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Looking back on Pólya's teaching of problem solving

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Abstract. This article is a personal reflection on Pólya's work on problem solving, supported by a re-reading of some of his books and viewing his film *Let Us Teach Guessing*. Pólya's work has had lasting impact on the goals of school mathematics, especially in establishing solving problems (including non-routine problems) as a major goal and in establishing the elements of how to teach for problem solving. His work demonstrated the importance of choosing rich problems for students to explore, equipping them with some heuristic strategies and metacognitive awareness of the problem solving process, and promoting 'looking back' as a way of learning from the problem solving experience. The ideas are all still influential. What has changed most is the nature of classrooms, with the subsequent appreciation of a supporting yet challenging classroom where students work collaboratively and play an active role in classroom discussion.

Key words and phrases: problem solving, mathematical discovery, heuristics, strategies, reflection, mathematical processes, Pólya.

MSC Subject Classification: 97D50, 97A30.

Introduction

Preparing this contribution has been a great pleasure, because I have taken the opportunity to look again at Pólya's work on problem solving after many years. Of course, I cannot say I have re-read his books, since they are books to work through, problem by problem, not to read like a novel or even like a mathematics textbook. Re-reading has shown me how Pólya's ideas have had a lasting impact on the goals of school mathematics and how he popularized some key aspects of teaching for mathematical problem solving. I have appreciated anew some of the richness and detail about mathematical discovery that I originally enjoyed in his work.

My path in mathematical problem solving and my love of doing mathematics began at secondary school, first through the pleasures of Euclidean geometry and other curriculum problems, buoyed with optional problems that my teachers sometimes suggested. It was natural that I would do mathematics at university. After my doctorate, I was employed in a teachers' college, and therefore, like Pólya, I became interested in the mathematics education of prospective teachers and what was most important for them to learn. University curricula generally aim to produce research mathematicians or technical users of mathematics in engineering or scientific work, but the needs of teachers are different. Stacey (2008) discusses this in detail. Within a few years, my research and professional interest included the teaching of mathematics in schools, and how students of all ages could experience a richer, more useful, and more engaging mathematics. At all levels, it is essential to create a good balance between attention to teaching the content of mathematics and attention to teaching about the processes of doing mathematics (which is solving problems). Getting the right balance has been an ongoing theme in the story of mathematics education over the past 50 years. Problem solving remains an elusive goal, but always the most important.

Pólya's work highlighted the importance of both the content and process aspects of mathematics. In his book *Mathematical Discovery* he explains:

> "Our knowledge about any subject consists of information and of know-how. If you have genuine bona fide experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of know-how to the student. What is know-how in mathematics? The ability to solve problems – not merely routine problems but problems requiring some degree of independence, judgement, originality, creativity." (Pólya, 1962, Vol. 1, p. viii)

Pólya goes on to identify the lack of attention to this as the 'worse gap' in the education of teachers and sets out his aim to address it through his work. Filling this gap has been a major theme of thinking and innovation in mathematics ever since.

Pólya's work addresses the essential ingredients for helping students of all ages learn to solve problems, summarized by Stacey and Groves (2006) as:

- genuine experience of solving non-routine problems (experience);
- developing good problem solving habits and problem solving strategies (strategies);
- thinking about and discussing these experiences (reflection);
- all in a supportive yet challenging classroom environment.

The following sections offer some observations on each of these. In summary, I conclude that Pólya's work has been very influential with regard to the factors of experience, strategies, and reflection, but that social change quickly led to classrooms promoting problem solving being organized very differently to the model used by Pólya.

Let Us Teach Guessing

The earliest clear memory I have of George Pólya was when I was a doctoral student in number theory and a part-time university mathematics tutor, watching the film *Let Us Teach Guessing* (Pólya, 1966) with my undergraduate students. So, in preparation for this article, I watched the film again, now on the internet. The film aims to show Pólya's 'attitude to teaching' with a demonstration lesson with a class at U.C.L.A. He leads the students to a solution of a wonderful problem. Into how many regions is three dimensional space divided by 5 planes in a general orientation? In doing so, he stresses a suite of points about mathematical discovery.

