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Pólya's influence on (my) research

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Abstract. In this article, I outline the influence of George Pólya's work on research in different areas and especially on mathematics education, namely heuristics and models of the problem-solving process. On a more personal note, I will go into some details regarding Pólya's influence on my own work in mathematical problem solving with a focus on the research project for my PhD thesis.

Key words and phrases: mathematical problem solving, George Pólya.

MSC Subject Classification: 97xxx.

Pólya's influence on various scientific disciplines

I want to start this article with a bold statement: *I believe that the importance of George Pólya and his influence on research cannot be overestimated.* If someone was unfamiliar with Pólya and his work and took a look at his Wikipedia page (Wikipedia: George Pólya (English), n.d.) in the summer of 2022, he or she might not be impressed. Compared to other scientists, politicians, or especially sports or pop stars, Pólya's entry in the famous online encyclopedia is quite short. However, this first impression should not obscure the huge influence Pólya had on not only one, but several scientific disciplines.

Regarding **mathematics**, Wikipedia states, "He made fundamental contributions to combinatorics, number theory, numerical analysis and probability theory." (ibid.) The page also mentions several theorems, conjectures, and inequalities, as well as "three prizes named after Pólya, causing occasional confusion of one for another." (ibid.) The German Wikipedia page on Pólya (Wikipedia: George

Pólya (German), n.d.) goes into more details regarding his mathematical work and, amongst others, also mentions his seminal book *Aufgaben und Lehrsätze aus der Analysis* (together with Gábor Szegő). Certainly, all of this does not happen to someone who has not achieved greatness in mathematics.

Mathematics, however, was not the only discipline Pólya influenced. His works on heuristics, initiated with his famous 1945 book, *How to Solve It* – I will go into details later in this article –, was a major influence in the disciplines of **computer science** and **artificial intelligence**. Allen Newell (1981) states, "Everyone in AI, at least that part within hailing distance of problem solving and general reasoning, knows about Polya." (p. 1) Newell, though, quickly relativizes Pólya's role, by saying:

> A neat phrasing of its theme would be *Polya revered and Polya ignored*. Polya revered, because he is recognized in AI as the person who put heuristic back on the map of intellectual concerns. But Polya ignored, because no one in AI has seriously built upon his work. In the coin of the AI realm, no one has built an AI system to realize the schemes investigated in Polya's works. (Newell, 1981, p. 1)

However, more recent books like Michalewicz and Fogel's (2004) *How to Solve It: Modern Heuristics* clearly show that Pólya's work still influences these scientific disciplines. The authors transport Pólya's ideas into the modern world, describing computer-based approaches to problems like the "travelling salesman".

Pólya's influence does not stop there; his *enumeration theorem* is used in **chemistry** and papers of his on *apportionment* were used in **politics**, in Swiss elections to be more precise (Wikipedia: George Pólya (German), n.d.).

In **psychology**, Pólya greatly influenced research on *metacognition* and *self-regulation*. When Pólya wrote his books on problem solving, starting with *How to Solve It*, he anticipated most of the later research on metacognition. Actually, you won't find the term "metacognition" in any of Pólya's books from the 1940s and 1950s, which might be surprising at first. But then you realize that the term was not coined before the 1970s by Flavell (1976), who referred to Pólya's work (see also Kaune & Cohors-Fresenborg, 2010; Konrad, 2005).

It does not stop there, even in **law education**, teachers advocate to use Pólya's four-step plan of the problem-solving process (see below) and his heuristics to enrich the Socratic method and to help solve difficult law problems – both in teaching and actual application of jurisdiction (Rhee, 2007).

Pólya contributed to the study of **history of mathematics**, for example, with his analyses of Euler's works (Pólya, 1954). Also, he was a major influence of

Imre Lakatos and a reviewer of his thesis (1976), *Proofs and refutations.* Lakatos, in turn, greatly influenced history as well as **philosophy of mathematics**.

