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Consequences of a virtual encounter with George Pólya

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Abstract. The consequences of a virtual encounter with George Pólya as a teacher are recorded. An instance of his influence on my mathematical thinking is recounted through work on one of the problems in one of his books.

Key words and phrases: problem solving, mathematical powers, specializing and generalizing, conjecturing and convincing, Moessner's process, looking back and reflection, Pólya.

MSC Subject Classification: 01A99, 11A05, 97-03, 97D50.

History

In 1968, in my second year as a graduate teaching assistant, in Madison, Wisconsin, my assignment was to teach one of 12 first year Algebra & Trigonometry course sections, at 7:45 am, five mornings a week, covering a pre-assigned chapter each week. The students had to pass the course to remain in the university, but otherwise would get no credit towards graduation.

On a Friday night at the beginning of the semester, the department showed the film *Let Us Teach Guessing* (Pólya, 1965). The following Monday I changed my teaching style, replacing straight exposition with engaging students through interactive questioning. On the next three days I reverted to exposition so as to complete the chapter, and on Friday, as usual, I reviewed the chapter, working with them on review exercises. Within a few weeks I was working with the students for the first three days 'my way', then expounding as deemed necessary in order to finish the chapter on the Thursday, and reviewing the chapter on the Friday by solving exercises with contributions from the students.

Years later I realised that the film had brought to the surface my experiences in High School at the hands of my teacher Geoff Steele, who had provided me with extra mathematical stimuli. Later still I discovered that he was not in fact trained as a mathematics teacher. He had been a local choirmaster who had turned to teaching mathematics, and it appears that he taught himself in order to keep ahead of me. There was something in the way he treated me which was released by Pólya's film and I have been unable to break the habit ever since.

When I finished my PhD in 1969, my wife and I set out to travel in Europe, eventually landing at the Open University, a distance teaching institution which had started up only a few months earlier. For its foundation course (first offered in 1971) the team had chosen *How To Solve It* (Pólya, 1945) as an inspiration, though not as a set book. I was asked to design and implement the one week residential summer school (provided on three university campuses simultaneously, for some 7000 students over 11 weeks). I chose the film as the core of the week's work, showing the film on the first evening and then having tutorials the following morning in which students worked on other similar problems (Mason, 1996) in what I hoped would be a similar fashion. I had assumed that all of the tutors we hired, being mostly current or in some cases retired university lecturers, would be familiar with the basic mathematical processes of specialising and generalising, conjecturing and convincing, only to discover that there was a job to be done to 'remind' tutors of these fundamental processes and to engage them in a manner which they could then emulate with the students, so as to bring those processes to the consciousness.

For the first few years, I directed the first summer school week at each site, in order to get things going. One year the film failed to arrive in time for the first week, but being very familiar with it, I was able to re-enact it with the students, from which I received the nickname 'son of Pólya'!

The Open University in conjunction with the BBC invited Pólya to make a second film (BFI, 1972), but because I was away at the time I knew nothing about it until it was completed, so I missed an opportunity to work directly with him. The second film did not however have the same charm and insight as the first. The original film continued to be shown and worked on some 30 times a year over a period of 25 years to perhaps some 150,000 students.

In the early 1980s when we were developing our second course in mathematics education, it became evident that part of each week's work (nominally five 2-hour sessions delivered by text) should be an evening of problem solving for the students, in order to refresh their motivation for teaching mathematics, remind them of core mathematical processes, and to experience some pedagogical actions which they could then implement in their teaching. The problem was how to select suitable tasks for our students. Leone Burton and I decided to outline a structured approach to mathematical problem solving so as to clarify for ourselves what we wanted students to encounter. The result of our planning was the outline for *Thinking Mathematically* (Mason, Burton, & Stacey, 1982), expanded and developed with the help of Kaye Stacey. The book was a re-casting of Pólya's book (1962) combined with insights from *How to Solve It*, expressed in a form which we hoped would be suitable for teachers in 1970's U.K., where teaching was just becoming an all-graduate profession. Its principal feature was to concentrate on the lived experience of mathematical thinking, strongly inviting readers to engage with problems, to reflect on what they experienced, and then to direct attention to salient aspects highlighted by each individual problem. This was in alignment with our approach to mathematics education as well.

