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Teaching Mathematics and Computer Science

Teaching undergraduate mathematics – a problem solving course for first year

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Abstract. In this paper we describe a problem solving course for first year undergraduate mathematics students who would be future school teachers.

Key words and phrases: Pólya, undergraduate mathematics, curriculum, problem solving.

MSC Subject Classification: 97B50, 97B70, 97D50, 97D60, 97F60, 97U30.

Developing a curriculum that covers mathematicians' thinking processes

Usiskin (2001) argues that teachers need to transmit the discipline of mathematics to their students. For example, teachers of mathematics at every level should be able to see the big ideas in mathematics (Charles, 2005). They need to know how mathematicians solve unfamiliar problems, how to solve unfamiliar problems themselves, and finally how to teach their future students authentic problem solving. They should know how mathematics can be used to model real world situations but must also guide their students to not only rely on their real-world intuitions, for example, in their understanding of infinity and probability. Future teachers ought to embrace the centrality of proof in mathematics and thus resolve to guide their students to comprehend proof as a distinguishing aspect of the subject of mathematics. They must themselves learn how to read and write mathematics and see that, like other subjects, self-directed learning in mathematics depends very much on personal reading and writing (Tay, 2001). See Pólya (1958) for his views on what a curriculum for prospective high school teachers should be like. When combined with the relevant pedagogical education, such an ideal pre-service teacher mathematics curriculum will produce what we term the mathematician educator (Tay, 2020).

Alcock and Simpson (2009) note that students take a number of years for "development from an action through a process to an object conception before they begin to use the concept at university, [...]. [A]t the university level, a similar development is necessary, but a much shorter time period is available" (p. 22). This fact highlights the difficulty that makes any well-intentioned pedagogy at university level flounder – there seems to be not enough time in class to slow down. Even though many well-intentioned (and enlightened) lecturers try to make a difference in their classes, they often find that they are alone in their department and that the process skill, for example, problem solving in the style of Pólya (1945), they teach in their class is not reinforced (or even contradicted) as the students go to other classes and move on to the next years of their programme. At the university, there always seems to be not enough time to teach reading and writing, and understanding and construction of proofs, when the knowledge content needs to be covered.

It thus seems necessary that if any meaningful attempt to incorporate all the disciplinarity training into an undergraduate mathematics teacher programme is to succeed, a curriculum review that must involve all who are teaching the curriculum should be conducted. Tyler (1949) proposed a basic model of curriculum design that we believe has stood the test of time. The three key aspects of Tyler's model we use are

(i) ascertaining the Learning Objectives,

(ii) determining the Learning Experiences that would attain the objectives, and

(iii) Assessment of the learning.

In the implementation of Tyler's model, planning requires first that the learning objectives of the curriculum be placed as column headers in a matrix with the courses in the programme as row headers so that the design can ascertain which course can be used to attain which objective. When ascertained, the design requires the development of learning experiences to achieve the objectives within the course. The various assessments of the objectives are also decided at the design stage.

The Mathematics and Mathematics Education Department of the National Institute of Education used Tyler's model for the curriculum review and design of its undergraduate mathematics programme from 2015 to 2019. In particular, important process skills such as reading, writing and problem solving (Toh et al., 2014; Ho et al., 2014) were thoughtfully spaced out over the four-year curriculum, and a lecturer would know from the start what he/she has to emphasise and assess, what has been done before his/her course, and what would follow further down the line. For example, emphasis and assessment of reading is spaced out as follows: reading of definitions (Year 1), reading of a short proof (Year 2), reading of definitions and proofs before a lecture (Year 3), reading of journal papers for honours dissertation (Year 4). As another example, a new course on mathematical problem solving was developed and implemented. In the rest of this paper, we shall describe some key features of this problem solving course, the underlying Pólya model, the connections of the problem solving course to other courses in the curriculum, and general student feedback.

The mathematical problem solving course

Figure 1 illustrates the current 'ideal' teaching approach. We teach a particular topic in Mathematics. Students practice with routine exercises (learning mathematics through practice). Some hard problems are given and the students have a deeper understanding of the concepts through struggling with the hard problems and obtaining the deep results (learning mathematics through problem solving). The learnt mathematics is now used to solve some real-life problems or applied to other mathematics topics (learning mathematics for problem solving). Typically, assessment consists of items resembling the exercises and the hard problems.

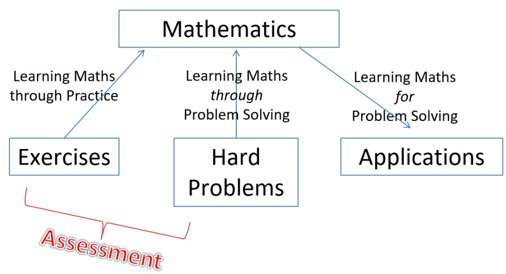

Figure 1. The current 'ideal'

Teaching and learning difficulties occur in the classroom. Students cannot, without much effort, solve the difficult problems. The conscientious teacher then decides to teach how to solve the difficult problems to prepare the students for the exams. The hard problems become themselves the object of study; specific techniques are taught for each type of hard problem (learning mathematics for problem solving). Curriculum time for deeper learning and exploration is often sacrificed as a result. Figure 2 illustrates the current 'compromise'.

