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Promoting a meaningful learning of double integrals through routes of digital tasks

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Abstract. Within a wider project aimed at innovating the teaching of mathematics for freshmen, in this study we describe the design and the implementation of two routes of digital tasks aimed at fostering students' approach to double integrals. The tasks are built on a formative assessment frame and classical works on problem solving. They provide facilitative and response-specific feedback and the possibility to request different hints. In this way, students may be guided to the development of well-connected knowledge, operative and decision-making skills. We investigated the effects of the interaction with the digital tasks on the learning of engineering freshmen, by comparing the behaviours of students who worked with the digital tasks (experimental group, N=19) and students who did not (control group, N=19). We detected that students in the experimental group showed more flexibility of thinking and obtained better results in the final exam than students in the control group. The results confirmed the effectiveness of the experimental educational path and offered us interesting indications for further studies.

Key words and phrases: design of digital tasks, feedback, formative assessment, integration of multivariable functions, tertiary education.

MSC Subject Classification: 97D40, 97U70, 44A45.

Introduction

This paper concerns two routes of digital tasks for engineering students, designed within the frame of formative assessment, and aimed at fostering their approach to calculate double integrals. Multivariable calculus is a basic course for engineering students, and it often represents an obstacle in their academic career (Kashefi et al., 2011; Çekmez, 2021).

Often university students, mainly freshmen attending applied studies, react to the increasing abstraction of contents with a rote procedural behaviour, neglecting to deepen the conceptual meanings of the topics at stake. The integration of functions of one or more variables is perceived by students as one of the most difficult topics of calculus courses (Kiat, 2005; Mahir, 2009; Pino-Fan et al., 2018), because typically it is not sufficient to apply procedures to correctly calculate integrals. Hence, "*more research is needed to understand the way students think about and work with integrals in an applied context*" (Larsen et al., 2017, p. 539). Moreover, within the integration of two-variable functions, specific difficulties can arise, concerning the handling of subsets of $\mathbb{R}^2$. Indeed, solving a multiple integration problem requires not only knowledge and technical skills, but also decision-making processes: students have to know and correctly apply integration techniques, but also be able to recognize and suitably describe the domain of integration, evaluate different available strategies, and finally choose among them the most convenient one. In this respect, we identify two phases in the solving process of a problem requiring the calculation of double integrals. The first phase concerns the setting up of the problem and asks for taking decisions on the modeling (calculation of an area, of the coordinates of a center of mass, of an inertia moment, etc.), the identification of the integration domain and of its available representations, in order to apply Fubini's Theorem and calculate the original integral as iterated integrals. In this phase, which has a strategic nature, coordination and conversions between graphical and analytical representation of the integration domain turn out to be very useful or even necessary (Duval, 2006); moreover, flexibility of thinking is required (Xu et al., 2017). The second, more procedural phase, envisages the actual calculation of the integral, making use of the correct reduction formulae given by Fubini's Theorem.

Behind the cognitive difficulties related to the mathematical content, in some educational situations, obstacles of psychological and sociological nature arise, related to the transition from secondary school to university (Durand-Guerrier et al., 2018; Di Martino & Gregorio, 2019). Generally, in our country, engineering students encounter the topic of integration of multivariable functions during their first university year, when they pass from the classroom environment to the university context, facing a radical change of didactic contract (Gueudet & Pepin, 2015). Here, the high number of students, their heterogeneous backgrounds and the detached relationship with the teacher can create difficulties to freshmen, who may experience a lack of recognition of their identity (Fazey & Fazey, 2001).

Following these considerations, we believe that it is necessary to support students in overcoming their difficulties in setting up the calculation of double integrals, by enacting impactful interventions at all levels of learning. On the one hand, actions are needed for a better understanding of the mathematical content, and the design of the tasks in this perspective is essential (Breen & O'Shea, 2010); on the other hand, the student's perception of the university environment and teaching must be improved.

Digital environments play a significant role in this respect (Descamps et al., 2006; Albano, 2011; Leung & Baccaglini-Frank, 2017). From the cognitive perspective, they can stimulate the learners through different channels, allowing exploration and visualisation of mathematical contents; from the metacognitive perspective, students' monitoring of their learning can be fostered by automatic feedback and guiding questions. Finally, the interaction with digital activities could favour the valorization of the individual and the cooperation among peers.

This research is set in the stream of studies focused on the use of technology to favour the secondary-tertiary transition by fostering a university teaching/learning tailored to individual needs (Bardelle & Di Martino, 2012; Sosnovsky et al., 2013; Silverman & Hoyos, 2018; Alessio et al., 2019; Lepellere et al., 2019; Cusi & Telloni, 2019, 2020; Telloni, 2020, 2021). In particular, we aim at giving a contribution in favoring meaningful learning of a topic that has received little attention by the literature, i.e., the setting up and calculation of double integrals. The formative assessment framework (Wiliam & Thompson, 2008; Black & Wiliam, 2009) provides us the key theoretical element to design two routes of digital tasks, RT1 and RT2, aimed at valuing the individual learning needs and identity through response-specific feedback and hints. The RT1 is focused on the description of subsets of the plane, which is the first useful step to set up the calculation of a double integral, and on the conversions between graphical and analytical representations of subsets in the plane, in Cartesian or polar/elliptic coordinates. The design of the RT1 has been discussed in detail in Alessio et al. (2019).

The RT2 is focused on the description of subsets of the plane as normal domains, defined later (p. 114). This is related to the applicability of Fubini's Theorem and the identification of the correct limits of integration of the iterated integrals to which a double integral can be reduced. The processes which lie at the basis of the calculation of a double integral require conceptual understanding and the development of decision-making skills and flexibility of thinking. Indeed, there are no general rules to establish what representations are available for a general integration domain as a normal domain, and to identify the most effective one. Both RT1 and RT2 (from now the RTs) have been implemented by using GeoGebra[1] and given to students through a Moodle platform[2]. Based on the RTs, we carried out an educational path with engineering freshmen at Università Politecnica delle Marche (Ancona, Italy).