Pólya's influence on mathematics education

Finally, I reach the part where I am able to evaluate the influence of Pólya the best: **mathematics education**. Pólya's studies of and reflections about *mathematical problem solving* and *heuristics* had and still have a huge influence. Generally, as the term "problem" is often used for mathematical tasks of any kind (e.g., word problems), Pólya's steps or phases for solving problems are widely known amongst mathematics educators, teachers, and even students. More specifically, in the meaning of working on "non-routine problems" (cf. Schoenfeld, 1989, p. 87 f.), mathematical problem solving is a central part of mathematics curricula around the world (e.g., KMK, 2004; NCTM, 2000; MOE, 2006) as well as the conceptual frameworks of studies like TIMSS or PISA (OECD, 2003). And in this context, it is Pólya who is widely acknowledged to be the "father of problem solving (with regard to mathematics)" (Heinze, 2007, p. 16, translated by BR). Schoenfeld summarizes Pólya's contributions like this:

> [...] when Polya published *How to Solve It* in 1945, [...] the study of heuristic was indeed as good as forgotten. *How to Solve It* was 'an attempt to revive heuristic in a modern and modest form,' offering what might be considered a guide to useful problem-solving techniques. [...] *How to Solve It* was followed by the two volumes of *Mathematics and Plausible Reasoning* (1954) and later by the two volumes of *Mathematical Discovery* (1962 and 1965), in which Polya elaborated on the theme and on the details of heuristic strategies. Once nearly forgotten, heuristics have now become nearly synonymous with mathematical problem solving. (Schoenfeld, 1985, S. 22 f.)

Regarding the education of mathematics pre-service teachers at German universities, we conducted a survey amongst educators who work at German universities (Rott & Kuzle, 2017). There are not many seminars or lectures dedicated specifically to problem solving in Germany. However, all pre-service teachers of all school types (primary school, lower and upper secondary school, vocational schools, and special education schools) attend lectures like "Introduction into mathematics education", and within those lectures, process-related competencies like arguing or modeling play an important role. As stated above, as an integral part of the German curricula, one of those process competencies is mathematical problem solving (KMK, 2004) and, as far as I know, Pólya's ideas are taught at every German university in such lectures. However, I have to reduce the meaning of this statement. Even though most – if not all – students come into contact with Pólya's four-step plan (see below), dealing with Pólya's ideas is often restricted to a superficial analysis of those four steps. The more thorough analyses of the mathematical problem-solving and discovery process (Pólya, 1962) are not regularly covered in German teacher education.

Taking a look at international research, it is obvious that the early problem-solving research in the 1960s and 70s in the field of mathematics education is explicitly linked to Pólya. Representatives of researchers from this time period from the USA are Kilpatrick (1967), Lucas (1972), and Kantowski (1974) – in the case of Kilpatrick, Pólya was even one of the reviewers of his thesis. Regarding more recent works, in the 2007 ZDM special issue *Problem Solving Around the World: Summing Up the State of the Art* (Törner et al., 2007), Pólya is cited in every article but one. And in the ICME book by Liljedahl et al. (2016), Pólya is the most cited author.

In Germany, since 2014, we have an official working group of the Society of Didactics of Mathematics (GDM: Gesellschaft für Didaktik der Mathematik) of which I am currently one of the leaders. This working group regularly organizes conferences and within the presentations at those conferences, Pólya is one of the most mentioned authors there as well.

Pólya's influence on my personal research

Now, I want to proceed with some personal experiences and Pólya's influence on my own research. From 2001 to 2006, when I studied to become a mathematics teacher, we did not have many lectures or seminars on mathematics education and were not required to read a lot of mathematics education literature – but Pólya was one of the few authors I came into contact with during that time. Later, from 2008 to 2012, I have deepened my studies in mathematics education and I wrote a PhD thesis in this scientific field. The topic of my thesis was mathematical problem solving (Rott, 2013). The two authors that influenced my PhD studies the most were – not really surprising – Alan H. Schoenfeld and George Pólya. This is, of course, no coincidence; almost all researchers in the field of mathematical problem solving in Germany draw heavily on the works of Pólya.

If you take a look at the chapter on problem solving of any mathematics education handbook in Germany, Pólya's work is used and cited very prominently.