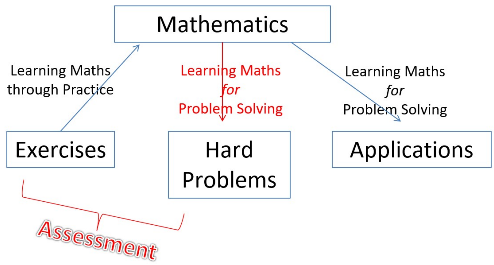

Figure 2. The current 'compromise'

The taught specific techniques fail the students' time, and again, when they encounter unfamiliar problems that do not exactly fit the mould. The robust solution to this conundrum is to teach the students generic mathematical problem solving skills which they will use to tackle hard problems in any course (learning about problem solving). (We use here the well-known three conceptions of problem solving attributable to Schroeder and Lester (1989): teaching for mathematics problem solving, teaching about mathematics problem solving, and teaching mathematics through problem solving.) Figure 3 illustrates the location of these skills and points to a need for a specialised course to learn about problem solving.

The Problem Solving course spans 36 hours over an entire semester. It is a compulsory course for all first year undergraduate students who are offered Mathematics as an academic subject. We situate it in the first year, because we think it is important that students learn about problem solving early so they can gain 'dividends' from it in the subsequent years of mathematics learning (and learning to teach mathematics, since they are also preparing to be mathematics teachers) at the university.

The first half of the course is devoted to 'generic' problems (see *Lockers Problem* in this paper for an example of such a problem). They are generic in the sense that the mathematical content required to access and attack the problem is not specialised to any particular branch of mathematics at the university level.

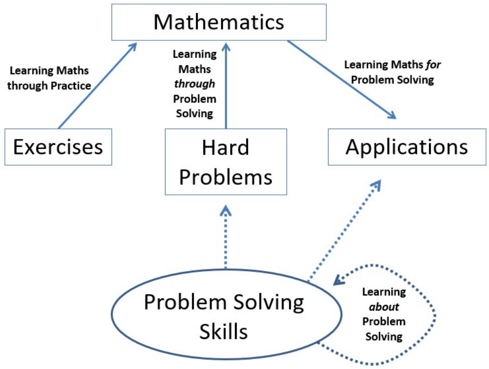

Figure 3. The robust solution

In fact, the associated mathematical content required to solve these problems is deliberately set to be at common school mathematics level (e.g., in the case of the Lockers Problem, the knowledge required is basic, such as that of factors and multiples which are taught in Singapore at the Secondary levels) so that the bulk of students' efforts – and concomitantly, the tutor's emphases – are directed at learning about problem solving.

During the first few lessons of the course, such generic problems will be posed in class; typically, students will be given opportunities to struggle with these problems so that their mental frame becomes adjusted to the nature of the course: that it is not primarily about learning new mathematical content as in other university level mathematics courses; rather, it is about learning useful skills and dispositions for dealing with unfamiliar problems. Such experiences of struggle are also meant to motivate them to learn new tools that will help them become better problem solvers. So, roughly from the third lesson onwards, we will formally introduce Pólya's (1945) model and heuristics as well as Schoenfeld's (1985) framework (a more detailed description of these is given in the next section). For the rest of the course, through solving more problems, they will learn to become more adept at the use of these strategies in a range of problem types.

In the second half of the course, the problems posed are those used in other concurrent undergraduate mathematics courses (see Section on authentic undergraduate problem solving for an example of such problems). The reasons for doing so are: students can immediately see the usefulness of these problem solving tools in the learning of undergraduate mathematics content and thus strengthen their motivation to learn them well; also, it is part of the "robust solution" mentioned earlier (see Figure 3) where students should experience the use of strategies learnt in hard problems and applications that are found in undergraduate mathematics topics.

Since this course was conceived, we have had five runs of it. Overall, we maintain the same structure as described in the previous paragraphs but we tweaked some aspects: (1) some students get easily put off when they find themselves repeatedly unable to solve problems posed to them, despite attempting the strategies taught. We instituted group collaborative problem in addition to the original individual solving mode. Through a combination of collaboration and individual work, students are given a runway to build confidence in problem solving. The collaborative component is gradually weaned off towards the middle of the course as we want them to internalize the strategies to take through all the four stages of Pólya by themselves instead of being over-reliant on building upon the ideas of others. (2) In the earlier runs, from the students' products, we notice that they lacked practice in presenting their solutions in a rigorous way. We thus included class presentations in later runs to give them the opportunity to show their solutions in the context of more formal public discourse. Also, in the second half of the course where they tackled topic-specific problems, there is greater emphasis in the clarity of their written solution. (3) We have over the years collected an increasingly larger pool of problems. (Good problems – in the sense that they tend to draw the interest of students and is rich in terms of allowing multiple routes of attack for students of different content levels – are not easy to construct. So, we obtain these problems through books, internet, and colleagues, including visiting scholars from overseas universities.) This has allowed us options to match problems with specific areas to target for the students' learning about problem solving. For example, the *Sum of Digits* problem (see discussion of it later in this paper) provides a concrete experience of the importance of a careful reading of the problem to Understand the Problem (Stage 1 of Pólya). Often, students solved the Sum of Numbers problem instead of Sum of Digits. They then realised that there was indeed a practical need to loop back to (re-)Understand the problem even when they had 'progressed' to later stages (see Figure 4).