The overall didactical aim of our educational path is to lead students, when facing a problem that involves double integrals, to:

- be able to represent the integration domain and acquire awareness about the usefulness of its graphical and symbolic representations;
- be able to recognize and describe the limits of integration in order to write a double integral as iterated integrals;
- be able to choose the most effective description of the integration domain towards the calculation of a given integral.

The research and educational questions we address in our project are the following:

- Could interactive digital tasks, properly implemented on an online platform, foster the approach to setting up the calculations of double integrals?
- Could suitable online tasks help the students in identifying the available representations of a planar region and in choosing the most efficient one towards a specific objective?
- How does the interaction with the digital tasks change the students' attitude and anticipatory capabilities along the educational path?

In this paper, we address the first two issues, while the third one is left for future work.

The structure of the paper is as follows: in the next section, the essential elements of the theoretical and analytical background are introduced. In Section 3, we describe the methodology we adopted, describing first the task design of the RT2 and then the implementations of the RTs with students. In Section 4, we report the analysis of the outcomes of the cases of study. We conclude the paper with a general discussion on the results and an outlook for future works.

[1] geogebra.org
[2] moodle.org

Theoretical and analytical background

For the essential elements which guided the design of our RTs from a cognitive point of view, namely the common misconceptions in Multivariable Calculus and the difficulties with conversions between different semiotic registers, we refer to Alessio et al. (2019), in particular, Section 2 and the bibliography. Here we present the formative assessment framework (Black & Wiliam, 2009), and integrate it with the characterization of "*the good solver*" (Lompscher, 1975; Schoenfeld, 1982, 1992), considering the activities in which the students are involved as problem solving ones. These theoretical elements have been used for the design of the digital tasks and as lenses to analyse the students' outcomes.

Formative assessment (FA) is a method of teaching/learning such that "*evidence about student achievement is elicited, interpreted and used by teachers, learners or their peers, to make decisions about the next steps*" (Black & Wiliam, 2009). According to the model proposed by Wiliam and Thompson (2008), there are five key strategies of FA:

(A) clarifying and sharing learning intentions and criteria for success;

(B) engineering effective classroom discussions and other learning tasks that elicit evidence of students' understanding;

(C) providing feedback that moves learners forward;

(D) activating students as instructional resources for one another;

(E) activating students as the owners of their own learning.

An extension of this model (Cusi et al., 2017) highlights the role of technology in promoting FA purposes through three main functionalities:

(a) sending and displaying tasks and solutions;

(b) processing and analysing data;

(c) providing an interactive environment.

Even if the FA model is mainly conceived for a classroom environment, we designed our RTs for tertiary level students in this frame. Indeed, in tune with earlier works (Ní Fhloinn & Carr, 2017; Alessio et al., 2019; Cusi & Telloni, 2019; Ţîru, 2019; Cusi et al., 2020), we think that FA could offer a significant opportunity to increase the interactions of the students with the teacher and their peers at the university level, as well as students' awareness, autonomy and focused attitude in learning mathematics. In our project, the FA strategies and the functionalities of technology are used as critical tools to foster an aware approach to setting up the calculations of double integrals.

The tasks we proposed to our students are designed so that the necessary conceptual knowledge is strengthened and multiple approaches to the solution are available. Tasks of this kind are "*a unique challenge for formative assessment*" (Bhagat & Spector, 2017): according to FA strategy (B), they allow to elicit evidence of students' understanding and could make the learners move forward in the learning through appropriate and timely feedback. Moreover, these tasks can be intended as "*problems*", although they are not exactly "*tasks that cannot be solved by direct effort and will require some creative insight to solve*" (Liljedahl et al., 2016). Indeed, even if the students have learned some resolution schemes on integration problems over a general domain, they still have to face the question of determining the correct integration order with the proper limits of integration and the most effective one.

For these reasons, in this study we refer to the literature about problem solving and, in particular, to the characterization of the good mathematical problem solver proposed by Schoenfeld (1992) as a component of the framework to be integrated with the FA.

Schoenfeld (1992) identifies the following factors for success in mathematical problem solving: knowledge, strategies, control, beliefs and practices. Good solvers have extensive and well-connected knowledge; they ground their strategies on structural features of problems rather than on surface ones; and they display clear metacognitive capabilities. Finally, good solvers are often influenced by positive individual and environmental beliefs, and by the practices they are involved in as problem solvers. According to Schoenfeld (1982), a good problem solver is also characterised by flexibility of thought:

He will bring up a variety of plausible things: related facts, related problems, tentative approaches. All of these have to be juggled and balanced. He may make an attempt solving it in a particular way, and then back off. He may try two or three things for a couple of minutes and then decide which to pursue. In the midst of pursuing one direction, he may go back and say "that's harder than it should be" and try something else. Or, after the comment, he may continue in the same direction.

In solving problems, good solvers are able to explore strategies, try some of them, decide if to pursue a strategy or abandon it, and try another one. Good problem solvers have the courage to change their approach.

In tune with the studies by Pólya (2014), as a further element to characterise the good solvers' behaviour, we consider the following actions (Lompscher, 1975), which can be associated with specific heuristics.

- *Reduction*: the problem is reduced into its essential components; typical heuristic tools for this phase are visualisation aids, like graphs or tables.
- *Reversibility*: capability of moving back and forth along the stem of thoughts; a typical heuristic for this phase is working in reverse.
- *Consideration of aspects*: many aspects of the problem are considered and integrated, as a manifestation of connected knowledge; a heuristic corresponding to this phase is the application of symmetry or invariance principles.
- *Change of aspects*: the assumptions and perspectives are changed to find a solution to the problem.
- *Transferring*: the knowledge is transferred from one context to another.

