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Teaching Mathematics and Computer Science

The tradition of problem-posing in Hungarian mathematics teaching

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Abstract. Based on the literature, Pólya was influential in problem-posing research. The present paper draws attention to a book written with Pólya's collaboration, which has not yet received sufficient emphasis in the problem-posing literature. On the other hand, Pólya's impact on mathematics education in Hungary has been significant, including the problem-posing paradigm. Two works, published only in Hungarian, that rely heavily on problem-posing are highlighted. Furthermore, it is presented how problem-posing appeared in the Hungarian Complex Mathematics Teaching Experiment (1962-78) led by Tamás Varga.

Key words and phrases: problem-solving, problem-posing, George Pólya, Hungarian mathematics teaching, talent care.

MSC Subject Classification: 97D50.

Introduction

Many think of Pólya[1] as the "father of the modern focus on problem-solving in mathematics education" (Passmore, 2007, p. 44). In addition, characteristics of mathematics teaching in Hungary are often thought of as a tradition of talent management (Hersh & John-Steiner, 1993; Stockton, 2010) and problem-based curriculum development (Andrews & Hatch, 2000, 2001).

These two statements are related, as Pólya's influence is still evident in the teaching of mathematics in Hungary (Gosztonyi, 2016, 2020; Győri et al., 2020;

[1] In this paper, the author keeps the Hungarian spelling of Pólya's name in the text.

Máté, 2006). However, the reality in both cases is much more complex, exciting, and nuanced. In this paper, a common feature in the picture of Pólya and mathematics education in Hungary is emphasized: problem-posing. Specifically, that Pólya played an essential role in problem-posing research as a partially independent field in mathematics education is demonstrated. Moreover, the paper aims to show that mathematics education in Hungary has a tradition of problem-posing.

After writing about problem posing in general, the author presents Pólya's contribution to the field, bringing out a work (Pólya & Szegő, 1925) that has received perhaps little attention in the problem-posing literature. Then, from the history of the Hungarian problem-posing tradition, the author addresses the activities of four personalities (Rózsa Péter, Tibor Gallai, Tamás Varga, and Julianna Szendrei), mainly based on their works in Hungarian, which are little known in the international research community. Lastly, the author demonstrates that problem-posing is still a part of math education practice in Hungary, especially in talent care.

Problem-posing

Problem-posing is an essential element of mathematical literacy alongside problem-solving (Niss & Jablonka, 2014). Although problem-posing has been researched for decades (English, 2020; Koichu, 2020; Silver, 2013), scholars' understanding of problem-posing is inconsistent. Silver's (1994) concept has received wide attention. It involves creating new problems based on situations and reformulating existing ones. Problem-posing has been described by Stoyanova and Ellerton (1996) as the creation of personal interpretations of concrete situations and the structuring of concrete situations as relevant mathematical problems. According to Cai and Hwang (2020), problem-posing in mathematics education involves teachers and students conceiving (or reformulating) and conveying a problem or assignment in a particular pedagogical context. According to Papadopoulos et al. (2021), there are five types of problem-posing methods: generating new problems only; reformulating existing or given problems only; generating and reformulating problems simultaneously; posing questions; and modeling. Despite the diversity, one of two concepts of problem-posing appears in most research papers (Baumanns & Rott, 2021). According to the first concept, problem-posing includes developing a new problem or reformulating an existing problem before, during, or after problem-solving. According to the second concept, problem-posing is the process through which students interpret experiences as mathematical problems. In the present paper, the author uses problem-posing in both senses.

Kontorovich et al. (2012) suggest a framework for handling the complexity of students' mathematical problem-posing in small groups. The framework has five modules: (1) task organization, (2) knowledge base, (3) problem-posing heuristics, (4) group dynamics, and (5) aptitude. Although the conditions of problem-posing in the present article vary from those in Kontorovich et al.'s paper, the author believes numerous components of that framework are applicable in general. In the present article, problem-posing heuristics play a crucial role. Brown and Walter (1990) systematized the "what if not" problem-posing heuristic strategy. The function of the strategy is to pose new problems from already solved ones by varying the conditions or goals of the given one. Here, the problem-posing heuristic is guided by a specific question. The method consists of four steps, namely, (1) the listing of attributes of the original problem; (2) posing "what if not?" questions for attributes; (3) posing mathematical questions about the altered problem; and (4) problem analysis: discovery of the self-posed situation.