When I saw the film for the first time, I was excited to see such a lesson. Although I was a doctoral student, I had never attended such a class myself, at school or at university. We had not had classes on the "know-how" of doing mathematics. By emphasizing discovery, Pólya's lesson was tapping into what I liked so much about mathematics. As he does throughout his books, Pólya chose an excellent problem, with lovely patterns, and a nice twist when reasonable conjectures break in an intriguing way. Here was a mathematician who did not see clear presentation of preformulated ideas as the most important characteristic of good teaching (although the film certainly also demonstrates clear and engaging presentation). Instead, the focus of his class was on the process of doing mathematics. For me, this opened a different possibility of what a mathematics class might be, and how students might be taught about the process of making mathematical discoveries (discoveries for oneself – they do not have to be new) and demonstrated a few elements of what might be taught. I stored these ideas away, as I went on to finish my doctorate.

Just a few years later, I was working in a teachers' college, teaching mathematics to elementary and secondary teachers. When the opportunity arose in 1976 to introduce a first year mathematics course that did not have to fit into the normal sequence of content, my colleague Susie Groves and I were granted permission to run an experimental course in problem solving (Stacey, 1977; Stacey, 2017). Susie, with her PhD in algebra, shared my desire to give students the experiences of mathematical discovery that we both enjoyed. We wanted to show them that the mathematics they had learned at school, and would go on to teach, is useful in real-life situations, and to give them some confidence and experience in tackling non-routine mathematical problems with it. At that time, problem solving was still a fringe activity in teaching mathematics, but in the next few years, it attracted growing interest in Australia and elsewhere. In addition to Pólya's popularization of teaching both information and know-how (content and process) in pure mathematics, Henry Pollack (1968) was conducting an experimental course on mathematical modelling at Teachers' College (Columbia). Our "problem solving" course included both pure mathematics investigations and modelling real world problems. More broadly, the academic world, shaken by the student movements of the late 1960s and drawing on influences such as Dewey's active learning and Bruner's discovery learning, began to introduce some problem-based learning, for example, in medical education (Schmidt, 2012). The movement towards problem solving in school mathematics, first in teacher education, began.

This time of social change perhaps explains the huge difference between the role of the teacher in Pólya's demonstration lesson of 1966 and in the classroom teaching of problem solving just a decade later, in our (initially experimental) course and elsewhere. Pólya's film demonstrated a classroom that was strongly teacher-led, in a Socratic dialogue. Through expert questioning, Pólya leads the class step by step through the problem to the solution. There is no student-to-student discussion shown, students have only a few seconds of thinking time in the class to answer the questions, and it is only very late in the lesson that any student answers a question with more than a couple of words or provides even a brief justification. In contrast, it seemed obvious to us that a problem solving class would devote considerable time to students 'doing mathematics' by working on problems in class sometimes individually but more often in small groups. Instead of guiding the whole class step by step, we believed that the role of the teacher was to allow students to pursue their own paths within a challenging but supportive classroom atmosphere (Stacey and Groves, 1985/2010). When Pólya writes that "the best practice is offered in group work" (Pólya, 1962, Vol. 1, p. 211), he is writing about how prospective teachers can learn to explain solutions, rather than to solve problems (which he expected to be done individually).

This different role for the teacher has flourished as an ideal in schools and teacher education ever since. This is the case even where there is no explicit emphasis on the solving of non-routine problems and all the problems relate closely to the topic at hand: quality learning is now recognized as arising in classrooms where students have at least some time and space to do their own reasoning. A major role of the teacher is to assist individuals or groups as required, drawing the class together to facilitate students' learning from their experiences. Having more "thinking classrooms" is now a goal for mathematics education. The Australian curriculum illustrates this.