Scaffolding and assessment using Pólya's model and Schoenfeld's framework

It is generally accepted that problem solving proceeds in stages (Pólya, 1945; Schoenfeld, 1985). Some diagrams depicting Pólya's approach use arrows to indicate the four steps in solving a problem. In our diagram (see Figure 4), we use opposite headed arrows to indicate that a student may move from Devise a Plan (DP) stage to Understand the Problem (UP) stage, upon his or her realizing being "stuck" in devising a plan and recognising the need to go back to understand the problem, and then back to Devise a Plan. This is an instance of control, a refinement by Schoenfeld of Pólya's model. Likewise, the student may backtrack from Carry out the Plan (CP) stage to DP or to UP when "stuck" or dissatisfied with the progress in solving the problem. To us, the students' control of the problem-solving endeavour is an important part of their mathematical problem solving experience. We include in our instruction this "knowing what to do when stuck" as an exercise of control over the affective aspect of problem solving such as managing frustration, garnering perseverance, and so on. The cognitive control can be found in problem solving prompts which Pólya suggests in the form of questions at each stage, for example, in UP, "What is the condition?"; in DP, "Do you know a related problem?"; in CP, "Can you see clearly that the step is correct?"; in CE (Check and Expand stage), "Can you see it at a glance?".

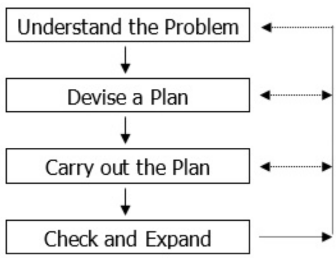

Figure 4. Pólya's model – 'Look Back' replaced by 'Check and Expand'

Note that we replaced Pólya's original phrase 'Look Back' with 'Check and Expand' to make it clearer to our students the dual aspects of checking and building on the solution (via alternative solutions and posing new problems). Also, we believe that back flows can skip prior stages such as a back flow from Check and Expand straight to Understand the Problem, but forward flows should normally proceed in sequential stages.

We elaborate a bit more on Schoenfeld's (1985) framework. We see Schoenfeld over-girding Pólya's model with four components, viz., Cognitive Resources, Heuristics, Control, and Beliefs. These components help us realize that Pólya's model or the teaching of heuristics is not all there is. The problem solver needs also to address the availability of resources and exercise cognitive and affective control over the problem solving process. Attention is also given to winning over the hearts of the students, because if they believe wrongly that they cannot solve unfamiliar problems, then they would not engage in problem solving even though Pólya's model is taught to them. The combined frameworks thus allow the students to have a structure through which they can focus when difficulties arise in solving a problem.

Prior to the conceptualization of the mathematical problem solving course, the approach of using Pólya's Model and Schoenfeld's Framework were first trialed in two undergraduate courses, (1) Differential Equations (DE) course; and (2) Introduction to Number Theory (NT) course (Toh et al., 2013; Toh et al., 2014). Two different conceptualizations of the teaching of the problem solving approach were experimented in the two courses, in which the infusion of problem solving in (1) involved a re-structuring of the course lesson delivery, but in (2), there is no change in the course structure.

The course structure of DE was re-designed to include eight hours of mathematics "practical" lessons in which the undergraduate students were given the opportunity to apply the Pólya's model and Schoenfeld's Framework to tackle one DE problem within each lesson, with the instructor discussing how Pólya's four stages were applied to solve the problem, ending with extending the problem to create another new problem for homework. These eight hours of practical lessons were taken from a part of the mathematics lectures. It was reported in Toh et al. (2013) that the students had positively learnt to handle non-routine problems on DE and were able to consider alternative solutions to a given DE problem even during the examinations.

In (2), the infusion of problem solving into NT did not entail a re-design of the structure. As the instructor did not want to introduce any structural changes to the NT course in order to ensure that the usual course content continued to be covered substantially. Thus, he introduced problem solving through the use of the main theorems in NT as a context for problem solving. Comparatively, the NT course is relatively heavier in content than DE, thus it is not surprising that the instructor chose not to alter the course structure.

Comparing the two structures (1) and (2) above, it appeared that generally, Model (2) was more sustainable for a general content-heavy undergraduate mathematics course. Thus, weaving problem solving into undergraduate mathematics courses using the model of (2) continued to be applied to other undergraduate courses such as the Introduction to Calculus, the first undergraduate calculus course offered at the university. This consideration of not compromising on the rigor of the undergraduate courses was another reason for conceptualizing the Problem Solving course (as described in the preceding section). The Problem Solving course is the place where students learn about problem solving and attempt challenging problems from the other undergraduate courses.

It is no doubt that the undergraduate students (pre-service teachers) would eventually encounter problem solving and Pólya's problem solving model in the mathematics pedagogy modules (e.g., Kaur & Toh, 2011). The difference between the problem solving taught in the pedagogy modules and in the Problem Solving course was that the problems used in the former tend to be set at elementary and high school mathematics, which are not authentic problems for the undergraduate students.

Teaching of problem solving

We used numerous generic problems in the Problem Solving course to introduce the students to Pólya's model of problem solving. In this section, we will describe two of them, Lockers and Number of Digits, to explain what we emphasised and what the students discovered about the Pólya's method.