At the time, I felt that mathematical problems were independent of human beings, and so I did little to record the sources of many of the problems which were gleaned from a variety of places, often modified or developed for our own use. One of the problems that I found in Pólya (1954, pp. 117–118), is to explain and justify a phenomenon which Pólya had probably garnered from an article by Moessner (1951), but not mentioned. I retitled it *Pólya Strikes Out* (Mason, Burton, & Stacey, 1982/2010, p. 169). The next section traces the development of Moessner's phenomenon.

In the late 1980s I received a letter from Pólya asking me for the source of *Pólya Strikes Out*, so I duly looked it up and indicated where I had found it. That was the closest I came to an actual encounter with Pólya himself, while being imbued with his thinking.

Moessner, Pólya, Guy and Conway, and long strike out

The basic phenomenon as presented by Pólya is the following:

- Write down the natural numbers in a long row.
- Select a positive integer, say 3.

- Cross out every third number in the sequence, then write below each number (except of course the ones crossed out) the cumulative sum of the numbers not crossed out so far.
- Now repeat the process with the new sequence, crossing out every other number, then forming cumulative sums. The crossed-out numbers border a triangle of uncrossed numbers to their left, and when this triangle reaches a vertex, highlight those numbers (and stop performing cumulative sums and crossing out using them).

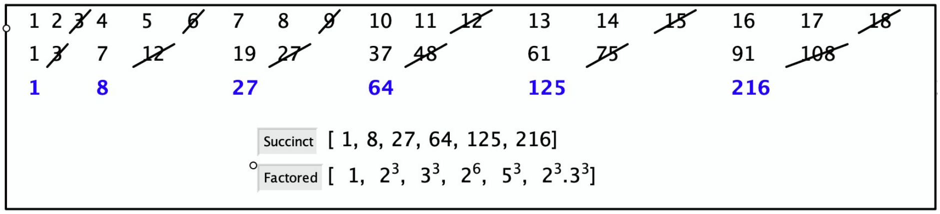

Figure 1. Moessner phenomenon producing cubes

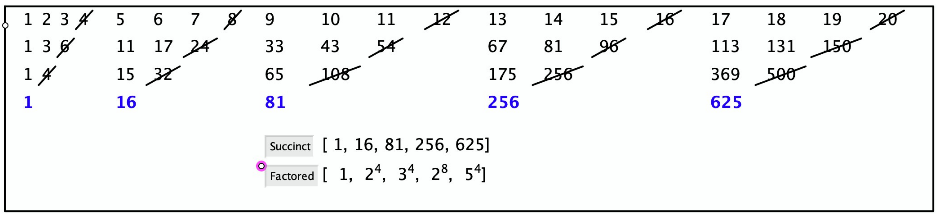

Figure 2. Moessner phenomenon producing 4-th powers

My natural propensity is to try to place problems in as general a context as possible. My working hypothesis has always been that generalisation is usually both possible and desirable. My experience suggests that generality encompasses more and more apparently different manifestations, offering both insight into and comprehension of 'what is really going on', and appreciation of a growing web of relationships.

However, Moessner's phenomenon as presented by Pólya defeated me. I could see why it worked for the sum of the k-th powers of the positive integers but I could not see how to generalise it. Then I encountered *The Book of Numbers* (Conway & Guy, 1996), with a two page development of the Moessner phenomenon (ibid., pp. 63–65). I was astonished at the generality they had uncovered, but could not even see how to begin to justify their claims, much less put them into a general context beyond what they had done.