Methodology

In this section, we first describe the essential features of the design of the RTs, which is a key element of our educational methodology, since each task is structured according to the expected approach by the students. Then, we illustrate in detail the implementation of the RTs to the case study involving Engineering students attending the course of Analisi Matematica 2. Finally, we describe our research methodology.

Design of the RTs

Both RTs have been designed on the basis of a preliminary analysis along the cognitive dimension (common difficulties afflicting the teaching/learning of multiple integrals), the epistemic dimension (theoretical fruitfulness and implications of the content at stake) and the didactic dimension (educational opportunities offered by the digital environment with respect to the mathematical topic).

The design and implementation of RT1, where we focused on multiple representations of subsets of the plane, have been discussed in Alessio et al. (2019).

In this paper, we focus on the design of the RT2, concerning the description of subsets of the plane as *normal domains* in Cartesian or polar coordinates, and the choice of the most effective representation towards the calculation of a double integral as iterated integrals through Fubini's formulas.

We recall that a planar subset $D \subset \mathbb{R}^2$ is normal in the x-direction if there exist two continuous functions $\alpha(x)$ and $\beta(x)$ such that $D = \{(x, y) \in \mathbb{R}^2 \mid$

$x \in [a,b]$, $\alpha(x) \leq y \leq \beta(x)\}$. In this case, the integral over D may be calculated as iterated integrals, first in the y variable, and then in the x variable: $\iint_D f(x,y)\,dxdy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right) dx$.

D is normal in the y-direction if there exist two continuous functions $\gamma(y)$ and $\delta(y)$ such that $D = \{(x,y) \in \mathbb{R}^2 \mid y \in [c,d],\ \gamma(y) \leq x \leq \delta(y)\}$. In this case, first we integrate in the x variable, and then in the y variable: $\iint_D f(x,y)\,dxdy = \int_c^d \left(\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right) dy$.

We notice that describing a set as a normal domain means to understand if it is possible to apply Fubini's Theorem, identifying correctly the limits of integration of the iterated integrals.

RT2 includes six digital tasks (T1–T6) requiring short open-ended (numerical or symbolic) answers and one final open problem (P). The choice of open-ended questions is motivated by the goal of avoiding trial and error approaches; it was suggested by the outcomes of a pilot study carried out on the RT1 (Alessio et al., 2019). As said, all the tasks in RT2 are implemented using GeoGebra. They graphically give a planar region, and the students are required to provide its analytical description as a normal domain in Cartesian coordinates (in the x- or y-direction) or in polar coordinates. The tasks also provide the option of asking for some hints, if the student experiences difficulties and wishes to do that. For each action by the student, automatic feedback returns, which is facilitative and strongly dependent upon what the student did (Shute, 2008). The provided feedback is of different kinds (interrogative, visual, dynamic, ...); in the case of a wrong answer, the region selected by the student is shown, allowing him/her to adjust the previous answer and deepen the covariation of the limits of integration and the graphical representation of the set at stake. For these characteristics, the feedback is not only on the task, but also on the process to solve the task and on self-regulation (Hattie & Timperley, 2007). When a task is correctly performed, the following one becomes available.

The first three tasks of RT2, T1, T2 and T3, are about the identifications of normal domains (one in the x-direction, one in the y-direction but not in the x-direction, the last again in the x-direction but not in the y-direction). In these tasks, the definition of normal domain is recalled, and the student is required to enter the functions with their intervals of definition, i.e., the limits of integration, which identify the domain as x-normal or y-normal. These tasks want to strengthen the knowledge of the concept of normal domain, showing that this definition may not apply even in simple examples. The task T4 deals with a domain with circular symmetry: the student should choose the direction (x or y) in which the given region is a normal domain and describe the functions identifying it. After the task is solved, the program shows that the region can also be described using polar coordinates and precisely as a normal domain in the θ-direction. The region in T5 is neither a normal domain in the x- nor in the y-direction, but it can be described as a normal domain in polar coordinates (in the θ-direction). The last task, T6, displays a region that can be described as a normal domain in both x- and y- directions, and also by using polar coordinates. In this case, the student, in addition to finding the limits of integration, is required to indicate the most efficient representation of the domain between the three available to evaluate a given double integral. Through a suitable system of checkboxes, the calculation of the integral according to the different representations of the domain is provided, so that the student could detect the most efficient one. This supports the idea that, to foster the students' flexibility and autonomy in problem solving, comparing methods is more efficient than presenting them in sequence (Rittle-Johnson & Star, 2007). The final problem (P) provides the analytical representation of a planar region and requires the calculation of its area, together with a justification of the adopted strategy.

The tasks of RT2 have an increasing level of difficulty, up to the final problem, which calls not only for a deep understanding of the representation techniques, but also for the metacognitive ability to compare different methods of solution.

Because of space limitations, we describe in detail the design of three tasks of RT2 and the final problem[3].

T2. A set D is shown on the right side of the screen (Figure 1). It is a normal domain in the y-direction, as stated on the left side of the screen. The student is required to insert the functions $\gamma(y)$, $\delta(y)$, and the interval $[c, d]$ identifying D as a normal domain.

If a wrong description of the region is submitted, the student is provided with visual feedback: the region corresponding to the given answer, if not empty, is shadowed (Figure 2).

This feedback is *about the task* and about *self-regulation* for the student, who can correct his/her answer by taking advantage of the feedback information. Moreover, it is response-specific, since it strictly depends on the answer given by the student. In our opinion, this is one of the most significant opportunities offered by technology with respect to a paper and pencil approach concerning the

[3] At the website https://math-diism.univpm.it/progetto/, the complete route of tasks RT2 is available.