Moreover, simple questions concerning, for example, the number of solutions or the generalization of the result, also belong to problem-posing heuristics (Baxter, 2005). For example, such questions may include the following: How many? Is it always true? Is there a pattern? How do we know we have them all? Is there a largest (or smallest) value?

Pólya and his role in the problem-posing research

Pólya's impact is often highlighted in works on problem-posing, mainly regarding his books (1945, 1954, 1981). Silver claims in his seminal work (1994) that Freudenthal and Pólya were among the first researchers to show that posing problems is an integral part of teaching mathematics, referring to (Freudenthal, 1972) and (Pólya, 1954). Leung (2013) stresses that problem-solving is often based on "great problems" at its heart, and this aspect of problem-posing has been viewed as important by many authors, including Pólya. Cai et al. (2015) point out that problem-posing strategies have a long history; in many cases, these strategies rely on Pólya's four-step model. Brown and Walter (1990) based their "what if not" technique on Pólya's "look back" stage. Gonzales (1998) complements Pólya's four-step method with a fifth step called "posing related problems." Cai and Brook (2006) also embedded generating, analyzing, and comparing alternative solutions into the "look back" phase.

Kilpatrick (1987) examines the different phases of Pólya's problem-solving model. During the planning phase, a student could use Pólya's heuristic advice to see if changing the conditions of the problem or breaking it up into more problems makes the original problem easier to solve. Auxiliary problems are frequently generated and solved throughout problem-solving processes (Cai et al., 2020). Davis (1985) also points out that newer reformulations of the problem may be needed in the process of problem-solving: "problem formulation and problem solution go hand in hand" (p. 23). After a problem has been solved, the solution or the problem itself can suggest additional problems in the "look back" phase. The learner also looks back at a problem they have solved improperly or incompletely to see if resolving it in a different method leads to a better answer. If the student becomes an "autonomous problem formulator" (Kilpatrick, 1987, p. 202), he/she can ask questions, such as "How many?", "What is the most/least?", "What makes this work?" (Mason et al., 2010, pp. 165–166). Problem-posing becomes second nature after a period of practice with these types of questions; in Marion Walter's words, one appears to view the world through "problem-posing colored glasses" (Baxter, 2005, p. 122).

Pólya, in his book (1954), highlighted the importance of analogies in mathematics learning. Kilpatrick (1987) claims that Pólya "showed that analogy can be a fertile source of new problems" (p. 208), especially how mathematicians used them to uncover new concepts (Lee & Sriraman, 2011). Likewise, "when mathematicians engage in the intellectual work of discipline, it can be argued that self-directed problem-posing is an important characteristic," Silver (1994, p. 22) argues.

Pólya has developed several examples to illustrate his principles. Perhaps one of the most frequently cited examples is the Pythagorean theorem, which he has dealt with on several occasions (Pólya, 1948, 1954, pp. 15–17). This well-known example illustrates the role of special cases, the general case, and analogies in problem formulation (Walter & Brown, 1977).

Another book was written with Pólya's participation (Pólya & Szegő, 1925), which is rarely cited in works on problem-posing but which influences the Hungarian problem-posing tradition. This includes the organization of problems into problem sequences, where successive elements of the problem sequence can be considered variants of the preceding members. The role of problem sequences in the Hungarian tradition is discussed in detail by Gosztonyi (2016), albeit without referring to (Pólya & Szegő, 1925).

Pólya and Szegő (1925) paid particular attention to the relative arrangement of the book's challenges (Schoenfeld, 1987; Tamarkin, 1928); the problem solver is continuously asked to pay attention to *how* and *where* he is questioned. Much advice will appear later in the book *How to Solve It.* For example, the authors emphasize the role of generalization in the preface (p. VII):

> A more general statement may be easier to prove than a more particular; in such cases, the most important achievement consists precisely in setting up the more general statement, extracting the essential, and realizing the complete picture.