<u>The Lockers Problem</u>

The new school has exactly 343 lockers numbered 1 to 343, and exactly 343 students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all the lockers. The second student will then enter the school and close every locker with an even number. The third student will then 'reverse' every third locker; i.e., if the locker is closed, he will open it, and if the locker is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 343 students in turn have entered the building and reversed the relevant lockers. Which lockers will finally remain open?

The sheer verbosity of the problem would force anyone to spend some time trying to Understand the Problem. Our students were mostly willing to engage in this process. However, some did not know how to engage, spending time reading and rereading the problem. The teacher at this stage would suggest the use of heuristics such as Act It Out, Draw a Diagram or Use Smaller Numbers.

Some students would quickly conjecture that the prime numbered lockers would remain open at the end. The teacher would encourage them to be clear about the basis for their conjectures and suggest that they proceed to Devise a Plan. We would further suggest the heuristic use of Solve a Simpler Problem. Students reducing the problem to just ten lockers and acting it out by drawing a simple list would discover that Lockers 1, 4, and 9 would remain open and make the conjecture that the square numbered lockers would remain open with some basis.

Students would return to the original problem and pause to decide what the new plan should be. After some prompting using the heuristic Restate the Problem, students would follow this sequence of questions:

- Which lockers will remain open?
- Which lockers are 'touched' an odd number of times?
- Which lockers have an odd number of factors?
- Prove that only the squares have odd numbers of factors.

Thus, the Number Theory part is revealed and the students use various techniques to prove the last statement.

At the last stage, students over the years managed to pose a number of interesting new problems:

- Student n reverses lockers with numbers which are multiples of n, except Locker n itself.
- Student n reverses lockers with numbers which are factors of n.
- If there are n lockers altogether, how many lockers will remain open?
- Student p reverses lockers which are multiples of p if p is prime, otherwise, nothing is done.

Sum of Digits Problem

Find the sum of the digits of all the numbers $1, 2, 3, \ldots, 999$.

For this problem, we would like to highlight the various interesting solutions of students over the years. Students are always encouraged in Stage 4 to Check and Expand. Expand includes posing new problems and coming up with alternative solutions. The Lockers Problem shows the capacity of the students to pose new problems. Here, we show the capacity of the students to come up with alternative solutions.

Solution 1

We first express all the numbers as 3-digit sequences, for example 7 as 007. We add 000 to make 1000 numbers without affecting the total. We tabulate the numbers as in Figure 5.

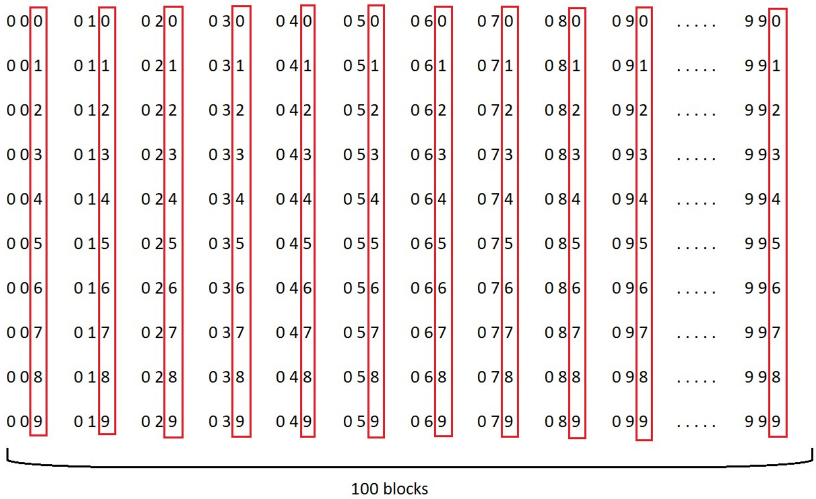

Figure 5. A table for 000 to 999

Now add all digits in the units place. Each vertical block contains $0, 1, 2, \ldots, 9$. Since there are 100 blocks, the sum of all digits in the units place is

$$100 \times (1 + 2 + 3 + \cdots + 9) = 4500.$$

We continue by adding all digits in the tens place. To do this, we take horizontal blocks of 10. Each horizontal block contains $0, 1, 2, \ldots, 9$ in the tens place. There are 10 horizontal blocks for each row. Since there are 100 blocks in the 10 rows, the sum of all digits in the tens place is

$$100 \times (1 + 2 + 3 + \cdots + 9) = 4500.$$

By considering the numbers in order by a hundred each time, i.e, 000 to 099, 100 to $199, \ldots, 900$ to 999, we see that there are 100 each of $0, 1, 2, \ldots, 9$ in the hundreds place. Thus, the sum of all digits in hundreds place is

$$100 \times 1 + 100 \times 2 + \cdots + 100 \times 9 = 4500.$$

In conclusion, the sum of the digits of all the numbers $1, 2, 3, \ldots, 999$ is

$$3 \times 4500 = 13500.$$

Solution 2

We will count the number of appearances of each digit.

