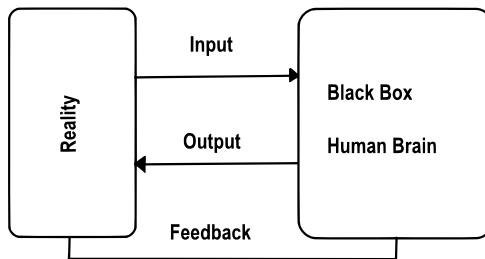


Figure 4. Stimulus response scheme of Skinner

The *Operant Conditioning* of Skinner had a great impact on the Methodology of Teaching Mathematics. The *Programmed Instruction (PI)* has had great influence in the design of computer-based learning scenarios (Pfleger, 2002). Rolf

Arnold, Thomas Prescher and Waltraud Amberger discussed PI as a special method of teaching in the context of technology (Arnold, Prescher & Amberger, 2010). In their presentation they indicate the importance of multimedia-based representation, the so-called *EIS principle* of the US American psychologist Jerome Seymour Bruner (1971): thereby *E* stands for *Enactive*, *I* for *Iconic* and *S* for *Symbolic* representation.

In 1968, the *Zentralstelle für Programmierten Unterricht und Computer im Unterricht an bayerischen Gymnasien* (*Central Office for Programmed Instruction and Teaching Computer Science at Bavarian Gymnasia*) was established. Many programs of mathematics covering, among others, the topic of rational functions, systems of linear equations and determinants, trigonometry, stochastics, differential and integral calculus or mathematical economics were published by several different authors in the 70s (Brockmann, 2015).

The online learning system *bettermarks learning mathematics successfully* used worldwide by 400000 students in the school year 2020-2021 provides a large number of tasks in mathematics. It can be used for teaching as well as for private lessons. The tasks are organised by the concept of Programmed Instruction (Figure 4).

Great influence on mathematics education has had the Swiss biologist and psychologist Jean Piaget. He may be seen as the originator of *cognitive developmental psychology* (*cdp*) with his *Genetic Epistemology* (1953). Therein Piaget describes the development of knowledge as the building of, or modification of already existing schemata as active learning by doing. The change of the quality of knowledge getting older Piaget indicates as development from the *Concrete* to the *Abstract* in his stage model.

The cognitivism of Piaget has had a great impact on the concept of the *constructionism* in the context of computer science. It was founded by the US American scientist Seymour Papert. He stated the following words:

[...] It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, where it is a sand castle on the beach or a theory of the universe (Papert 1991).

Great influence on mathematics education has had the focus of Piaget on active learning to the methods of teaching mathematics.

The contribution of Andreas Filler from the Humboldt University of Berlin is cited as an example. In his *Introduction to Mathematics Education* (2015/2016),

he expands the concept of Piaget by the aspects which Hans Aebli, a student of Piaget, had added.

Filler discusses the *Operative Principle* and its postulations for an *Interdisciplinary Teaching (IT)*. Thus IT should be identified by

[...] Establishing, Recognising and applying relations, dependences and correlations like

- Inverting something (reverse task, trial task) (*reversibility*)
- Swapping something (*commutativity*)
- Creating adjacent tasks (*associativity*)
- Taking tasks apart
- Merging of parts to bigger aggregates
- Describing different approaches
- Different sequential arrangements of single steps
- Taking a detour
- Summing up advantageously
- Variation of data
- Variation of representations (manifold variation between enactive, iconic and symbolic level).

Around the 60s of the last century, the US American psychologist Abraham Maslow characterised human motivation in a hierarchically structured concept, the so-called *Maslow Pyramid*, in the context of *20th Century Humanism*. Its meaning was interpreted by Heinz Bachmann, a Swiss mathematician, in the context of teaching (Dorfmayr, 2019, S. 10) as can be seen in Figure 5.

The considerations of Maslow have played a major role in *social psychology (sp)*.