In the present paper, the author does not promise to explain how even a single chapter is put together, but he provides a series of problems to illustrate the essence of the authors' method (Figure 1). The theme of Section 1 of Part 1 is "Additive Number Theory, Combinatorial Problems, and Applications." The setup is based on three problems: *the change problem*, *the postage stamp problem*, and *the weighing problem.*

The change problem (Problem 1): *In how many different ways can you change one dollar? That is, in how many different ways can you pay 100 cents using five different kinds of coins, cents, nickels, dimes, quarters and half-dollars (worth 1, 0, 10, 25, and 50 cents, respectively)?*

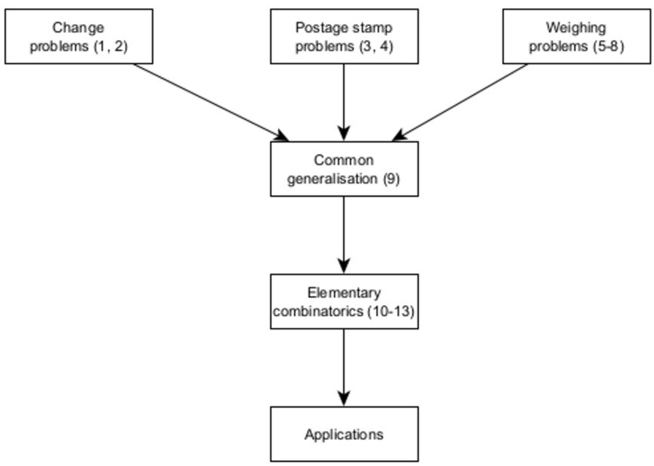

Figure 1. The structure of problems in the first section of the book by Pólya and Szegő (1925)

The postage stamp problem (Problem 3): *In how many ways can you put the necessary stamps in one row on an airmail letter sent inside the U.S., using 2, 4, 6, 8 cents stamps? The postage is 10 cents. (Different arrangements of the same values are regarded in different ways.)*

The first weighing problem (Problem 5): *Someone has a set of eight weights of 1, 1, 2, 5, 10, 10, 20 and 50 grams. In how many different ways can 78 grams be composed of such weights? (Replacing one weight with another one of the same value counts in a different way.)*

The elementary formulation of the starting problems is immediately followed by *generalizations*, e.g., in the case of the change problem, one looks for the number of solutions of the Diophantine equation $x + 5y + 10z + 25y + 50v = n$ in nonnegative integers (Problem 2). Later, the authors arrived at the most general question, which is:

9. Generalize the preceding examples by replacing the particular values of the coin, stamp and weight with $a_1, a_2, \ldots, a_l$.

The point of this general problem is that all previous solving methods were similar and based on a series expansion of certain rational polynomials; this method can be the "kernel" (Katona, 2020) of the problem sequence. The following problems are elementary combinatorial problems, which the authors approach from the general method. For example,

13. Consider the general homogeneous polynomial of degree n in the p variables $x_1, x_2, \ldots, x_p$. How many terms does it have?

Further problems arise in applying problems that have already been solved. For example:

20. [Prove that] Each positive integer can be decomposed into a sum of different positive integers in as many ways as it can be decomposed into a sum of equal or different odd positive integers.

Overall, it can be concluded that Pólya's principles laid down in *How to Solve It* and other later works appeared much earlier in his oeuvre, and that by sequencing problems, he added a new dimension to problem-posing that merits further research.

Tibor Gallai and Rózsa Péter

In 1949, Tibor Gallai and Rózsa Péter published their mathematics textbook for 9th-grade students (Gallai & Péter, 1949), which opened a new period in Hungarian mathematics education (Gosztonyi, 2016). After the formalism of the previous decades, the textbook broke with everything that could lead to formalism in teaching mathematics (Győri et al., 2020). The authors were prominent figures in Hungarian mathematics. Tibor Gallai (1912-1992) worked in combinatorics and graph theory. Rózsa Péter (1905-1977) was a researcher in the theory of recursive functions, but her book *Playing with Infinity* (Péter, 1961) brought her international fame as well.

Of the authors, little is known about Tibor Gallai's views on mathematics teaching. He wrote an article in Hungarian on methodological issues in university education for the journal *Felsőoktatási Szemle* [Higher Education Review], of which he was also a member of the editorial board (Gallai, 1952). However, the author is unaware of Gallai's theoretical work on public education.