Consider the digit "1". There are 100 occurrences of "1" in the units place – $001, 011, 021, \ldots, 991$; 100 occurrences of "1" in the tens place – $010, 011, \ldots, 019, 110, 111, \ldots, 119, \ldots, 910, 911, \ldots, 919$; and 100 occurrences of "1" in the hundreds place – $100, 101, \ldots, 199$. In total, there are 300 copies of "1". Similarly, we can verify that every digit appears 300 times.

In conclusion, the sum of the digits of all the numbers $1, 2, 3, \ldots, 999$ is

$$300 \times (1 + 2 + 3 + \cdots + 9) = 13500.$$

Solution 3

Consider the 3-digit sequence $a_1a_2a_3$. Suppose $a_3 = i$, for some $i = 0, 1, 2, \ldots, 9$. Then the number of such sequences is $10 \times 10 \times 1 = 10^2$. Similarly, the number of sequences with $a_j = i$, for some $i = 0, 1, 2, \ldots, 9$, $j = 1, 2$ is 10^2. Hence the number of i equals 3×10^2 and the sum of digits is $3 \times 10^2 \times (0+1+2+\cdots+9) = 3 \times 10^2 \times 45$. This form allows us to obtain the solution of the general problem "Find the sum of the digits of all the numbers $1, 2, 3, \ldots, 10^n - 1$" as $45n10^{(n-1)}$.

Solution 4

We will count the sum of the digits ten by ten and start by counting the sum of the digits of all numbers $1, 2, 3, \ldots, 99$ (see Table 1).

Range	*Sum of the digits*
1 – 9	45
10 – 19	55
20 – 29	65
30 – 39	75
40 – 49	85
50 – 59	95
⋮	⋮
90 – 99	135

Table 1. Sum of digits for 1 to 99

Range	*Sum of the digits*
1 – 99	900
100 – 199	1,000
200 – 299	1,100
300 – 399	1,200
400 – 499	1,300
500 – 599	1,400
⋮	⋮
900 – 999	1,800

Table 2. Sum of digits for 1 to 999

We have an arithmetic sequence with common difference 10 because to obtain the next term, we add one to the digit in the tens place in the previous term. Since there are 10 numbers in every row, the difference becomes 10.

The sum of the digits of all the numbers $1, 2, 3, \ldots, 99$ is

$$45 + 55 + \cdots + 135 = \frac{(45 + 135) \times 10}{2} = 900.$$

We can continue to our problem of finding the sum of all digits of all the numbers $1, 2, 3, \ldots, 999$ (see Table 2 above).

Using a similar argument as above, we have an arithmetic sequence with common difference 100. The sum of the digits of all the numbers $1, 2, 3, \ldots, 999$ is

$$900 + 1000 + \cdots + 1800 = \frac{(900 + 1800) \times 10}{2} = 13500.$$

Solution 5

There are 500 pairings (x, y) satisfying $x + y = 999$ for $0 \leqslant x < y \leqslant 999$. Let $x = \overline{abc}$ and $y = \overline{def}$, we must have $c + f = 9$, $b + e = 9$ and $a + d = 9$, which implies $a + b + c + d + e + f = 27$. Thus, the sum of the digits of all the numbers $1, 2, 3, \ldots, 999$ is

$$27 \times 500 = 13500.$$

Teaching through problem solving

Though certain canons of modern mathematics are universally agreed to be important to the wholistic education of a mathematics major, current curricular designs at tertiary level do not pay sufficient attention to their systematic inclusion in the courses. One of the major reasons is that most university courses are operating at a modularized manner, and thus individual courses tend to focus only on the mathematical content and techniques specific to the course. One of these canons is the concept of *countability* – a set is countable if there is an injective map from it into the set of natural numbers. Most lecturers would have taken this concept as an essential background knowledge but not separately dedicate time to teach it. We advocate to teach such mathematical canons by invoking the process of problem solving. In other words, we focus on creating interesting mathematical problems with the intention of teaching the problem solver some key mathematical concepts embedded in these problems.

To illustrate this point, we give an example of a problem called the *Groundhog Problem* which was suggested by a Simon Fraser University professor, Peter Liljedahl in 2017. This problem has since been used in our Problem Solving course and has proven to be effective for teaching first year students the concept of countability.

Groundhog Problem

A groundhog has made an infinite number of holes roughly a metre apart in a straight line in both directions on an infinite plane. Every day it travels a fixed number of holes in one direction. A farmer would like to catch the groundhog by shining a torch, but only once a night, into one of the holes at midnight when it is asleep. What strategy can the farmer use to ensure that he catches the groundhog eventually?

It is not difficult to mathematise this problem by conjuring a real line with the integers marked on it, denoting the Hole Number h. Then we fix the constant (integral-valued) velocity of the groundhog as v, and call up n as the Day Number.

An enthusiastic and engaged problem solver will inevitably have to grapple with the following sub-problems:

- Which hole was the groundhog situated on Day 0?
- What is the constant velocity v?

Since we do not know the answers to the above two important questions, how can one select which hole to inspect on Day n?

Solution

Two commonly applied heuristics is to (i) simplify the problem and (ii) make a systematic list. In this case, the students are invited to make suitable assumptions centred around the two questions to simplify the problem at hand.

