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Realizing the problem-solving phases of Pólya in classroom practice

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Abstract. When teaching mathematical problem-solving is mentioned, the name of Pólya György inevitably comes to mind. Many problem-solving lessons are planned using Pólya's steps and helping questions, and teachers often rely on his heuristics even if their application happens unconsciously. In this article, we would like to examine how the two phases, *Making a plan* and *Looking back*, can be realized in a secondary school mathematics lesson. A case study was designed to observe and analyse a lesson delivered using cooperative work.

Key words and phrases: teaching mathematics, problem-solving, making a plan, looking back, cooperative work.

MSC Subject Classification: 97B10, 97C70, 97D40, 97D50.

Introduction

Our study describes a mathematics lesson where students worked in cooperative groups and analyses how Pólya's two phases were used during the lesson. We would like to emphasize that the four phases mentioned by Pólya strongly depend on each other, since, without *Understanding the problem*, a *Plan* can neither be designed nor carried out, and there is definitely no *Looking back* on the solution. However, this article focuses on the phases: *Making a plan* and *Looking back*. The reasons for selecting these two phases are that *Making a plan* is difficult for the students, even for the more talented ones, and *Looking back* is often neglected in the problem-solving process, not only in Hungarian mathematics education but also in other countries. Accepting these, we concentrate on *Making* *a plan*, and *Looking back* from the teacher's and students' points of view, and observe how we can realize them in classroom teaching.

Theoretical base

Finding the solution idea, making a plan

Finding the way that leads to the solution and *Making a plan* is the hardest phase in the problem-solving process. It is both hard to find a correct solution method for a given problem and hard to teach students how to find a good solution plan. Pólya formulates more than 25 questions, and pieces of advice for making a solution plan, but these are mostly too general for students. (Pólya, 1977)

On the other hand, this phase contains only three words, "ATTACK! STUCK! AHA!" in Mason's problem-solving model (Mason, 1982). In Nokes' problem-solving model this phase includes the following guidelines:

> Focus on deep features of the problem, access other relevant knowledge! Search for operations: schema, relevant knowledge, facts, procedures, constraints. Operation application: Apply appropriate procedures given the context. Engage domain-specific forward-working strategies (Nokes, 2010).

In a mathematics lesson, the main questions related to *Making a plan* are: "When does the teacher need to help the students?" and "How much help should they be given?". The situation is quite complex in the classrooms, as different students may have rather different struggles and questions while solving problems.

Looking back in the problem-solving process

In his original *Looking back* phase Pólya included the following guidelines for helping. *Looking back* at a problem and its solution:

- Examine the solution obtained.
- Can you check the result? Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

According to Pólya, students can consolidate their knowledge and develop their ability to solve problems by "looking back" at a completed solution and by reconsidering and re-examining the result and the path that led to it.

Recent research in Singapore involving problem-solving also indicates that while Pólya's stages remain a prominent working model, few studies report successful implementation of *Looking back* (in the form that goes beyond mere checking as discussed above) among the participants. Against the continuing curricular emphasis of Pólya's fourth stage, the lack of literature on successful enactments of *Looking back* in actual classroom settings signals an urgent need for more research to be directed in this area. Moreover, teachers need to be supported by instructional materials that will aid their attempts at focusing on *Looking back*, instead of leaving them to work out their lesson resources all by themselves. (Leong et al., 2011)

Cai et al. (2006) extend Pólya's thoughts on *Looking back*, they propose three approaches to encourage the students:

(1) *Generating, analysing and comparing alternative solutions.*

(2) *Posing new problems.*

(3) *Making generalizations.*

Cooperative work

People often think that cooperative work is identical to arranging students into groups and giving them problems to solve or tasks to carry out. The major difference between simple group work and cooperative work is that in the case of cooperative work the success of the groups depends on the ability of the members to support each other. To become a successful team, the members have to trust each other and rely on each other (Kagan, 2004). To ensure that the workload is shared equally among the participants in a cooperative lesson the following four principles should always be present: positive interdependence; individual accountability; equal participation; simultaneous interactions (Johnson & Johnson, 1994).