The Conway–Guy version uses a more complicated crossing out rule for determining the initial sequence to be crossed out, which enables them to predict the final sequence. Here are some examples:

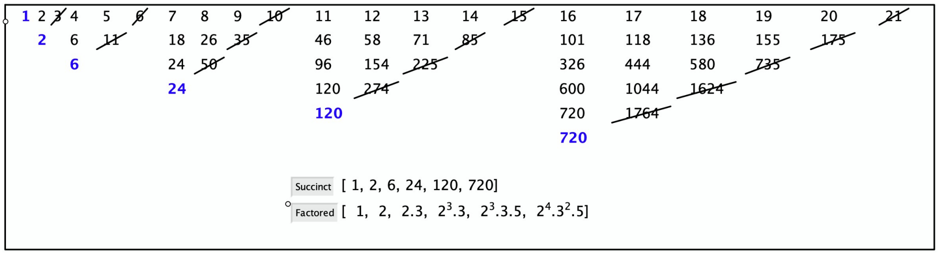

Figure 3. Moessner phenomenon for crossing-out sequence of triangular numbers starting from the natural numbers

In Figure 3 the rule for highlighting and performing cumulative sums is harder to state: each uncrossed number in the current row is involved in the cumulative sums, but once highlighted, it no longer takes part.

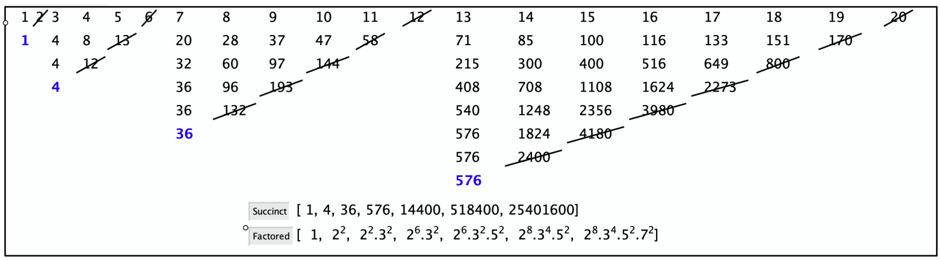

Figure 4. Moessner phenomenon for crossing-out sequence leaving successive odd numbers as gaps, starting from the natural numbers

The highlighted numbers form the final sequence. I choose not to report Conway & Guy's explanation for the sequences resulting from using their generalised crossing-out rule, so as not to interfere with readers' own exploration.

Having embarked on this writing and thinking, I searched for other writing on Moessner's process, discovering a chain of Calvin Long's articles on the subject written for different audiences (Long 1986, 1982, 1966), and articles by

Person (1951), Paasche (1953, 1954/55), & Slater (1983). Karel Post (1990) provides a way of depicting the crossing-out process using directed graphs, such that determining the final sequence is a matter of counting directed paths.

Of course it is to be expected that someone else will have 'been there first', but what matters to me is the personal lived experience. I don't want to look at someone else's resolution, at least until I have exhausted my own efforts. If what matters is the thinking, rather than the result, especially for students, then the posting of resolutions on the internet acts against the better interests of students who may be tempted to 'get an idea' by searching the internet rather than using their own resources.

In *Thinking Mathematically* we chose not to provide resolutions, only suggestions and for the in-text problems, some indications of thinking processes which we found useful for that problem, as a way of drawing it to the attention of readers. As with many of the problems in *Thinking Mathematically*, the advice to try to generalise by making changes, here to the striking out procedure, or the starting sequence, was intended to encourage others to go beyond what we were able to accomplish!

In 2010 Kaye Stacey and I added a new chapter recasting the 'processes' of mathematical thinking as the use of natural powers which everyone possesses and which underpin mathematical thinking (Mason, Burton, & Stacey, 2010). Techers and educators seemed to respond more positively to the notion of powers rather than the 1980's language of processes. We also added some further problems, and provided a glossary linking the many problems to common elements of mathematical curricula.

Constructing a striking out applet

Although I thought that I had comprehended the Moessner phenomenon, it was only when I started to construct an applet in order to help me comprehend and even possibly extend Conway & Guy's version that I realised I had not been sufficiently specific about how the striking-out is done. Furthermore, whereas in my notes I could manifest the striking-out process without really attending to the details, trying to instruct a computer proved to be unexpectedly challenging. It took me three tries before I was able to get a basic applet functioning, without bells and whistles.

The first issue is what to store: the initial number sequence of course, and the terms to be crossed out, or rather their position in the sequence, leading to acknowledging the importance of storing both the numbers and their positions.

Then there was the small matter of highlighting the isolated terms contributing to the final sequence. Finally, there was the matter of deciding how the crossing-out sequence changes from row to row, and which terms to highlight.