Rózsa Péter worked in primary teacher training and mathematics teacher education for many years. In addition to her joint textbook with Gallai, she wrote a university booklet (Péter, 1948) and participated in developing the curriculum for classes with a special mathematics program. Péter was acquainted with Pólya's work, and there are some direct indications of this in her works. First, as early as 1948, four years after the first edition of *How to Solve It* and one year before the release of the book she wrote with Gallai, she recommended Pólya's book as reading material in her university booklet for primary school prospective teachers (Péter, 1948). At that time, Pólya's book had not yet been translated into Hungarian, and it is unlikely that Péter's students would have read in its original form, so its inclusion certainly reflects the impression Pólya made on Péter. Second, in an obituary (Péter, 1976) written on the death of her famous contemporary and colleague, László Kalmár (1905-1976), Péter draws parallels between the "Pólya–Szegő" problem book and Kalmár's teaching method (p. 730).

> As for teaching mathematics, Kalmár already knew [during the university years of Péter and Kalmár, i.e., in the 1920s] one of the basic ideas behind the modernization of mathematics education today: that it is effective if students discover almost everything through a suitable series of problems. However, unfortunately, the excellent book by György Pólya and Gábor Szegő, which introduces analysis in this way, was not known at the time;

> Kalmár was the living "Pólya–Szegő" of his fellow students; he introduced the most diverse problems of mathematics with such a series of problems.

The spirit of the textbook (Gallai & Péter, 1949) is entirely in line with the ideas of Pólya. In this paper, only one problem has been chosen from this textbook, which is included as part of the supplementary material (Geometric maximum and minimum problems, geometric inequalities), so it is intended for higher-ability students.

Problem: *Find the shortest path from one point to another if you have to touch a point on a line.*

The solution depends on the position of the points, but for now, only the case where the line does not separate the points (Figure 2) will be considered. The analysis of the problem itself is an excellent manifestation of the heuristic method, with the transformational approach as the source of intuition. However, in the following, the author will only deal with the "look back" phase and the problem sequence generated within it.

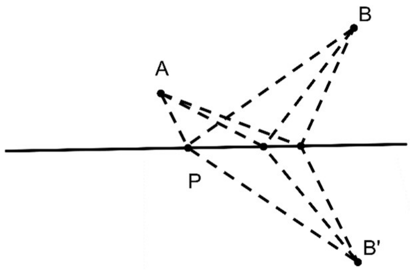

Figure 2. The "shortest path" problem in (Gallai & Péter, 1949)

The authors suggest several problem variations, all of which can be formulated using the paradigm of the "what if not" method. *What if not...*

I) *we must touch not one line but two.*

II) *we are not looking for the shortest path but the longest.*

III) *we investigate not* $PA+PB-t$ *but* $|PA-PB|$*, i.e., the difference between the segments to the line. What are its maximum and minimum?*

Particularly noteworthy are questions where there is no extremum. E.g., for Problem II, where the set of path lengths is not bounded from above, and the

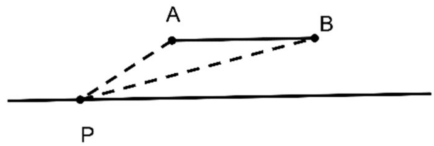

Figure 3. A special case for problem variation III

answer is easy to justify by elementary tools. Concerning Problem III, the question about $|PA - PB|$ is more challenging. While $|PA - PB| \leq AB$ is always valid, there is no maximum if AB is parallel to the given line (Figure 3). The mathematical background for solving the problem goes beyond the mathematical knowledge of ninth-graders. The authors do not hide this fact nor make the student believe that a naive explanation is a proof, but they draw attention to how attractive the phenomenon is (p. 397).

> We suspect that the further away P is, the larger the distance difference will be. This suspicion is true, but we will not go into the lengthy proof. We only want to point out the interestingness of the result. As P gets further away, the difference in the distances in question becomes increasingly greater; however, it cannot be arbitrarily large, since it must always be less than the length of section AB. [...] It seems odd that there is no maximum, since, in everyday life, we are used to other things.

Several pedagogical principles can be discovered in the above example. First, the authors take a *vertical approach* to the curriculum. The issue could be addressed by using coordinate geometry, where the position of P can be defined by a variable t on the line, which could be the x-axis. The investigation of the function $d(t) = |PA - PB|$(monotonicity, extrema, limit at infinity) is the mathematization of the problem. The necessary mathematical tools are not yet available to the students, but the problem is understandable to them. Second, the authors build the student's interest in *cognitive conflict*: the mathematical facts do not necessarily correspond to our intuitive experience (that there is always a maximum in a finite set of numbers). Both phenomena, vertical curriculum design and cognitive conflict, are prominent in the literature on mathematics education (e.g., Freudenthal, 1972; Fujii, 2014).