Arranging students into groups of four is the best option, as four people can be rearranged into two pairs if necessary. Furthermore, nobody feels neglected or left out. Grouping can be random, but it might depend on the students' ability or the students' attitude to mathematics (Burns, 1990).

In a cooperative lesson, not only the classroom arrangement but also the teacher's role is altered. Instead of playing a leading role in delivering the lesson, the teacher becomes an observer or a coach who consults students when necessary and provides help for groups who need it. It is the teacher's responsibility to maintain a productive atmosphere in the classroom and to ensure that any occurring obstacles are dealt with in the learning process (Crabill, 1990).

Some remarks about the Hungarian situation

Hungarian Mathematics textbooks usually contain one worked example after demonstrating the new material. Generally, the problems are closed problems and will end with the obtained result without looking back or discussing the solution method. In the case of geometrical construction problems, we might find the *Looking back* phase in the form of different solutions and conditions of the solutions. In the lower grades, *Looking back* means checking the result, and answering the question.

Methodology

A case study investigates one single case and aims to generalize based on experience (Tight, 2017). In this case study, we examine the mathematical problem-solving of secondary school students focusing on the steps of problem-solving described by Pólya (1977) with special attention to the phases of *Making a plan* and *Looking back*. Teaching children to solve mathematical problems has always been one of the main tasks of mathematical education. To become successful problem solvers in their future lives they need to experience the solution of many problems, thus storing solution methods in their long-term memory (Lovell, 2020).

Although all four steps suggested by Pólya are equally important and build upon each other, in this article we would like to focus on *Making a plan* and on *Looking back*.

Questions

Based on the observations of the teacher related to one particular mathematics lesson, we wanted to answer the following questions:

(1) Which methods do students use to try to solve the given problems? How do they try to *Make a plan*?

(2) How is the *Looking back* phase realized in the lesson?

Participants

The class where the observation took place consisted of 16 students all of whom were 15-16 years old boys. They were students of a class that has a higher number of foreign languages lessons and a higher number of mathematics and physics lessons, preparing the students for higher education in science-related majors. The weekly number of their mathematics lessons was four. Following these, we can say that the participants have a positive attitude to mathematics and are motivated and willing to solve mathematical problems. The majority of the class is talented in mathematics with two students whose achievement is outstanding. However, some students are average, sometimes struggle with understanding the more complicated ideas. The half-term grades were the following: seven students achieved a 5, six students achieved a 4, and three students achieved a 3 (5 is the highest achievable grade and 1 is the lowest achievable grade).

The lesson

The topic of the observed lesson was interior angles and the number of diagonals in polygons. Students already had some prior knowledge of the topic. The previous topic covered the sum of the interior angles in a triangle. The learning method applied in delivering this lesson was cooperative group work, which gave an opportunity for the teacher to have a better insight into students' problem-solving strategies and thinking methods.

Lesson outline

The topic of the lesson: interior angles of polygons, number of diagonals in polygons.

Previous lessons: properties of triangles and quadrilaterals; the sum of their interior and exterior angles.

Warm-up exercise: Decide whether the following statements are true or false:

(a) Every regular polygon has line symmetry.

(b) The number of symmetry axes of an n-sided regular polygon is $2n$.

(c) The number of symmetry axes of an n-sided regular polygon is n.

(d) The number of symmetry axes of a $2n$-sided regular polygon is n.

(e) None of the regular polygons are centrally symmetric.

(f) Every regular polygon with an even number of sides has central symmetry.

Working format: individual thinking, students decide what the correct answer is. Discussion and checking are frontal.

Main task: Students work in small groups, which were formed at random, on the solution to the following problems:

(1) Calculate *the sum of interior angles* in a pentagon, a hexagon, and a heptagon. How did you work out the sum? What do you notice? Can you generalize your idea and using what you discovered to calculate the sum of interior angles in an n-sided polygon?

(2) Determine *the number of diagonals* in a pentagon, a hexagon, and a heptagon. How did you work out the total? What do you notice? Can you generalize your idea and, using what you discovered, to calculate the number of diagonals in an n-sided polygon?