Experience

Pólya firmly established that learning about problem solving must be embedded in experience of solving problems. My favorite quote about this is:

> "Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice. [...] if you wish to learning swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems." (Pólya, 1962, Vol. 1, p. v)

To embed learning in experience, his books build their discussion of how to solve problems around example problems, often presenting the steps towards solution through a sample dialogue between a teacher and a student/class. He presents case histories of solutions, describing essential steps and disclosing the moves and attitudes prompting these steps. (Pólya, 1962, Vol. 1, p. vi). This was different to other books on mathematics for a general audience at the time, which presented interesting mathematical topics and discoveries, perhaps with mathematical exercises (e.g., Kasner & Newman, 1949) or books such as Hadamard's (1945) that discussed aspects of mathematical invention based on reports of mathematicians. When John Mason, Leone Burton and I wrote *Thinking Mathematically*

(1982/2010), the natural choice was to similarly embed discussion of doing mathematics in the experience of solving problems: readers were encouraged to solve the problems themselves, not just read solutions, and our points about the process of thinking mathematically were made in response to resolutions of those problems.

A further key feature of Pólya's books is that he chooses his problems carefully so that student solutions are likely to demonstrate what he wants to show about heuristics. Whilst there are often many ways to do a problem, and there is no intention to tie-down what approaches a student might take, it remains the case that some problems illustrate some heuristics much better than others. Good problem solving materials follow this principle.

Strategies

Pólya's writing on problem solving above all presents his study of the means and methods of problem solving – heuristics. The books all contain very detailed accounts of the use of heuristic moves, illustrated by multiple problems and always stressing that heuristics never guarantee a successful solution. A key aspect is his list of questions (sometimes more like suggestions) for problem solvers to use when they reach an impasse. Pólya says his questions are selected to be "natural, simple, obvious, just plain common sense" (Pólya, 1957, p. 3). He recommends that teachers use his selected set of questions frequently, so that students will grow independent, coming to ask them of themselves when appropriate. This approach has been widely followed (see, for example, Bell, Binns, Burkhardt et al., 1984).

Pólya provides a structure for his questions and suggestions through his famous four phases of problem solving: understanding the problem, devising a plan, carrying out the plan and looking back. The four phases and their associated questions are summarized in a table at the start of *How to Solve It* (1957/1973), and the table also appears on the inside front and back covers of the NCTM's 1980 yearbook – the year when the NCTM declared that problem solving should be the focus of school mathematics in the 1980s (Krulik & Reys, 1980). This focus was quickly adapted in Australia, my country, and this led to many changes in mathematics teaching, curriculum, and assessment (Groves & Stacey, 1988). However, from the 1990's, emphasis on the goal of problem solving (beyond completion of curriculum-related exercises) has steadily reduced, and the goal of learning through problem solving took over. For example, the Australian curriculum from its inception around 2010 until 2021 positioned problem solving as one of four mathematical proficiencies, which were said to "describe how content is explored and developed [and] provide a meaningful basis for the development of concepts". Fortunately, the new version (9.0) of the curriculum now being implemented more explicitly recognizes the importance of the goal of students using mathematics to solve problems, both routine and non-routine and in mathematical and real world contexts (Australian Curriculum, Assessment and Reporting Authority, 2022). Perhaps this recognition will lead to more importance given to the capacity to use mathematics to solve a wide variety of problems, beyond standard exercises. This gives a better balance between learning to solve problems and learning mathematical content through solving problems.

In the teaching of the 1980's, Pólya's four phases became both the most used and the most abused part of Pólya's work. We know that it is beneficial to increase students' metacognition for problem solving – they need to be aware of where they are in the problem solving process and to monitor their actions (Mason et al., 2010, Chapter 7). Indeed, in *Thinking Mathematically* (Mason et al., 2010), we also hoped to increase students' metacognitive awareness with a similar structure of three phases (Entry, Attack, Review), around which we gradually revealed the advice by discussing problems. We associated "memorable" questions/suggestions with each stage. For example, the Entry phase was associated with the questions "What do I **know**?", "What do I **want**?" and "What can I **introduce**?".