Similarly, both phenomena play a role in the work of our next author, Tamás Varga, who also incorporated an "accepting attitude toward unfamiliar cognitive problems" (Klein, 1980, p. 47) into the principles of the Hungarian Complex Mathematics Teaching Experiment between 1963 and 1978.

Tamás Varga

Tamás Varga (1919-1987) was a leading figure in Hungarian mathematics education in the second half of the 20th century. His significance is due to the reform project called the "Complex Mathematics Teaching Experiment," which Varga spearheaded between 1963 and 1978, which led to the national curriculum in 1978 and still influences Hungarian mathematics education today. The reform offered a new curriculum, strategy, material, resources, and teaching styles. Varga focused on students' intellectual, physical and social activity, problem-solving, and inquiry (Gosztonyi, 2020).

Varga's principles included the application of problem-posing in the teaching process. In his work, problem-posing primarily based on open situations was observed. He writes about the situation as a problem source (1987, p. 29):

> Nobody disputes the importance of ready-made math problems in schools, both with and without texts. However, an important recognition is that it does not permeate school practice enough today: open-ended situations in which pupils recognize and formulate a mathematical task are needed, often several different ones. Life does not present mathematical problems in textbook language but situations. It provides the raw material for the tasks.

This idea is not far from what Pólya writes about ready-made problems and problems seen in the environment (1981, p. 157).

> Problems with a background connected with the world around us, or with other domains of thought, and problems involving plausible reasoning, challenging the judgment of the students, have more chance to lead them to intellectual maturity than the problems that fill the textbooks and serve only to practice this or that isolated rule.

The next task illustrates Varga's idea from the 5th-grade textbook written under his guidance.

What does Table 1 say about the number of cars and motorcycles? Formulate questions! (Eglesz et al., 1979, p. 159)

year	Number of cars	Number of cars owned by private individuals	Number of motorcycles
	Numbers are all year-end values		
1960	31000	18000	236000
1965	99000	83000	391000
1966	117000	100000	445000
1967	145000	126000	472000
1968	164000	141000	513000
1969	192000	167000	556000
1970	240000	213000	611000
1971	291000	261000	667000
1974	491000	462000	726000
1975	580000	551000	729000

Table 1. Number of cars and motorcycles. Reconstruction of the table from (Eglesz et al., 1979)

The table contains actual data from a statistical yearbook. Some possible questions are also described in the teaching manual (Eglesz et al., 1981), e.g., asking students to estimate data for years not included in statistical yearbooks. The handbook also encourages teachers to formulate questions themselves and pupils to explore different situations themselves (p. 52):

> To practice reading tables and graphs, it is a good idea to get a statistical pocketbook and look up different problems. Then, have the children try to explain a graph or table independently. [...] In this way, the pupil can see the close connection between reality (the environment and its phenomena) and mathematics.

In his joint paper with Halmos (Halmos & Varga, 1978), they define the cognitive goals of mathematics education in terms of a dual system of thinking operations (synthesis and analysis) and student behavior (receptive-reproductive and productive). The article also provides an illustrative example of a problem-posing situation for 15- to 18-year-old students. This example is also crucial because problem-posing arises as a modeling task.

What time do you need to leave for school to arrive on time?

The situation is given but not precisely explained. The student must formulate the problem in concrete terms and describe the circumstances. For example, interpret what it means "to arrive on time." A possible question might be the following: *We cycle to school. The journey time is 15-20 minutes, depending on traffic. What time should we leave so that we have at least an 80% probability of arriving at 7:55?* This particular problem formulation is not mentioned in (Halmos & Varga, 1978), but the author of the present paper has used it in this form several times.

The student can obtain the empirical distribution diagram of travel times by measuring the time on the transport device used or by building a model of the problem. Then simplify the problem so that the delay is 0, 1, 2, 3, 4, or 5 minutes relative to the 15-minute minimum travel time. Students could test the question on the basis of a uniform distribution, for example, by throwing dice, where the number is thrown minus 1 is the delay. One can set up a model where the probability of the extreme cases is smaller than the probability of the intermediate cases. Halmos and Varga propose the repeated addition of random numbers (0 or 1 in the simplest case) thrown by a pocket calculator. This is considered a "didactic" model, because the result obtained from the calculator (currently more like from a computer simulation) can be compared with the calculated probability, which is an easy combinatorial problem for the student to approach. The probability of a k-minute delay is:

$$p(k) = \frac{\binom{5}{k}}{2^5}, \quad k = 0, \ldots, 5.$$

In the end, Halmos and Varga refer to statistical distribution functions. The above mathematical model of the problem is based on the binomial distribution.