Nearly at the same time, the US American psychologist Benjamin Bloom presented his *taxonomy* of cognitive development (Bloom, 1956).

The US American psychologists Lorin Anderson and David Krathwohl together with a group of pedagogical psychologists und educators reviewed the taxonomy of Bloom (2000) as can be seen in Figures 6 and 7.

The taxonomies have had great influence on the development of *educational standards in different subjects*. In mathematics they constitute the discussion of a *competence-oriented teaching* (Göldi, 2010; Fuchs, 2021).

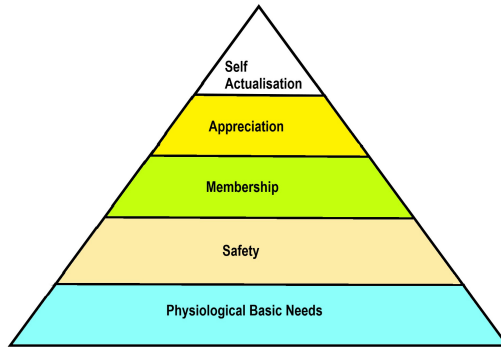


Figure 5. Pyramid of Maslow in the interpretation of Bachmann

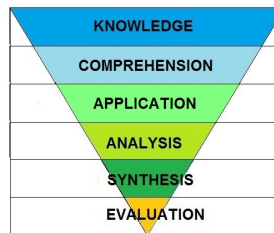


Figure 6. The cognitive levels of the taxonomy of Bloom

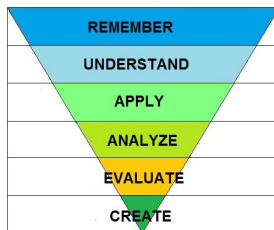


Figure 7. The cognitive levels of Anderson and Krathwohl

The nucleus of the discussion was the following definition of competence by the German psychologist Franz Emanuel Weinert addressing the taxonomy of Anderson and Krathwohl. He defines competences as follows:

Cognitive abilities and skills the individual have available or are familiarized with them to solve specified problems as well as the motivational, volitional and social preparedness and capacity to use the solutions of the problems in variable situations successfully and responsibly (Weinert, 2001).

The output in mathematics was the *competence model in Mathematics* (presentation adapted Siller & Fuchs, 2009).