Simplifying the problem and making a systematic list

Assume that the groundhog was at Hole 0 on Day 0, and that the groundhog moves with a positive constant velocity, i.e., it moves v holes per day to the right. Since we do not know what the value of v is, we make a systematic list of the possible integral values of v, i.e., we enumerate all possible positive integers: $1, 2, 3, 4, \ldots$. For convenience, a groundhog that moves at a velocity of v is called a v-groundhog. What needs to be done is to first suppose this is a 1-groundhog. Then on Day 1, it would have moved to Hole 1, and thus the farmer should inspect Hole 1 on Day 1. What if it is not there? Well, the farmer will then suppose that it is a 2-groundhog. Because it is Day 2, a 2-groundhog would have moved to Hole 4 (i.e., 2×2). Again, if it is not at Hole 4 on Day 2, the farmer would move on to the next possibility, i.e., suppose it is a 3-groundhog, and so would go ahead to check Hole 9 on Day 3. In fact, the farmer can draw up an "Inspection Schedule" (as shown in Figure 6): n denotes the Day Number, v denotes the velocity of the groundhog, a denotes the initial Hole Number on Day 0, H_n denotes the Hole Number on Day Number n. Figure 6 shows how the "Inspection Schedule" is carried out as the farmer moves from hole to hole as the days pass by. Whatever the velocity of the groundhog is, it would eventually be reached on the Day whose value equals to that velocity; whence the farmer would be able to locate the groundhog at Hole Number v^2.

Inspection schedule for a = 0 and v > 0

n	v	H_n	= ?
1	+1	1 × (+1)	1
2	+2	2 × (+2)	4
3	+3	3 × (+3)	9
4	+4	4 × (+4)	16
5	+5	5 × (+5)	25

Inspection enacted for a = 0 and v > 0

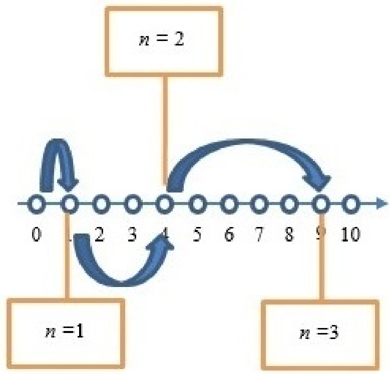

Figure 6. Making supposition and making a systematic list for $a = 0$, $v > 0$

Although the simplified problem has now been solved, the original problem is far from being settled. What if we do not know that the groundhog moves to the right (i.e., $v > 0$)? Here we still assume that the groundhog is at Hole 0 on Day 0; after all we do not want to make things too complicated now.

There appears to be very little problem here as well. Like before, all we need to do is to enumerate the possible velocities (which range over all the integer – positive, zero and negative) by listing them out this way: $0, 1, -1, 2, -2, 3, -3, \ldots$. In other words, we are "counting off" all the integers ensuring that we do not miss out any of the integers along the way. Following this enumeration scheme, the farmer can again draw up an "Inspection Schedule" (Figure 7). Following this new schedule, the farmer can then carry out his inspection of the holes like in Figure 7.

Inspection schedule for $a = 0$ and $v \neq 0$

n	v	H_n	= ?
1	+1	$1 \times (+1)$	1
2	−1	$2 \times (-1)$	−2
3	+2	$3 \times (+2)$	6
4	−2	$4 \times (-2)$	−8

Inspection enacted for $a = 0$ and $v \neq 0$

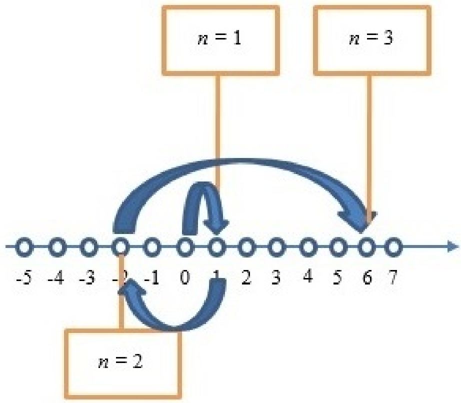

Figure 7. Making supposition and making a systematic list for $a = 0$, $v \neq 0$

By now, the observant problem solver would have noticed that the ‘formula’ for the Hole Number H_n to be inspected on Day n is really not so important. The most pivotal part to solving the above partial problems is to be able to ‘keep track’ of the possible velocities (the first case, ranging over natural numbers; and the second case, ranging over integers) by counting them one by one without missing out any. This is precisely the property of countability, i.e., both the set of natural numbers and that of integers can be put on a one-to-one correspondence to the set of natural numbers: one reads off this bijection between the elements under column heading ‘n’ and those under column heading ‘v’ (see Inspection Schedules in Figures 6 and 7). This kind of matching or one-to-one correspondence is shown clearly in Figure 8.

The idea here is that one makes purposeful use of the problem solving opportunity to teach the problem solver the embedded mathematical concept of countability. When the problem solver notices the need to enumerate the set of natural numbers (this is trivial) or the set of integers (this is slightly easier and manageable), he or she inevitably re-invents the concept of countability. Though the formal definition of countability might not (and need not) be formulated,

Case 1 $a = 0, v > 0$	$v \in \mathbb{Z}^+$	+1	+2	+3	+4	+5	+6
		↕	↕	↕	↕	↕	↕
	$n \in \mathbb{N}$	1	2	3	4	5	6
		↕	↕	↕	↕	↕	↕
Case 2 $a = 0, v \neq 0$	$v \in \mathbb{Z}$-{0}	+1	-1	+2	-2	+3	-3

Figure 8. Numbering off elements in $\mathbb{Z}^+$ and $\mathbb{Z} - \{0\}$ without missing out any elements

it is nevertheless applied to settle the problem completely. We proceed to illustrate this.