For checking the results and discussing the solution methods, the whiteboard is divided into four sections, one for each group (Figure 1):

Sum of interior angles:	Number of diagonals:	Sum of interior angles:	Number of diagonals:
Sum of interior angles:	Number of diagonals:	Sum of interior angles:	Number of diagonals:

Figure 1. Whiteboard picture

After the time given for group work is up, one spokesperson of each group presents their generalized formulae on the board (Figure 1) and describes how they reached their conclusion in a few words. As a final step, there is a frontal discussion. We compare the formulae and explain possible differences.

Time filler activity: How many sides does a regular convex polygon have if one of its exterior angles is 24°? (They already know the sum of exterior angles of polygons.)

Lesson outline

In this section, we analyse the students' work by relying on the observations and the reflections of the teacher, and the notes they made in their exercise books.

After each group had a chance to present their work, the following was written on the board (Figure 2):

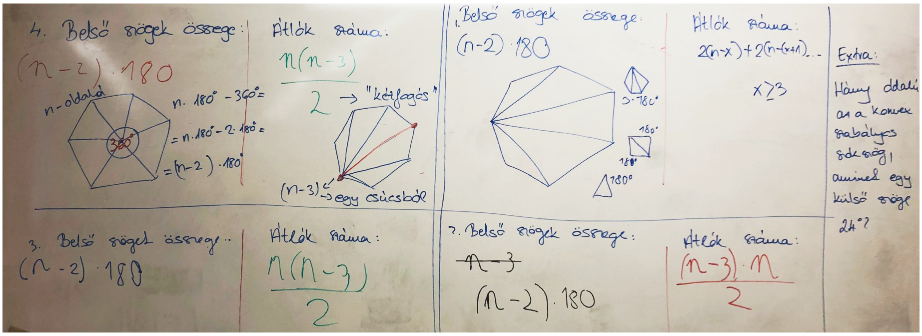

Figure 2. The whiteboard with the formulae

Solution methods to find the sum of interior angles of polygons

<u>Solution 1</u>

Students divided the polygons into triangles so that one vertex of each triangle was the centre of the polygon (they used regular polygons). They realized that using this method the number of triangles they obtain equals the number of sides of the polygon. They also noticed that if they calculate $180° \cdot$ *the number of sides*, they have to subtract $360°$ from the sum because the triangles "meet" in the middle forming a full rotation, but this does not count in the sum of interior angles.

<u>Solution 2</u>

Students joined one vertex of the polygon to other vertices thus dividing the polygon into triangles. They noticed that using this method the number of triangles they obtained is two less than the number of vertices. Following this way of thinking, the sum of interior angles can be calculated as $180° \cdot$ (*the number of sides* $- 2$), see Figure 3.

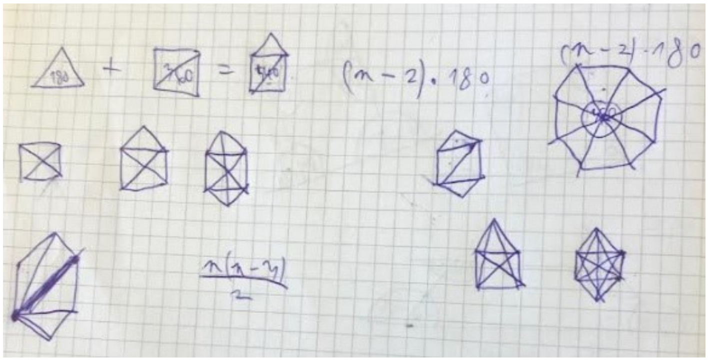

Figure 3. Student's attempt to generalize the sum of interior angles in polygons

Solution 3

Students calculated the sum of the interior angles of a pentagon (540°), a hexagon (720°) and a heptagon (900°), and they noticed that as the number of vertices increases by one, the sum of the interior angles increases by 180°. This method can lead to the introduction of mathematical induction. In the observed lesson, there was not enough time to have such a discussion.