Among others, Varga's colleague, Julianna Szendrei, who was renowned both in Hungary and abroad, continued Varga's legacy after Varga's death.

Julianna Szendrei

Julianna Szendrei (1948-2013) worked closely with Tamás Varga in the Complex Mathematics Teaching Experiment and was involved in curriculum development. Moreover, she was an outstanding figure in teacher training in Hungary. Her scientific interests are wide-ranging, including issues of proof and reasoning (Szendrei-Radnai & Török, 2007) and general theoretical questions in mathematics education (Boero & Szendrei, 1998). From 1988-1992, she was Vice-Chairman of the CIEAEM International Commission for Mathematics Education, and in 2003, she became its president.

In a dialog-form book (Szendrei, 2005), she expresses her views on the teaching of mathematics to teachers, parents, and the general public. This work reflects Pólya's considerable influence. Concerning Pólya's "look back" phase, the imaginary partner asks a question about the value of problem-posing (p. 130):

> *Would it not be better to solve another problem instead?*
>
> This is a difficult choice. Solving a challenging, new problem is a joyful way to engage students. To "chew over" the one already solved requires a great deal of mental effort on their part – an activity that is worth keeping as brief as possible at first and only gradually increasing the time. The time invested will pay off handsomely and is essential for progress.

The practical implementation of her principles is presented in the booklet (Radnainé Szendrei, 1988) that was published in the series of Középiskolai szakköri füzetek [Booklets for math circles for secondary school students]. This particular problem collection was designed for secondary school students (grades 9-12) in vocational training, i.e., for students who were interested in mathematics but (generally speaking) not exceptionally gifted in mathematics. The conception of the elaboration includes the principle of "searching for generalizations." For example, the author writes in the introduction:

> We "play" with a problem even after finding a solution. By examining the problem from several angles and searching for generalizations, the aim is to integrate the newly acquired knowledge more deeply into the solver's previous knowledge and to give the reader a complete picture of how mathematics is done.

In this short quotation, two essential insights were formulated: on the one hand, through problem variation, the student's schema for each mathematical knowledge and concept was extended, and on the other hand, the student "cultivated" mathematics, i.e., behaved as a mathematician.

The following example illustrates the nature of the booklet.

<u>Problem</u>: *Prove that the sum of squares of any five consecutive integers is always divisible by* 5.

The problem is a task from the final round of the National Competition of Vocational Schools in 1982. One can quickly solve the problem using algebraic operations after understanding and mathematizing the text. Let us denote the middle number by a! The sum in question is:

$$(a-2)^2 + (a-1)^2 + a^2 + (a+1)^2 + (a+2)^2 = 5(a^2+2).$$

The problem involves obvious generalizations and variations of problems. For example, in the book, Szendrei poses two problem variations related to the number of terms and divisibility.

Problem: *For which n natural numbers are the following statements valid: $S = (a+1)^2+(a+2)^2+\cdots+(a+n)^2$ is divisible a) by 5, b) by n, where a is an integer?*

The 5-remainders of successive square numbers form a periodic sequence: $1, 4, 4, 1, 0, 1, 4, 4, 1, 0, \ldots$, from which it follows that S is divisible by 5 for each a if and only if n is a multiple of five. Moreover, by calculating S, one obtains $S = n \cdot a^2 + an(n+1) + \frac{n(n+1)(2n+1)}{6}$. It follows that the answer to the second question is that S is divisible by n, if and only if n is a multiple of neither 2 nor 3.

Do these variations fulfill what Szendrei promised? According to Schoenfeld's well-known system (Schoenfeld, 1985), mathematical problem-solving performance is influenced by mathematical resources, heuristics, control, and belief systems (p. 15). Here, the author will only deal with the first three components. The mathematical background of the original problem was the knowledge of algebraic manipulations, and the heuristic background is also simple: "calculate." Control has not been particularly concerned. Therefore, what is the added value of these variations, and why are they worth considering? First, the logical structure of the variations makes them challenging for the targeted student. The task is determining the *set* of n-s such that for *every* a, S is divisible by 5 or n. For example, for $n = 7$, *there exists* a such that S is divisible by 7, but not all a has this property. Similarly, the second statement is not true even for $n = 2$ (counterexample). This analysis can help to understand the logical structure of the task as part of the control process. The logical structure of the original problem is obscured because of the mechanical solution, while in the problem variations, Szendrei exploits this potential.