Countability of the set of pairs of natural numbers and non-zero integers

The original problem seems intractable because we do not know the two quantities, i.e., what is the starting Hole Number on Day 0, a, and the velocity of the groundhog, v. But the earlier simplified problems disclose the key to solving it, i.e., one must know how to enumerate the set over which the unknown ranges. For the original problem, the unknown can be seen as a pair (a, v) of elements – the first one, a, is an integer, and the second one, v, a non-zero integer. Thus all one needs to do is to enumerate all the possible such pairs (a, v) in the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ – count these pairs off without missing any one of them. We thus look for a way to match a given natural number n uniquely to a pair $[n] := (a, v)$ comprising an integer and a non-zero integer v, in a bijective manner (i.e., a one-to-one manner). Instead of giving the explicit formula to describe this one-to-one correspondence, the students can display their counting scheme for the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ using diagrams such as those in Figure 9.

Final solution

Thus on Day n, the unique starting hole-velocity pair is $[n] := (a, v)$, and hence the Hole Number H_n is calculated as $a + nv$. Like in the previous two simplified problems, the farmer can draw up an "Inspection Schedule" in the form of his favourite counting scheme (see Figure 9), and apply the formula for H_n to inspect the designated hole on Day n.

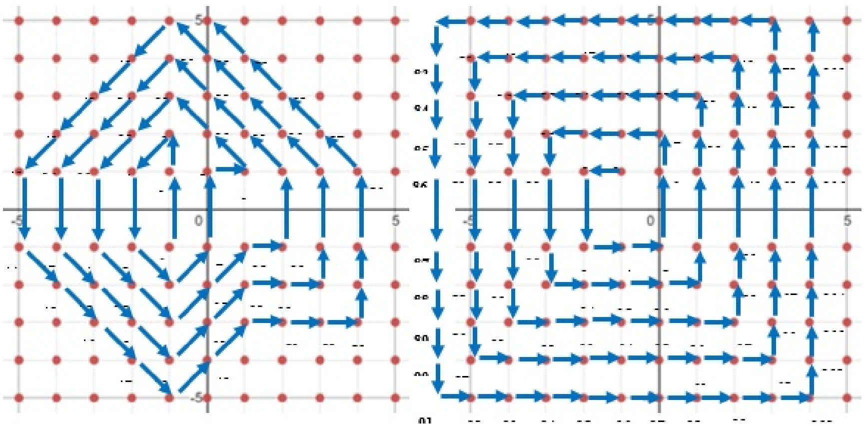

Figure 9. Two counting schemes for the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

Authentic undergraduate problem solving

As mentioned in the second section of this paper, in the second half of the Problem Solving course, the students were assigned authentic problems from concurrent mathematics courses to convince them of the efficacy of the problem-solving approach in their learning of mathematics content. One of the concurrent courses that they were reading was Number Theory, which covered the concepts of divisibility, Diophantine equations, prime numbers and Euler's generalization of Fermat's Little Theorem and its application to the RSA cryptography system. Integral to the discussion on the latter topic of Euler's generalization and the RSA cryptography system is Euler's Totient Function, $\phi(n)$. In previous iterations of the Number Theory course, the instructor would introduce the totient function to the students, discuss and prove various properties satisfied by the function, the most salient of which are

- its evaluation at prime powers: $\phi(p^k) = p^k - p^{k-1}$; and
- it is a multiplicative function: $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime.

These two properties would allow the evaluation of the totient function at any integer, through its prime factorization. However, the proof of the multiplicative property is not straightforward and compounded with the fact that this is the last topic of the course prior to the examination, students do not usually have enough time to gain a good understanding of the topic.

To address the situation, the following problem was posed to the students in the Mathematics Problem Solving course one or two weeks before the topic was to be taught in the Number Theory course.

Euler's Totient Function

For $n \geqslant 1$, $\phi(n)$ *denote the number of positive integers that are less than or equal to* n *and are relatively prime to* n*. Given integers* a *and* b *that are relatively prime, what is the relationship among* $\phi(a)$, $\phi(b)$ *and* $\phi(ab)$*?*

The intent was to allow students sufficient time to explore properties of the totient function on their own. Through the heuristic of Solve a Simpler Problem – such as actual working out on small primes, they would be able to discover (whether individually or as a whole class) the special case that $\phi(pq) = \phi(p)\phi(q)$ when p and q are distinct primes. And in doing so, they would discover that $\phi(p^2) \neq \phi(p)^2$ and hopefully arrive at the correct evaluation of the totient function at prime powers. In this way, when the topic is eventually introduced in Number Theory, the students were already familiar with the totient function and were more ready to learn about the results leading towards the RSA cryptography system.

