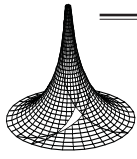

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tmcs@science.unideb.hu
http://tmcs.math.unideb.hu

Teaching
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Thoughts on Pólya's legacy

Alan Schoenfeld

There is a saying, "the older I get, the smarter my parents become." What it means, of course, is that the more we learn, the more we appreciate the wisdom of our forebears. For me, that is certainly the case with regard to George Pólya.

There is no need to elaborate on Pólya's contributions to mathematics – he was one of the greats. See, for example, Gerald Alexanderson's (2000) edited volume *The Random Walks of George Pólya*, or Pólya's extended obituary (really, a 53-page homage) in the November 1987 *Bulletin of the London Mathematical Society* (Chung et al., 1987). Pólya was one of the most important classical analysts of the 20th century, with his influence extending into number theory, geometry, probability and combinatorics.

My concern, of course, is with mathematics education and what is called "problem solving." Within mathematics education, problem solving is what Pólya is known for – or more better put, Pólya is the giant on whose shoulders we stand. What mathematics educators focus on most is *How to Solve it* (Pólya, 1945). There's good reason to do so: In *How to Solve it* Pólya expanded on the notion of *heuristic strategies* as a key aspect of mathematical thinking. The bulk of the book is Part III, a short definition of heuristic, within which we find Pólya's discussion of "modern heuristic," "the mental operations typically useful in solving problems." Pólya notes that the study of modern heuristic "could exert some good influence on teaching, especially on the teaching of mathematics" and that the current volume is just a first step toward the realization of his program.

Indeed, the program was anticipated by his problem books with Gábor Szegő (Pólya and Szegő, 1925, Volumes I and II) and expanded substantially by the two volumes of *Mathematics and Plausible Reasoning* in 1954 and the two volumes of *Mathematical Discovery* in 1962 and 1965.

There is a huge amount of substance in those books! Anyone who works through the problems in the two books with Szegő has learned a substantial amount of analysis. Likewise, there is a huge amount of mathematics to be learned by working through the other volumes. Here, however, I want to focus on the spirit of the work rather than the substance. And that spirit has to do with inquiry, not just in mathematics but in all fields. Pólya called it "plausible reasoning"; today we might call it "thinking mathematically," as did Mason, Burton, and Stacey (1992), or just plain sense-making. What Pólya was really after was how we make sense of the (mathematical) world. That includes problem solving, and problem posing (e.g., *Let us teach guessing*, 1966), but it's bigger than that.

Over the past few months, as I reread Pólya's books while thinking about this introduction, I was struck by how much I had missed or forgotten – and just how ambitious Pólya's intentions were. To be honest, I might have skipped the preface to *Mathematics and Plausible Reasoning* the first time I worked through it; I focused on the math. But the preface does more than introduce the book, it indicates Pólya's grand plan.

Pólya notes two kinds of mathematical reasoning: "We secure our mathematical knowledge by *demonstrative reasoning*, but we support our conjectures by *plausible reasoning*." (p. v). He goes on to say that "anything new that we learn about the world involves plausible reasoning, which is the only kind of reasoning for which we care in everyday affairs."

Yes, the volumes are about mathematics. But, Pólya argues, mathematics is a wonderful discipline in which to learn (in my language) sense-making. Patterns are there to be found. But, one has to have one's eyes open. And, of course, noting a pattern is one thing; pursuing it is something else. What seems to be reasonable? And, if something seems to be the case, why might it be the case? As Pólya notes, "proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference." Pólya was writing about nurturing a spirit of inquiry, a spirit of discovery, a spirit of sense-making.

Much has changed since Pólya wrote the words quoted above. My entry into educational research was stimulated by my reading *How to Solve It* and Pólya's other books. The short version of the story is that I believed the ideas, as did

others; but they didn't seem to work in practice because they weren't quite specific enough to help teachers make the ideas come alive in classrooms. That led to studies of problem solving, and studies of mathematical thinking. As I look at my current work, which focuses on teaching for robust understanding (see https://truframework.org/), I realize in hindsight that in some ways it represents a fleshing out of Pólya's sense-making agenda. The same is the case for research on problem posing, on intuitive mathematics, and more. Changes in the field, changes in the tools available to us, and the accumulation of knowledge over many decades, allow us to make both theoretical and practical advances, as indicated in this volume. In many ways, however, they can be traced to the intuitions and ideas of George Pólya, the giant on whose shoulders we stand.

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ALAN SCHOENFELD
UNIVERSITY OF CALIFORNIA, BERKELEY (USA)

E-mail: alans@berkeley.edu

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