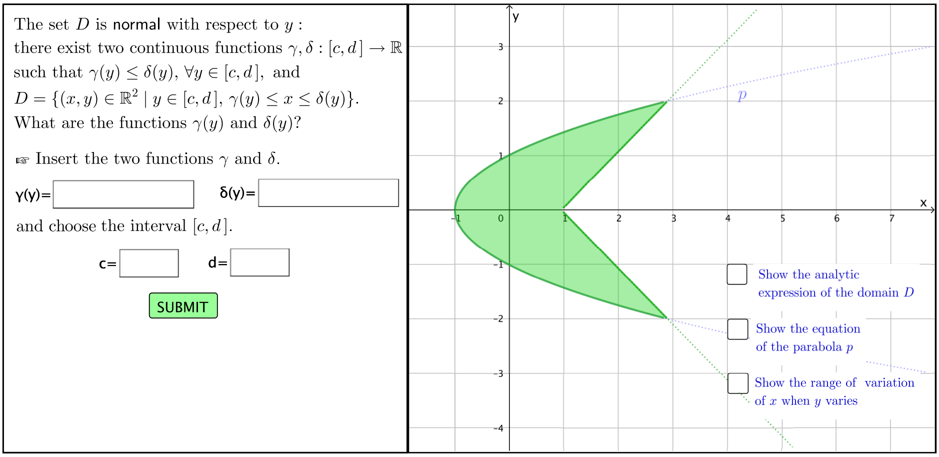

Figure 1. The starting screen of task T2

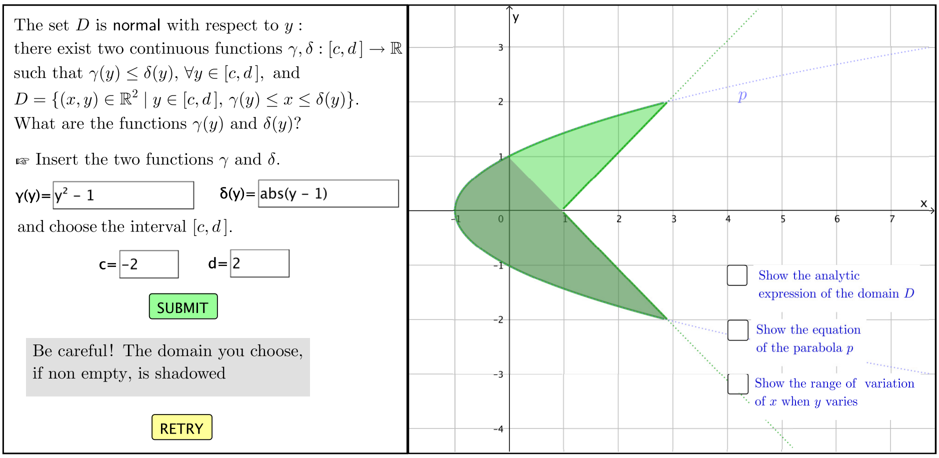

Figure 2. The visual feedback within the task T2

same task: the students can see dynamically the covariation of the analytical and the graphical representation of a normal domain.

When the description of D is correctly performed, the question *“Is D a normal domain in the x-direction?”* appears. If the student selects the wrong answer *“Yes”*, interrogative feedback in verbal language appears, inviting him/her to dynamically explore the opportunity of describing D as a normal domain in the

x-direction by moving a slider (the segment linked to the slider becomes red when it divides into two parts, see Figure 3).

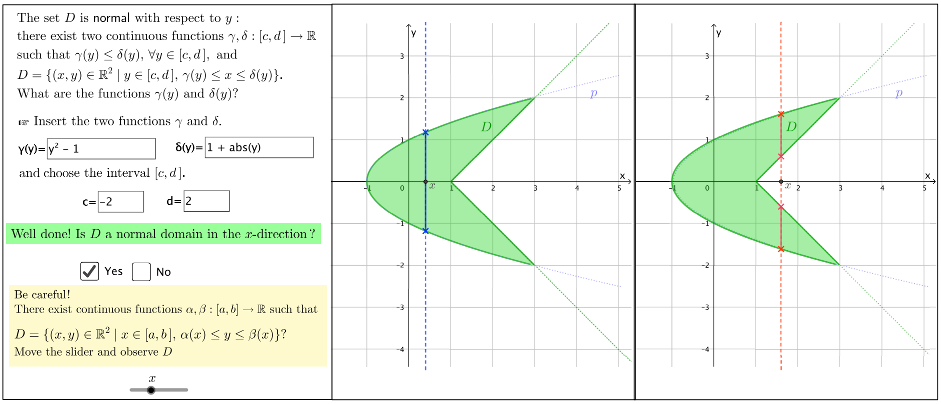

Figure 3. The dynamic exploration allowed by the slider in task T2

Finally, if the answer is *"No"*, a feedback appears, explaining that the domain is not x-normal, but it is the union of two x-normal domains, which are automatically shown (Figure 4). This feedback should make clear to the student that there are more possible ways to describe the given domain, for example, either as a y-normal domain or as a union of x-normal domains.

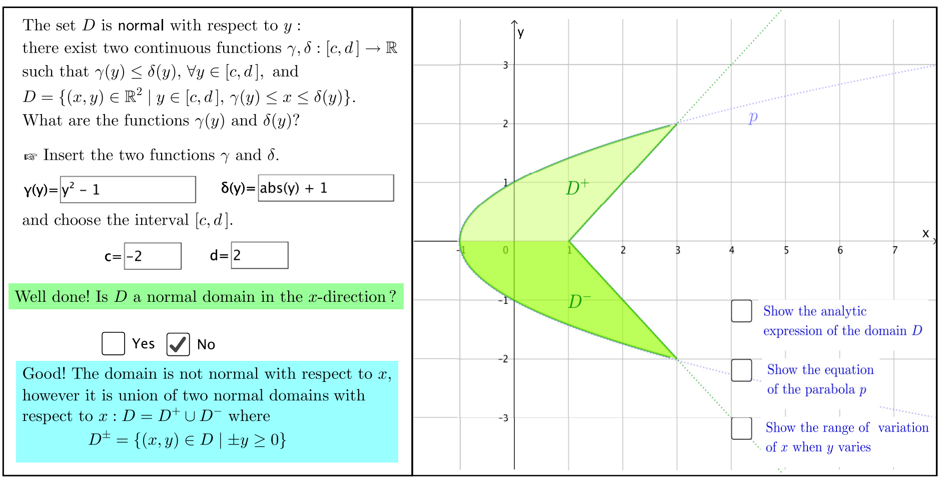

Figure 4. The final screen of task T2

This task should activate FA strategy (B), since it aims at eliciting evidence of the students' levels of learning; strategy (C) through different kinds of feedback, given also by the hints; strategy (E), since the task, together with the whole RT2, is a self-assessment tool based on the interaction computer-learner and aimed at fostering the student's awareness of his/her own learning progress.