Second, the mathematical background broadens it and the more sophisticated algebraic knowledge of the sum of squares complements the simple algebraic manipulation.

Third, calculating with remainders is a method that is often used in divisibility problems and can be considered part of the heuristic knowledge.

In the 2000s

Problem-posing is still a living tradition in Hungary, although it would be an exaggeration to say that it is common in classrooms.

The author first highlights a problem book (Fazekas & Hreskó, 2006)[2]. The possibility of rich generalizations characterizes the problems intended for gifted seventh- and eighth-graders. The book's unique feature is that students prepared the solutions and problem variations presented under the guidance of their teachers. The material shows that problem-posing, based on a given task, can be a living and working classroom practice for gifted children. Next, the author selects a typical problem from the book.

"Always square numbers." Show that in all bases greater than four, 441_a *is a full square.*

The solution is direct: $441_a = 1 \cdot 1 + 4 \cdot a + 4 \cdot a^2 = (2a+1)^2 = (21_a)^2$. A pupil asks the question (recorded in the book) whether there are any other "always squares," i.e., numbers that are square numbers in all possible number systems. The "How many?", "How do we know we have them all?" questions appeared, which is part of the classroom culture. They show other three-digit "always squares": 100, 121, 144, 169, and point out that 225 is not an "always square number." They said there is no two-digit "always square," but the general problem is left open in the book, thus providing additional material for later work.

Their teachers gave more problem variations. For example,

Are there any numbers that are not squares in any number system? ("Never square numbers")

The problem is challenging even for two-digit numbers if one wants to determine all the "never square" numbers. If the number is ab_k, then the question is about the squares in the arithmetical sequence $a \cdot k + b$.

The second example is connected again with talent care. Mathematics camps are a long-established and perhaps the most compelling part of talent management in Hungary (Győri et al., 2020). Student problem-posing as a method is a committed element of the Pósa camps (Juhász, 2019). Bóra (2020) presents a series of problems from the camp, the starting point of which was provided by the session leader, but the students formulated further questions. The basic principle of the process is that the initial problem is followed by a brainstorming

[2]The title, literally impossible to translate into English, contains a reference to the imaginary country of "Bergengócia" in Hungarian folk tales.

session, where the simpler questions are answered immediately. Then, they discuss what makes a new question good. For example, whether the posed question is well defined. Does it add a new element to the problem?

Further research

Further research is needed to investigate current classroom practices of problem-posing in Hungarian classrooms. As a step in this direction, the author of this paper and his coworker Eszter Kónya have added problem-posing to the problem-based learning paradigm and looked into how problem-posing can be used to aid in teaching mathematics in a typical school setting (Kónya & Kovács, 2019, 2022; Kovács & Kónya, 2021).

Conclusion

The author concluded that Pólya's principles in *How to Solve It* and other later works appeared much earlier in his oeuvre, namely, in (Pólya & Szegő, 1925). From the point of view of problem-solving, this fact has been pointed out by several scholars, see, e.g., (Schoenfeld, 1987). However, by sequencing problems, this book also added a new dimension to problem-posing that merits further research.

Two types of problem-posing in 20th-century Hungarian mathematics teaching practice were investigated. First, problem-posing based on a problem that has already been solved can be seen in a textbook (Gallai & Péter, 1949) and a book on problems for talent care (Radnainé Szendrei, 1988). Second, situational problem-posing appeared in the principles of the Hungarian Complex Mathematics Teaching Experiment led by Tamás Varga.

Some of the relationships and interactions between the people presented in this paper have already been studied by researchers, such as Gosztonyi, who investigated the relationship between Rózsa Péter and Tamás Varga (Gosztonyi, 2016, 2020) and the influence of Pólya on Tamás Varga (Gosztonyi, 2016; Máté, 2006). In this paper, the author noted that, in addition to Tamás Varga, Pólya had a significant influence on Julianna Szendrei, and he also listed some indications that Pólya influenced Rózsa Péter, especially by (Pólya & Szegő, 1925).

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