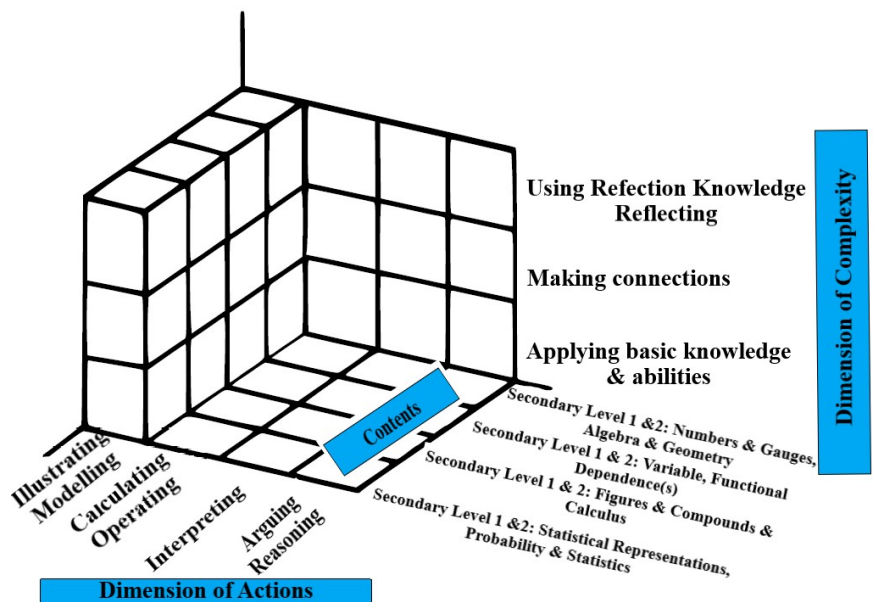


Figure 8. Competence model in Mathematics

In the 60s of the second half of the 20th century, the two US American psychologists Abraham Maslow and Carl Rodgers made the way to *humanistic psychology* (*hp*). The discussion of *self-concept*, referring to a perception of the self of each individual in relation to several characteristics, is a central issue of *hp*. Maslow proposed that human behaviour is corresponding to his pyramid (Figure 5) influenced by *basic needs* followed by *safety*, *love*, *esteem* and finally *self-actualisation*. In the approach of Rodgers, self-concept is based on introjected standards rather than organismic evaluation, and the keypoint in the development of personality is *self-acceptance*.

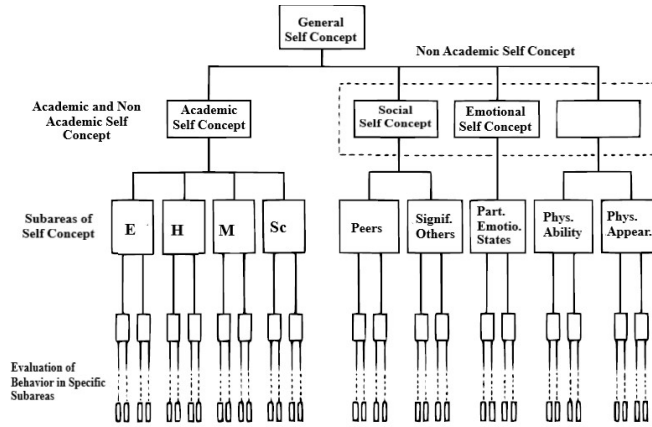


Figure 9. Hierarchical model of self-concept

The relations of the different modules of the self-concept was illustrated in a *hierarchical model* by Richard J. Shavelson, Judith J. Hubner and George C. Stanton (1976, p. 413).

This concept was amongst others adopted by mathematics educators as a tentative explanation of mathematical achievements, such as in the publication of Jakob Kelz in 2019. Actually, the publication “Mathematics self-concept and challenges of learners in an online learning environment during COVID-19 pandemic” (2021) by Rex Bringula, Jon Jester Reguyal, Don Dominic Tan and Saida Ulfa investigated the relation between individual factors of online learners and their self-concept in mathematics in a mixed-method methodological approach.

A topic of *Differential Psychology (DP)*, *achievement motivation*, was investigated by the German psychologist Heinz Heckhausen, who belongs to the post-Second World War period, too. In the center of his model of motivation are the terms *situation*, *action*, *result* and *consequences* (Figure 10).

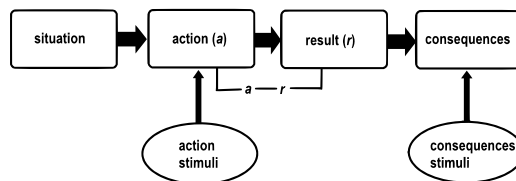


Figure 10. Achievement motivation model of Heckhausen

The discrepancy between action and result plays a decisive role in the pictured situation. Heckhausen may be considered as a pioneer of cognitive psychological researchers in the field of *experimental memory research*, an issue of *General Psychology*. General Psychology formulates scientific findings about human behaviour in general statements. The experiments of Ebbinghaus focus on the process of learning and forgetting. He won his outcomes like the *Learning Curve*, the *influence of repetition* or the *Plateau effect* from the famous *nonsense syllables experiment* (Gundlach, 1985). In 2009, Naomi and Victor Chudowsky presented and Nancy Kober edited the report of the results of the study of the Plateau effect in test scores using the results of tests in mathematics. Figure 11 shows the frequency of the Plateau effects (Chudowsky, 2009, p. 5).