Solution methods to find the number of diagonals of polygons

Solution 1

Students drew and counted the diagonals in pentagons and hexagons and tried to find a formula based on their results. From the students' work it is difficult to see which method they used, but in the frontal discussion it was clarified that to obtain the number of diagonals in a polygon, they calculated the number of diagonals that can be drawn from one vertex ($n-3$) and multiplied this by the number of vertices (n). This way every diagonal is counted twice, so the product must be divided by two. So, the formula for the number of diagonals is $\frac{n(n-3)}{2}$. One student noticed that this final step also appears in the problem of calculating the number of handshakes in a group with n people.

Solution 2

Another possible solution is starting from a quadrilateral which has two diagonals. In the next step, the quadrilateral is extended to obtain a pentagon (Figure 4). Thus, the number of diagonals increases by three: one of them is the side of the quadrilateral, the other two are line segments joining the new vertex to the original vertices of the quadrilateral.

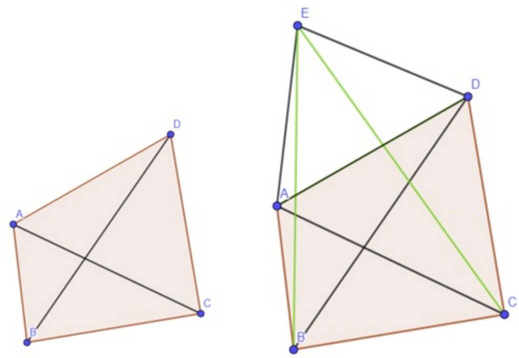

Figure 4. Extending polygons

When extending the pentagon into a hexagon the number of diagonals increases by $4, \dots,$ etc. Based on this reasoning, we notice that in the case of an n-sided shape the number of diagonals equals $2 + 3 + 4 + \cdots + (n - 2)$ which equals to $\frac{n(n-3)}{2}$. One of the groups worked with the sum of diagonals, and we can see the following in their work: $n - 3 + n - 3 + n - 4 + n - 4 + n - 5 \cdots$ (Figure 6). The explanation for this method was that they calculated the number of diagonals from each vertex in case of an n-sided shape, decreasing the number as they went along, but the group was unable to find a general formula for the total number of diagonals.

To sum up, the final formulae written by the students were almost the same, although the ways leading to the formulae were different. When asked to present their solution plans, in the whole class discussion, the students struggled with using mathematically correct expressions and they found it difficult to present their ideas and methods logically. Furthermore, as it can be clearly seen from their work, students did not present their ideas consistently and their notes are often hard to understand. However, every student participated actively in the problem-solving process and, as a result of group work, different plans were made and many ways of thinking were present. One of the students commented on the lesson saying: “we did not write much, but we learned much more this way.”

Since working in groups is slower and requires more time than a frontal presentation, in the next lesson the formulae had to be revised and the application had to be practiced through exercises.

Answering the questions

In this section, we would like to answer our initial questions. First, we would like to make some general comments about the lesson. The instructions for group work were explained in frontal work. The teacher explained the task to every student without giving any further instructions regarding the solution method. After asking students to arrange themselves into groups, they were requested to start discussing their ideas in small groups. In the beginning, some students raised their hands needing clarification. As the questions from the different groups were similar, the teacher decided to answer them in front of the whole class. Although the wording of the problems already suggests a problem-solving method, it was emphasized in a whole-class discussion that, first, groups have to try to work out the sum of interior angles and the number of diagonals for concrete cases. Then, based on the method they used, they have to find a general formula or describe a general idea with the help of which the sum of interior angles and the number of diagonals can be calculated for any polygon. During group work, the less abled students asked for further guidance Those students who were uncertain about the correctness of their work asked for confirmation. To help these students, the teacher gave advice or instructions only to those who were asking for it. Instead of telling students how to proceed, the teacher used helping or guiding questions.