The first thing I had to do in my PhD was to properly define the term "problem". As stated above, this term is used with varying meanings. One of the clearest definitions comes from Pólya himself:

> [D]er [...] wichtigste Unterschied [zwischen Aufgaben] ist der zwischen Routine- und Nichtroutineaufgabe. Die letzteren verlangen vom Schüler ein gewisses Maß von Kreativität und Originalität, die Routineaufgaben dagegen nicht. [...] Die Grenzlinie zwischen beiden Arten von Aufgaben mag nicht besonders scharf sein; aber die Extremfälle sind klar erkennbar. (Pólya 1980, S. 4 f.)
>
> One of the most important differences among tasks is the one between routine and non-routine tasks. The latter require a certain degree of creativity and originality from the student, whereas the routine tasks do not. [...] The borderline between the two types of tasks may not be particularly sharp; but the extreme cases are clearly recognizable. (Translation by BR)

One part of my PhD studies dealt with heuristics or problem-solving strategies in students' processes. I tried to identify heuristics in the students' actions and their communication. Thus, I observed the students' problem-solving attempts and took notes whenever they tried something. Only after reading Pólya's works, I was able to recognize our students' attempts of understanding and solving the problems as heuristics, namely working backwards, considering a special case, drawing an auxiliary line, or making a sketch, etc. The coding scheme I had developed (Rott, 2018) was built on ideas by Kilpatrick (1967) and especially by Koichu, Berman, and Moore (2007), who both were heavily influenced by Pólya's works. Actually, without Pólya, this line of research would not exist the way it does today. His *Short Dictionary of Heuristic* (the second part of *How to Solve It*) laid the foundation for lots of collections, lists, and taxonomies of heuristic strategies in the literature.

In my PhD study, when we designed the learning environments and selected the problems for our students so that they would have a need and be able to work heuristically, our main inspirations were Pólya's books and, especially, the German translation of *Thinking Mathematically* by John Mason, Leone Burton, and Kaye Stacey (1982), who were, of course, also inspired by Pólya.

Another part of my PhD research was developing an empirical process model for describing and analyzing problem-solving processes. Apart from analyzing students' processes, I did an extensive literature review on models of the problem-solving process (Rott, 2014; Rott et al., 2021). There are a lot of (slightly) different frameworks by different authors developed for different purposes. Nearly all of those models can be traced back to Pólya's (1945) four consecutive steps of solving a problem:

(1) Understanding the problem;

(2) Devising a plan;

(3) Carrying out the plan; and

(4) Looking back.

(The few models that do not refer to Pólya are the ones built on Poincaré's (1908) descriptions of his own problem-solving processes, highlighting incubation and illumination phases.) Looking back at this review, I was surprised to see one single author having so much influence on an entire field of study up to this very day.

The models that do build on Pólya's work can clearly be identified as such. All of them use similar phases with variations like the following (cf. Rott et al., 2021): (a) Wilson and his colleagues (Wilson et al., 1993; Fernandez et al., 1994) criticize the linearity of Pólya's phases (i.e., first step 1, then step 2, etc.). They propose a model with the same four phases as Pólya did with the explicit possibility for problem solvers to go back and forth between those phases from any one phase to another. Additionally, they highlight the importance of metacognition for the transitions between phases. While (b) Mason, Burton, & Stacey (1982) combine the second and third of Pólya's phases, suggesting that they can hardly be distinguished empirically, (c) Schoenfeld (1985, Chapter 4) adds a fifth phase, further differentiating Pólya's second phase into planning and exploring. These three models are exemplary for the multitude of models of the problem-solving process in mathematics education – most of them are phase models with direct links to Pólya's ideas.

In this regard, I want to add a thought to Pólya's model and its variations: Did Pólya not know that problem-solving processes do not all proceed in a linear way? Did he not care that sometimes planning is hard to observe and not all students have a plan right away but need to explore the problem space? As a very experienced and successful mathematician, Pólya knew all this, of course. He even mentions skipping phases (Pólya, 1945). I suppose that he presented his phases in this specific order for didactical reasons; he did not want to discourage his readers by making problem solving look even harder. The purpose of his model was not to empirically describe (or do research on) problem-solving processes, but to guide his readers in becoming better problem solvers – he designed a normative model.

In such a normative way, Pólya's four steps are sometimes even used in German mathematics schoolbooks and then might look like this (translated and altered for copyright reasons by BR, see Figure 1).