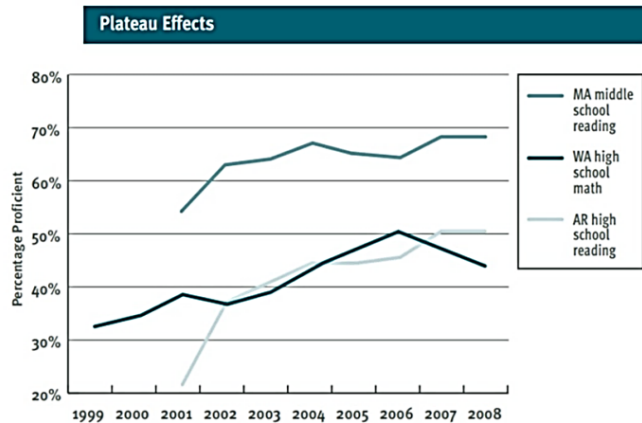


Figure 11. Frequency of the Plateau effects

Nearly chronologically parallelly, the US American psychologist Edward Lee Thorndike presented his theory about *primary mental abilities*. As this research focusses on inter and intraindividual differences in human behaviour, in this case the disposition of intelligence, it must be seen as an issue of *personality* or *Differential Psychology*. One of the factors, namely the Space Factor in Thorndike's theory has had great influence on the research of special perception in Geometry. An example is the publication of Günter Maresch and Sheryl A. Sorby in the *Journal for Geometry and Graphics* (2021, p. 279):

The findings of Ebbinghaus were utilised by mathematics educators similarly. They undertook his outcomes but modified their research questions in the

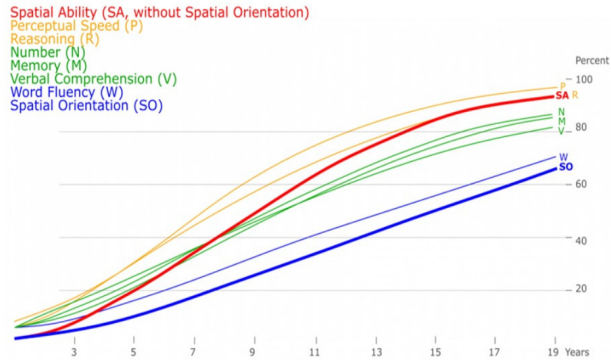


Figure 12. Progress curve of eight intelligence domains. Seven primary intelligence factors according to Thurstone (Spatial Ability (SA) and Spatial Orientation (SO)

context of mathematics education. Some examples are listed consequently. Indicators characterised as the *status of an important policy issue* by the authors are presented in the publication *The Learning Curve* (1996). An example for an indicator which was investigated in this context is *mathematics proficiency* (Suter, 1996, p. 8), see Figure 13.

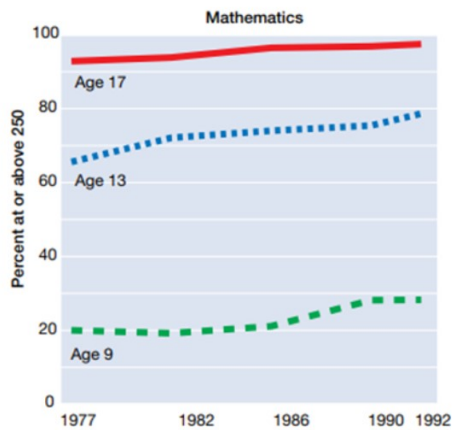


Figure 13. Science and mathematics proficiency: percent of students at or above anchor point 250, by age: 1977 to 1992

The influence of repetition to conceptual understanding in mathematics was investigated by Laila S. Lombibao and Santos O. Ombay. They published the outcomes in the International Journal of Science and Research (2017). One result is presented exemplarily in Table 1:

	Experimental Group N=35		Control Group N=35	
	Posttest	Retention Test	Posttest	Retention Test
Mean	94.56	95.01	69.31	69.70
SD	23.88	17.98	18.17	15.93

Table 1. Mean, standard deviation of students' conceptual understanding on the retention test on circles and plane coordinate geometry

The mode of achievement motivation of Heckhausen in the context of teaching mathematics was treated in the diploma thesis of Hanna Hirschmann (2020, p. 20ff).