(1) *Which methods do students use to try to solve the given problems? How do they try to* Make a plan*?*

The *Making a plan* phase was accomplished with the guidance of the teacher. To obtain information about the way students were trying to solve the given problems, the teacher monitored the work of the groups and after the lesson checked their exercise books. Furthermore, the final phase when students had to explain their work to the others also gave information about their ways of thinking. From the students' work presented above, it is clearly visible that the presentation of the ideas in the exercise books is rather messy. Furthermore, the drawings clearly show that students were focusing on regular polygons when trying to answer the questions. In some works, it is difficult to understand what the student or the group was trying to do. However, most exercise books contain some concrete polygons, where either the diagonals are drawn or the polygon is divided into triangles. Although the method is not straightforward, we can say that students worked out the answer for the concrete cases, then generalized their ideas for regular polygons. As the solutions were often referring to regular polygons, the teacher had to point out that these ideas can be extended to irregular polygons and she had to provide an explanation.

(2) *How is the* Looking back *phase realized in the lesson?*

At the end of the lesson, one student from each group had to present the formulae they created. This activity aimed to check whether the formulae were correct and to reflect on the solution method the groups used to answer the initial questions. This way we looked back at the solution methods of every group by emphasizing the "good" ideas. At the beginning of the following lesson, the formulae and the thinking methods were revised before applying them in actual problem-solving.

Conclusion

A case study aims to examine the complexity of a single case and tries to generalize based on experience. To analyse how Pólya's phases can be realized in a mathematics lesson, we have to consider that this lesson was taught using group work. Pólya's different phases can be implemented differently in an average, frontal lesson. In a frontal lesson, the guidance of the teacher is stronger, and her ideas are more dominant, especially with a class of less abled students. In this case, the teacher leads students through the *Making a plan* phase, but the plan is usually the one that the teacher has in mind. However, with a group of more abled students, it is possible to rely on their ideas and the students might come up with their own solutions. Based on experience, the best way to lead students through solution plans is using guiding questions – which was also suggested by Pólya (1977). Asking questions constantly helps students learn how to create solution plans on their own.

In an average lesson *Looking back* is often the most neglected part, usually due to lack of time. However, its importance is unquestionable. In a frontal lesson after presenting some examples with solution plans the teacher sets some tasks for the students that can be solved based on the presented examples. *Looking back* is realized when checking the answers of students. Here, not only the correctness of the result is checked, but the solution method is revised. Especially in the case when more students make a mistake in working out the answer for the set tasks. Using a final revision of the applied solution method at the end of the lesson also helps to deepen understanding. It is also a good opportunity to emphasize that the correct answer is important in problem-solving, however, the method that led to the solution is equally if not more important. If the following lesson is still based on solving similar problems, *Looking back* can be a starter to remind students how to tackle problems of a certain type. The *Looking back* phase usually includes collecting typical mistakes, too. This way we can teach students to finish problem-solving with a thorough revision of their work paying special attention to the points where something can go wrong.

Based on our study, we can say that realizing Pólya's phases in a mathematics lesson that is taught using group work is more time-consuming than in a frontal class. However, in this scenario, the teacher has more opportunities to help those students who are stuck in their *Making a plan* phase. Furthermore, fellow students can assist and support each other in finding the correct solution. Usually, in a frontal lesson there is one plan – that of the teacher – which makes the *Looking back* phase easier to carry out. In the observed lesson, the four groups had to explain their solution methods which took more time mainly because students had difficulty expressing their way of thinking in a mathematically correct manner. So, the teacher often had to help them.

Summary

All in all, teaching students to make their own solution plans requires a lot of practice and reinforcement and is a time-consuming process. Similarly, making students understand that the *Looking back* phase is equally important to the other phases of problem-solving and making them use it at the end of each problem-solving takes a lot of time. Moreover, it requires a consistent and conscious attitude from the teacher, who should reflect on their practice, learn from the experience gained in the lessons, building it in the next lessons. The teacher has to discuss the *Looking back* phase with the students until they are confident enough to reflect on their solutions on their own. The aim is to make students realize that the problem-solving process does not end when a correct solution is found and encourage them to apply *Looking back* after solving every problem.

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