T4. A circular sector D is shown on the right side of the screen, while on the left side the student can choose (by checking one of two boxes) whether D is a normal domain in the x- or the y-direction (Figure 5).

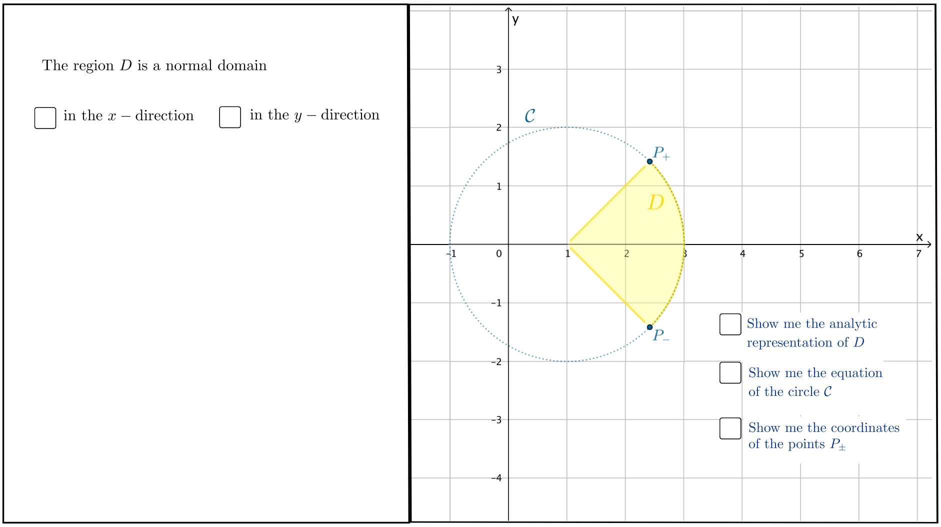

Figure 5. The starting screen of task T4

Since the region D can be described as a normal domain in both directions, a different screen appears for each choice: if "*normal in the y-direction*" is selected, the request of inserting the functions and the interval identifying D as a normal domain in the y-direction appears; if the other option is selected, the program requires inserting the functions (piecewise defined) identifying D as a normal domain in the x-direction (Figure 6).

The same hints of task T2 are available. If a wrong description of the region is submitted, the program signals the mistake and draws the region corresponding to the given answer, if not empty. When the description of D is correctly performed according to the initial choice, immediate feedback and also a brief explanation about alternative representations of D (as a normal domain in Cartesian coordinates in the other direction and as a rectangle in polar coordinates) appear. In particular, the description in polar coordinates is given (Figure 7). This is to enable students to compare different representations of the same domain. Task 4 is aimed at activating FA strategies (B) and (C) and (E).

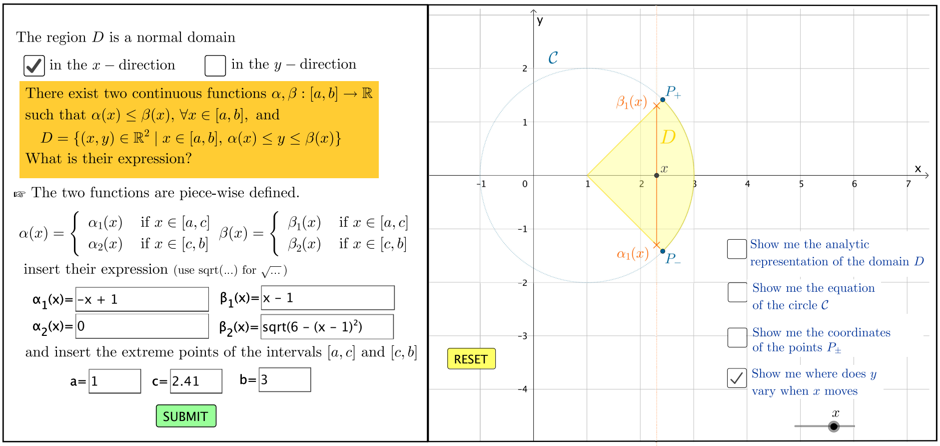

Figure 6. A screenshot from task T4

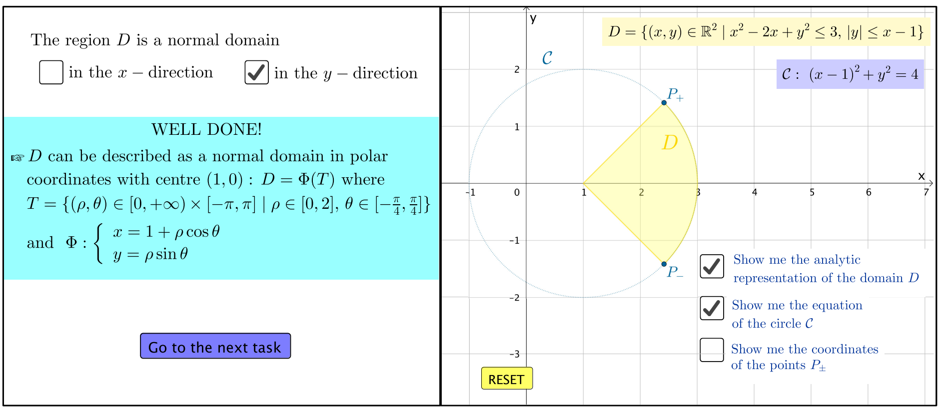

Figure 7. The description of the region D in polar coordinates

T6. The first part of this task is similar to the others, but in this case, we immediately clarify the goal: the calculation of the integral $\iint_D y\,dxdy$ as an iterated integral over a given domain $D \subset \mathbb{R}^2$ (Figure 8), perhaps after an initial change of variables.