Finally, we will discuss psychological theories which found their way into mathematics education in the second half of the 20th or starting 21st century. All of them are approaches in the context of technology. In the 80s of the 20th century, the Australian psychologist John Sweller presented his *Cognitive Load Theory (CLT)*. The theory concentrates on the architecture of human memory. CLT acts on the assumption that the cognitive load of the learners is divided into three areas, which Sweller called *intrinsic load* (induced by learning issues), *extraneous load* (produced by the learning environment) and *germane load* (caused by the learning process). All the three areas should exhaust the memory, see Figure 14.

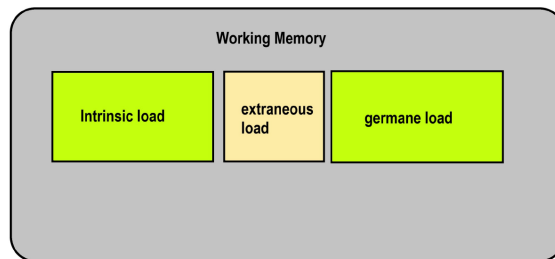


Figure 14. Satisfying use of the Working Memory

Figure 15 shows an undercharging use of the Working memory.

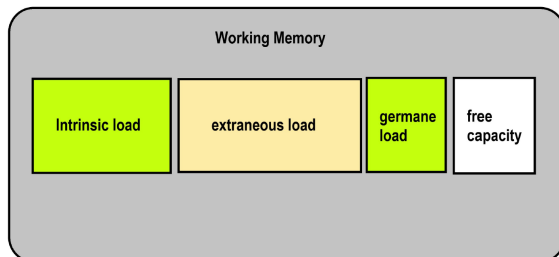


Figure 15. Undercharging use of the Working Memory

Joseph J. Dhlamini, an educator from the University of South Africa discusses the modelling/problem solving process in mathematics in the context of CLT (2016).

The Cognitive Load, in particular the working memory, is strongly connected with the *memory structure of the model* of Alan Baddeley (born 1934) and Graham Hitch (born 1946), which they presented in the 70s of the last century.

The US American psychologist Richard Mayer published the results of his research in multimedia learning in his *Multimedia Learning* (2009) in the starting 21st century. His research mainly focussed on the explanation of scientific and mathematical concepts. Fuchs and Simon Plangg express each principle in the book *Lehr- und Lernmedium Computer (Teaching and Learning Medium Computer* (2022, S. 13-16) as follows:

- The *Coherence Principle* states that visual or acoustical informations which are indeed interesting but irrelevant for the learning target reduce the knowledge acquisition, namely stimulating material without any educational merit affects learning efficiency.
- The *Modality Principle* indicates that audiovisual representations of depicted and textual linguistic informations foster knowledge acquisition in a greater extent than only one representation. Hence the use of spoken text is better than only written text.
- The *Spatial Contiguity Principle* and the *Temporal Contiguity Principle* state that spatial neighbouring representations of depicted and textual informations. The concurrency of these presentations will foster knowledge acquisition more than successive performances.

- *The Personalization Principle* states that individual addresses as well as the use of *pedagogical agents* support the learning process.

Conclusion

Finally, the experiences of the author in teaching the theme as interaction of contents of Psychology and Mathematics Education in numerous courses at the Paris Lodron University of Salzburg, the Leopold Franzens University Innsbruck and the Johannes Kepler University over twenty years are cited as summary and proof of the coherence of the presentation. In the planning stage, the *Zone of proximal development* by Lev Vygotsky (addressed are courses organised to handle tasks more and more independently) must be invoiced. The students have been delighted by the orientation and the structuring via individual stations of Psychology. They have perceived the format as self-contained and motivating.

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