My aim was to provide myself with a tool which would

- start from a recognisable sequence (input as a finite sequence or a formula);
- display the struck-out and the highlighted terms;
- collect a succinct version of the final sequence;
- factor the final sequence.

Lessons learned

Programming the striking-out action brought to the surface again the power of trying to instruct a machine to do something that I think I can do myself, advice which Seymour Papert made explicit (1980). This aligns with the pedagogic frameworks we derived at the Open University to provide teachers with some structure to their planning of and reflections upon mathematics lessons. Based fundamentally on Bruner's three modes of (re)presentation, *enactive–iconic–symbolic* (Bruner, 1966), but informed by experience of various members of the Open University writing team (Floyd, Burton, James, & Mason, 1982), we identified triples such as

- *do–talk–record* (or *see-say-record*) as a reminder of the importance of trying to articulate what one has a sense of (*DTR* and *SSR*);
- *manipulating–getting-a-sense-of–articulating* as a spiral of developing sophistication exploiting specialising, using confidence-inspiring examples in order to *get-a-sense-of* a potential underpinning relationship or *generality*, and refining articulation of this until it becomes itself confidence inspiring and familiar (*MGA*);
- *educating awareness, training behaviour and harnessing emotion* based on Caleb Gattegno's observation that *only awareness is educable* (1987; see also Mason & Johnston-Wilder, 1984; R. Young, & P. Messum, 2011).

These triples act as reminders, and have found favour amongst teachers internationally, both as frameworks for reflection, and for preparing to teach in the future. I see them as an ongoing part of George Pólya's legacy, initiated by his concern for the education of teachers of mathematics.

Thinking Mathematically suggests that establishing a *conjecturing atmosphere* is essential in order to sustain and develop mathematical thinking. It goes further, suggesting three phases or orientations while justifying conjectures: *convincing yourself, convincing a friend*, then *convincing an enemy*, or *sceptic*, as David Tall (private communication) puts it more helpfully. These can be seen as instances of *MGA*, and as phases of work in trying to reach a justification of a conjecture which will stand up to scrutiny. In short, learning to reason mathematically, to prove.

Most significant lesson

The most significant and salient lesson for me in drawing these threads together has been to revivify yet again my sense of Pólya's fourth phase of mathematical thinking: *looking back*. For many years this remained a fairly bland and general injunction, and as Jim Wilson put it (private communication, 1977 or so), it is (and continues to be) the least effectively enacted of all Pólya's advice. In the following passage, Pólya was thinking about teachers working on problems themselves, in preparation for posing problems to their learners:

> The best time for such reflection may be when the solution has been obtained and well digested. Then you *look back* at your problem and ask yourself‘; “Where could I use this problem? How much previous knowledge is needed? [...] How could I present this problem? [...] All these questions are good questions and there are many other good questions — but the best question is the one that comes spontaneously to your mind. (Pólya, 1962, p. 210).

But there are many other dimensions to *looking back*, which is usually placed under the portmanteau term *re-flection*, and in its fullness, also includes *pro-flection* and perhaps even, therefore, *flection*, what Donald Schön (1983) called *reflection-in-action*, as distinct from *post hoc reflection-on-action*. Imagining yourself in the future (*pro-flecting*), explicitly invoking specific actions, makes an important contribution in order to ‘learn from experience’, for as is not sufficiently often noticed,

> one thing that we don't seem to learn from experience, is that we don't often learn from experience alone: something more is usually required.

That ‘something more’ involves constructing a personal narrative (Chi & Bassok, 1989). The construction of personal narratives, the articulating of the sense one has after engaging with some mathematical thinking, provides the foundation for further development. Bruner (1990) described humans as *narrative animals*;

identifying mathematical actions (such as adumbrated by Pólya) and mathematical themes which did or did not lead to progress makes an important contribution to developing mathematical thinking.

Over the more than fifty years since seeing Pólya's film, I have come to see that perhaps as much as half of what is called 'learning' arises from attempts to articulate and re-articulate what has been experienced, not only reflecting on what happened, but imagining myself in the future making use of current insights (Mason, 1994; 2002).

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