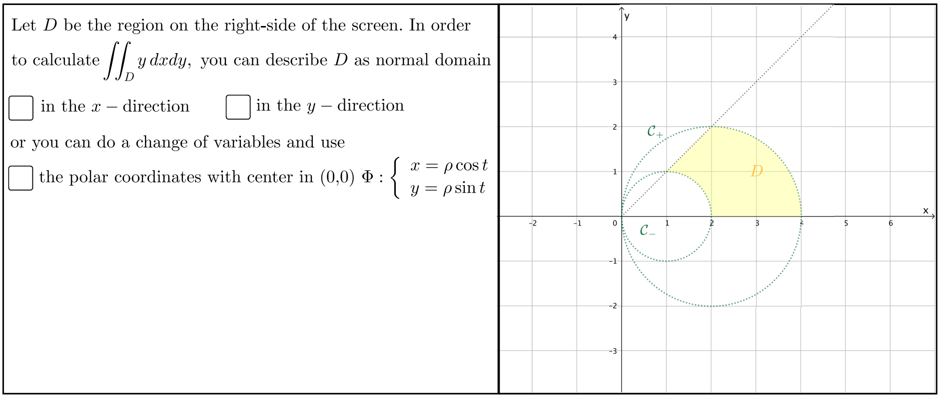

Figure 8. The starting screen of task T6

The student can choose to describe D as a normal domain in the x- or in the y-direction, or by using polar coordinates referred to the origin of the axes. Once the student has described D as a normal domain by inserting the functions, a further question appears on the right side of the screen: *"What is the best representation of D in order to calculate the integral?"* Now the student can see the calculations that are needed with different methods of solution (Figure 9), and thus realize by him/herself that different choices lead to different levels of complications. This part of the task is engineered so that FA strategies B and E are activated, to support the development of the student's awareness about the applications of definitions and techniques.

P. The student is required to calculate the area of the region D (shown in Figure 10), given only through the analytic representation: $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4,\ |x| \leq y \leq \sqrt{3}\}$.

This open-ended problem explicitly asks to describe at least two different methods to obtain the solution, to choose the most convenient one, to justify the choice, and to perform the essential steps needed to solve the problem. Different solution strategies are available: the region D can be described as a normal domain in the x- and y-directions, as a union of normal domains or by using polar coordinates; moreover, D is symmetric with respect to the y-axis, so its area is

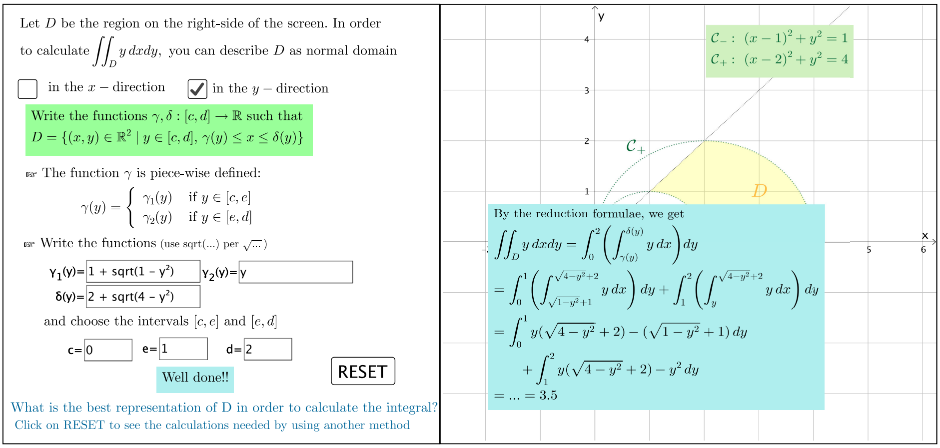

Figure 9. A screenshot from task T6

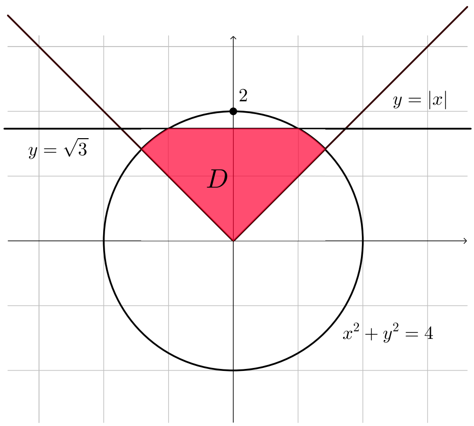

Figure 10. The region D of problem P

twice the area of $D^+ = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4,\ x \le y \le \sqrt{3},\ x \ge 0\}$. Finally, both D and D^+ can be viewed as unions or differences of sets.

According to the framework described in Section "Theoretical and analytical background", all typical actions of good solvers (*reduction, reversibility, consideration, change of aspects* and *transferring*) should be carried out for a successful approach to the problem. Students should activate knowledge and technical skills, but also flexible thinking and capability to anticipate the level of difficulty of the calculations required by each method.

The FA strategies (A) and (E) should be activated through problem P. In particular, the request clarifies that the choice of the most effective method is a criterion for success (strategy A).

The implementation of the RTs with the students

The educational path has been carried out with voluntary freshmen of Mechanical Engineering at Università Politecnica delle Marche (Ancona, Italy) attending the course of Analisi Matematica 2. The course was taught in the spring semester of the academic year 2017/18, after the preliminary course of Analisi Matematica 1; it included 72 hours of face-to-face lectures and 20 hours of practice sessions. Classical and digital resources were shared between the teacher and the students on the Moodle platform of the university. There were 134 students enrolled in the course and about 70 students (typically those who passed the preliminary exam of Analisi Matematica 1) regularly took part in the lectures. The main subjects treated during the course were: elements of differential geometry of curves; differential calculus for functions of several real variables; path and multiple integrals; smooth surfaces and surface integrals; conservative and irrotational vector fields; and ordinary differential equations. The final assessment was assigned through two partial written exams (mid-term and final) in the first available session or a single total written exam in the other sessions; when students passed the written exam, they faced a final oral exam.