General student feedback

As mentioned at the beginning of this paper, the Mathematics and Mathematics Education Department of the National Institute of Education carried out a curriculum review and design of its undergraduate mathematics programme from 2015 to 2019 using Tyler's model. Arising from the review, we proposed learning objectives for our undergraduate mathematics programme in six overarching objective domains: Content, Cognition, Problem Solving, Computation, Communication, and Disposition. In the domain of Problem Solving, we expect our mathematics undergraduates to possess the ability to pose, solve, and extend mathematical problems at the end of their four-year undergraduate programme. Unpacking this domain further, we expect our mathematics undergraduates to be able to: (i) solve non-routine problems, (ii) use suitable heuristics when solving mathematics problems, (iii) extend mathematics problems.

The recommendations of the curriculum review and design were first implemented for the cohort of mathematics undergraduates who enrolled in 2016, and have since been implemented for subsequent cohorts. In the 2016 cohort, there were twenty-nine undergraduates who read mathematics in their first year. Nineteen of them read mathematics as their first academic subject, which means that they proceeded to read mathematics in their second, third and fourth year. The other ten of them read mathematics as their second academic subject, so they did not study mathematics after their first year.

We conducted a survey of the 2016 cohort to gather feedback on their perception of achievement of the learning objectives in the six overarching domains in September 2019 when they were in the first semester of their fourth year of study. Their participation in the survey was voluntary. All nineteen undergraduates who read mathematics as their first academic subject and one out of ten undergraduates who read mathematics as their second academic subject took the survey. There were two to five questions in each of the six overarching objective domains, and the participants rated their responses to each question on a Likert scale of 1 to 4, 1 being strongly disagree and 4 being strongly agree. In respect of the objective domain of Problem Solving, the participants were asked to rate their responses to the following three questions: (i) "I am able to solve non-routine problems", (ii) "I am able to use suitable heuristics when solving mathematics problems", (iii) "I am able to extend mathematics problems". These three questions correspond to the three learning objectives in the unpacked domain of Problem Solving. The table below summarizes the responses of the twenty participants to the above three questions.

The results show that the majority who took part in the survey had very positive perception of their abilities in respect of solving mathematics problems. An overwhelming 95% of the participants were confident of using suitable heuristics when solving mathematics problems. This suggests that after completing the Problem Solving course, these participants had developed the capability of using heuristics when they attempted to solve a mathematics problem, irrespective of whether they could successfully solve the problem completely. On the other hand, 20% of the participants disagreed that they were able to solve non-routine problems. These participants probably had not been very successful in solving completely most of the non-routine mathematics problems they encountered in the Problem Solving course and subsequent mathematics courses in their second to fourth year. In response to question (iii), 85% of the participants agreed and strongly agreed that they were able to extend mathematics problems. Note that the question did not ask the participants whether were able to solve the extended mathematics problems. We infer that these participants had cultivated the habit

Questions	*Distribution*		*Means*	*Standard Deviation*
(i) *I am able to solve non-routine problems.*	1	0%	3.00	0.63
	2	20%		
	3	60%		
	4	20%		
(ii) *I am able to use suitable heuristics when solving mathematics problems.*	1	0%	3.20	0.51
	2	5%		
	3	70%		
	4	25%		
(iii) *I am able to extend mathematics problems.*	1	0%	3.10	0.62
	2	15%		
	3	60%		
	4	25%		

Table 3

of extending mathematics problems, even though they might not be able to solve the extended problems.

Besides the Likert scale questions, there were five open-ended questions in the survey that asked for participants' expectations of the mathematics courses, the skills or dispositions they considered the most important when learning mathematics, et cetera. None of the five open-ended questions specifically asked for participants' views on the Problem Solving course. Nonetheless, in response to some of the open-ended questions, a few participants stated their views on mathematics problem solving. For example, in response to the question "What would you consider as the skills or dispositions that are most important for you to learn to become a better student of mathematics?", four participants gave the following answers:

- "I think the mathematics problem solving heuristics have been extremely useful to frame my mindset when solving mathematics problems. It is applicable at all levels, including secondary school, and up until university level modules."
- "Problem solving skills and heuristics."
- "Heuristics, problem solving skills, and reading (with understanding) mathematical texts"

- "(1) being able to partition question into steps or parts or cases, (2) identifying simpler examples to find patterns before attempting the complicated question, (3) using diagrams or visuals or numbers to better find patterns and understand expectation of question, (4) using the above to verify if answer is correct, (5) persevering through tedious thought process or proving methods, (6) trying hard to understand where the mistake or gap is when incorrect, (7) understanding the uncertain concept or thought process by listening to others' explanation and questioning to clarify doubts, (8) looking for different methods in finding a solution and verifying if it is valid, (9) looking back to check and verify for any careless mistakes or gaps in answer."

In answering another question "What classroom activities helped you the most in learning mathematics? Can you give specific examples from any of the courses you took in this programme?", one participant wrote, "I found problem solving to be particularly helpful to help me throughout my mathematics journey in NIE." We note that two participants found problem solving to be very helpful when they tackled mathematics problems not just in the Problem Solving course, but in other mathematics courses up to their fourth year. Another participant literally listed the problem solving heuristics and stages, which suggests that he or she found them very important and helpful.

Although not all participants stated their views on problem solving when answering the five open-ended questions, the quantitative data indicate that the Problem Solving course helped most of the mathematics undergraduates develop their capacity in solving mathematics problems confidently. The views expressed by some participants also affirm the decision to offer the Problem Solving course in the first year of the undergraduate mathematics programme.

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