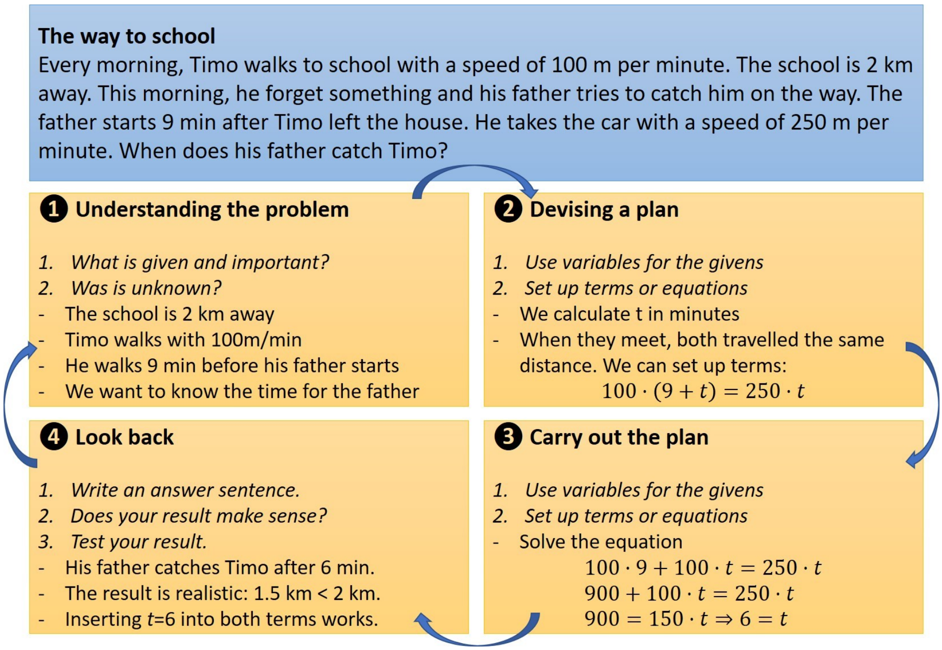

Figure 1. Pólya's four steps of problem solving in a German mathematics schoolbook

After this review of the literature on problem-solving models, I worked empirically, analyzing our students' processes with the goal to develop an empirically-based *descriptive* model. The main method that I adapted for this research was Schoenfeld's (1985, Chapter 9) *protocol analysis* framework. With this method, students' processes are parsed into episodes that are then characterized by means of episode types that resemble Schoenfeld's problem-solving model (see above). In the beginning, we had some difficulties in coding reliably (as predicted in Schoenfeld, 1992, p. 194). On the basis of my review, by assuming an analogy between Schoenfeld's (1985, Chapters 4 and 9) framework and Pólya's (1945) list of questions and guidelines that belong to his four-phase model, I was able to better operationalize the descriptions in my coding manual, resulting in codes with high interrater agreement.

Finally, in my PhD study, I analyzed the students' metacognition, especially in relation to transitions between the Schoenfeld episodes. This time, I used the framework by Kaune and Cohors-Fresenborg (2010). Trying to incorporate ideas from psychological research, it again was Pólya's writings that made the most sense and were best suited to adapt psychological research results for mathematical problem solving.

After my PhD, instead of observing students' processes in laboratory settings, I started to investigate mathematics lessons on the topic of problem solving with a focus on teachers' behaviors (Rott, 2020). At first, it was not easy to compare the results from various video studies and lesson studies, and especially to compare teachers' actions in vastly different lessons (different types of school, students' ages, selected problems, etc.). But again, Pólya came to the rescue. Even though participating lessons varied from grade 1 in primary school to grade 12 in upper secondary school with problems from simple arithmetic, over geometry, and combinatorics, to advanced analysis, every lesson contained phases in which students had to make sense of the problem given to them (either by themselves or explained by their teacher); students had to identify a plan (by themselves or with the help of the teacher); students worked on the problem (with or without aids from the teacher); and the solutions were presented and discussed with the whole class (with more or less focusing on different ideas or just the "correct" approach). Thus, Pólya's phases were perfect for organizing the data and comparing the teachers' ways of managing their students' processes.

I am still impressed by the visionary that Pólya was. In 1985, when he died at the age of almost 98 years, I had just turned 5 years old. His main thoughts, ideas, and results on problem solving were older than 60 years when I started to write my PhD thesis. Yet, to this day, Pólya's ideas are not outdated; on the contrary, Pólya still influences many scientific disciplines.

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