For the present study, both the RTs have been handed out to voluntary students in a computer room (and, for what concerns the RT1, also to a group of students at distance) through the official university Moodle platform, under the supervision of a teaching assistant. This methodology was aimed at observing the students' behavior to gain information about their typical approaches and to obtain suggestions for future improvements to the digital tool, according to the research-based design framework (Swan, 2013). However, the students' typical approach when facing the RTs is not discussed in the present paper, it is the focus of a forthcoming one.

The RT1, concerning the conversion between graphical and analytical representations of subsets of the plane, was proposed to all students attending the course in the last week of February, after the topic was treated in the course. 15 students chose to participate in the activity. The RT2, concerning the description of subsets of the plane as normal domains in order to calculate double integrals over them, was proposed to all students participating in the lectures in the last week of May, after the teacher concluded the theory of integration of two-variable functions. 24 students chose to participate in the activity. After the RTs were completed by the participant involved in this research, they were made available to all the students attending the course of Analisi Matematica 2 on the Moodle platform of the university, as a self-assessment tool. Students were free to access them without any constraints.

In order to promote both a meaningful learning and the students' awareness of it, we organised the cases of study in methodological-educational cycles, alternating activity phases with some argumentation and reflection phases. The design and the methodological cycle involving RT1 have been described in (Alessio et al., 2019). The methodological cycle including RT2 envisaged the following phases:

(P1) the first six digital tasks T1–T6 of RT2 were handed out;

(P2) the seventh task of RT2, i.e., the open problem P, was handed out, with the explicit request of written argumentations about different solution strategies and the choice of the most convenient one;

(P3) an anonymous questionnaire about the perception of the digital activity (ease of interaction, role of hints and feedback) was proposed.

The methodological-educational cycle exploited the functionalities of technology of *sending and displaying data* and of *providing an interactive environment* (Cusi et al., 2017).

Research methodology

In order to evaluate the outcomes of our educational path and give answers to the research questions, we collected different kinds of data: for students engaged with the RTs, a) screen-recordings of their interaction with the tasks, b) written argumentations concerning the solving processes for the problem in (P2), and c) answers to the questionnaire submitted in (P3). For all students attending the course, d) written final partial exams. In particular, data b) and d) allowed us to elicit evidence about students' approach to setting up a double integral and their capability of choosing a suitable strategy to solve problems involving double integrals. All the data have been quantitatively and qualitatively analysed: the researchers coded them separately, according to the theoretical framework and the research questions, and then they discussed the emerging themes until an agreement was reached (Sharma, 2013).

A quasi-experimental approach (Cohen et al., 2007) was used to compare the solving processes of all the (38) students who passed the midterm and final written exams of Analisi Matematica 2 at the first session. We divided them into two groups, of which we discuss in the following section: the first group, say, the experimental group (EG), was formed by students who decided to participate in at least one of the activities with the RTs. The other group, say, the control group (CG), was formed by students who preferred to not interact with the RTs (neither with the activity proposed in the computer room, nor independently via the Moodle platform). Both the EG and the CG consisted of 19 students. The groups are homogeneous: students in the EG and the CG attended most of the lectures and practice sessions, and had a comparable average score in the preliminary exam of Analisi Matematica 1.

Some effects of the educational path on the students' learning

In this section, we analyse the outcomes of the final written exam of Analisi Matematica 2 (first session), comparing the productions of students in the EG and students in the CG. We notice that this exam took place about one month after the involvement of the EG students in the interaction with the RT2, hence from their productions we can gather information about the permanence of the effects of the educational path on their learning.

In the final written exam of Analisi Matematica 2, we proposed the following problem: *"Determine the coordinates of the center of mass of the uniform planar object* $D = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \geq 4,\ \sqrt{3}|y| + 1 \leq x \leq 4\}$*"*. As usual, the problem requires a modeling, since students should translate the problem of finding the center of mass into suitable double integrals. Moreover, different strategies and techniques for finding the solution are available.

To correctly solve the problem, students needed to calculate the area of the domain, $A(D)$, and the coordinates of its center of mass, (x_G, y_G). $A(D)$ could be found by using elementary geometry, considerations about the symmetry of the set and/or double integrals; the coordinate y_G could be determined by symmetry without calculations, while for the x-coordinate of the center of mass, the calculation of an integral was needed, and it could be performed by exploiting different properties of integrals and/or the distributive property of the center of mass. The integration domain D was susceptible to different descriptions as a normal domain: in the y-direction, as a union of normal domains in the x-direction, or in polar coordinates (Figure 11). In this way, students were required to apply knowledge and technical skills, and they were also involved in decision-making processes.

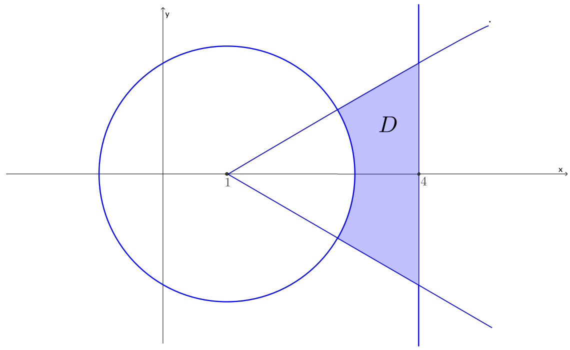

Figure 11. The domain D in the problem faced by the students of the EG and the CG

We analysed in detail and classified all the written solutions of the EG and the CG students to the above problem, according to the following questions:

1) Did the student draw the domain and in what way?
2) How did the student calculate the area of D?
3) How did the student calculate y_G?
4) How did the student calculate x_G?
5) Did the student change strategy during the resolution process?
6) How did the student set up the calculation of the integrals?