Unfortunately, it is often the case that an attractive and useful pedagogical device gets stripped of its real value when it is widely used. This happened to Pólya's four phases. Instead of guiding the solving of challenging problems, the four phases – perhaps displayed on a poster on the classroom wall without any of the accompanying hints – sometimes became a straight-jacket strapping down students' mathematical thinking. Even for word problems that deviated just a little from standard (so that devising a plan was simultaneous with carrying it out) some curriculum materials required students to write down something for each phase. The result was tedious busywork. Similarly, heuristics (such as draw a diagram) did not arise in class in response to working on a challenging problem, but sometimes became the topic of a set of exercises. Practices like these gave the teaching of problem solving a bad name, and I believe this poor implementation contributed to the decline reported above. In my opinion it is definitely worth spending some curriculum time focused on major strategies (e.g., look for a pattern, work backwards), but the need for the strategies and their benefits and use should arise from well-selected problems (Stacey & Groves, 2006). Pólya revealed his problem solving strategies through working on mathematical problems of considerable depth, but this was not easily translated to mass education.

Reflection

The "Looking Back" phase foreshadows a theme that is strong in today's vision of an excellent mathematics lesson: reflection. In *How to Solve It*, Pólya describes the benefits of looking back:

> "if you get into the habit of surveying and scrutinizing your solutions in this way, you will acquire some knowledge well-ordered and ready to use, and you will develop your ability of solving problems" (p. 36).

The suggested activities in this phase include checking, trying to improve the solution, scrutinizing the method that led you to the solution and trying to use it in other problems. However, when investigating how problem solving can be taught to young secondary school students for *Strategies for Problem Solving* (Stacey & Groves, 1985/2006), we found that the instruction to "look back" was too passive for most young students. When they get an answer to a problem, most students just want to move on, not look back. We therefore developed various techniques to assist in helping students to learn from experience of problem solving, making this an active time rather than a passive 'reflection'. Individually or in their problem solving groups, students should "look back" to write a clear explanation of what has been found, and also write down one or more problem solving strategies or hints that they found useful. Then they can "look forward", asking "What if. . . ?", to see how their discoveries might be used in other problems. These give young children something concrete to do. However, the key ingredient for reflection is nearly always teacher-orchestrated class discussion with strong student contributions drawing on their own solution processes. In this way, the teacher actively helps students to learn from experience.

A personal connection

In the 1960s, Professor George Szekeres, born in Hungary, was an inspiring teacher at the University of New South Wales. He was one of many European Jewish immigrants who fled to Australia around the time of the Second World

War, giving Australian mathematics a great boost. As a student in Budapest, George was one of a group of mathematics students that met weekly to discuss and solve the problems from Pólya and Szegő's famous book (1925). In 1933 a fellow student, Esther Klein, posed a problem to the group which she subsequently solved. George and his friend Paul Erdős were inspired by this and subsequently published the solution to a generalization. Working on this problem brought George and Esther together, and they were married for nearly 70 years. Erdős therefore called it the Happy Ending Problem. In its Preface, Pólya (1962, Vol. 1, p. xi) comments that *Mathematical Discovery* has strong similarities to his book with Szegő, with the problems organized in a similar way but changed to be more suitable for high school teachers and with more explicit discussion of methodological points. It is a nice connection that my teacher was inspired by a book from early in Pólya's career, and I was inspired by his late work.

Conclusion

My re-reading of Pólya's work has highlighted the richness of his analysis of the processes of mathematical discovery, and his pioneering work in demonstrating how the process aspects of mathematics can be brought to the forefront in a mathematics classroom. His work set out some fundamental principles: that whilst students learn about solving problems by solving problems, the teacher has an important role. This role involves suggesting problems that illustrate particular problem solving strategies so that students gain generalizable experience and providing a structure to help students understand the problem solving process and to reflect on their experience, and hence learn from it. Whilst the Australian classroom of today does not often include the teaching of problem solving strategies in any depth, it ideally is a place where students work on challenging problems, sharing their thinking, and reasoning in order to learn mathematics.

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