About the first question, we decided to classify the correct drawings into three categories, according to specific indicators: a drawing is *rough* if it represents the domain only qualitatively, without quantitative information; a drawing is *almost accurate* if it represents the domain with some quantitative information (coordinates of points or equations of curves), not sufficient to obtain its description as normal domain; a drawing is *accurate* if it contains all the quantitative information allowing the reconstruction of the analytical representation of the domain. In Table 1, we summarize the results of our classification of students' drawing of D. The EG displayed a clear tendency in drawing the integration domain and in doing it accurately, which was an explicit aim of our intervention.

The qualitative analysis of the solving processes in relation to the above questions 2, 3 and 4 showed a general awareness of the usefulness of drawing the domain by the EG and the strategic capability of making use of it. Indeed, some students simulated in their drawings the dynamic variation of a coordinate with respect to the other one, almost reproducing the dynamic hint of the RT2. Other students made several correct drawings of the domain D, with different levels of accuracy corresponding to different goals (initial exploration of the domain, exact description of D as a normal domain in particular directions); this is an expression of the reduction of the problem and of the corresponding heuristic (see Section "Theoretical and analytical background").

	No drawing or incorrect drawing	Rough drawing	Almost accurate drawing	Accurate drawing
EG	0	4	7	8
CG	3	10	4	2

Table 1. Domain drawing results

In Table 2, we indicate the number of methods used by students of the EG and the CG for calculating the area of the domain. The methods considered were: the additivity properties of integrals; elementary geometry; symmetry properties; iterated integrals on normal domains in the x- or y-direction; change of variables by using polar coordinates. Similar results were obtained by considering the methods used by students to calculate the coordinates of the center of mass. Students in the EG generally used more methods and referred to more theoretical elements than students in the CG. The qualitative analysis of the solving processes showed that usually EG students decomposed the problem (*reduction*), then they strategically chose and combined suitable methods and semiotic registers, considering and possibly *changing aspects of the problem* and *transferring knowledge.* This kind of behavior reveals flexibility of thinking and capability of taking a favourable point of view (see Section "Theoretical and analytical background").

	One method	Two methods	Three methods	More than three methods
EG	4	4	7	4
CG	10	4	4	1

Table 2. Number of methods applied to calculate $A(D)$

For example, many students of the EG used the symmetry of D, writing $A(D) = 2A(D^+)$, where $D^+ = \{(x, y) \in D \mid y \geq 0\}$, then expressed $A(D^+)$ by addition or subtraction of areas (i.e., $A(D^+) = A(T) - A(S)$, where T is a triangle and S is a circular sector, or $A(D^+) = A(P) + A(C)$, where P is a trapezoid and C is a curvilinear triangle), then they calculated these areas by using elementary geometry and possibly by integrals.

In Table 3, we report the number of students who changed strategy during the problem solving process. The change of strategy is a typical characteristic of the good solver (Schoenfeld, 1992). Note that we had the possibility of considering only those students who left traces of their first attempts. Typically, these students began to set up and calculate an integral by using a certain strategy (e.g. considering the domain as normal in the x or y-direction), and then changed their approach (e.g. describing the domain by using polar coordinates). In some cases, they explicitly wrote that the change of strategy was due to the difficulty of the calculations or to their developed awareness that the previous one "*is not convenient*"; when the exchange of strategy was not explained, students were asked directly about the reasons for their change of approach during the oral exam.

Students who changed their approach, mostly belonging to the EG, displayed a strong metacognitive control on the goal to be reached and the reversibility of thought, moving back and forth along their problem solving process. Indeed, they were able to go back when a certain approach turned out to be too difficult. On the contrary, CG students generally showed a certain rigidity: they tended to remain anchored to one or few methods, and kept using them even when they were not the most efficient solutions.

	Number of students who changed approach (out of 19)
EG	7 (36,84%)
CG	1 (5,26%)

Table 3. Change of approach results

Finally, in Table 4, we report the outcomes of the EG and the CG in the setting up of the integration problem. We divided students in the EG and the CG into three categories: students who correctly sketched the integration domain and identified the correct limits of integration (Group A); students who correctly sketched the domain and wrongly set up the calculations (Group B); and students who did not sketch or wrongly sketched the domain and wrongly set up the calculations (Group C). Moreover, we considered the percentage of students in Group A who did not conclude the calculation (Subgroup A*).

The analysis seems to suggest the effectiveness of the experimental approach adopted: more students in the EG than in the CG correctly set up the calculation of the double integrals and most of them used efficient strategies. Among students who correctly set up the calculation, the percentage of students who did not conclude the calculation is higher in the CG than in the EG, which may suggest that the strategies chosen by the CG were on average less effective than the strategies chosen by the EG. Furthermore, no students of the EG got the setting up of the calculation completely wrong, compared to about 16% in the CG.

	Group A	Subgroup A*	Group B	Group C
	Correct sketch and setting up of the calculation	Students in Group A who did not conclude the calculation	Correct sketch of the domain and wrong setting up of the calculation	Wrong sketch of the domain and wrong setting up of the calculation
EG	14 (73,68%)	4 (30,77%)	5 (26,32%)	0 (0%)
CG	10 (52,63%)	5 (50,00%)	6 (31,58%)	3 (15,79%)

Table 4. The result in the setting up the resolution of the problem for the CG and the EG

As a further element, we notice that among the 56 students who participated in the two partial exams, 18 did not pass the exams. Of them, 5/24 were students who chose to interact with the RTs and 13/32, a much higher ratio, were students who chose to not interact with the RTs.

Final remarks

This study is about an educational path devoted to promoting a meaningful approach to the setting up of double integrals by university students through their interaction with routes of digital tasks. The tasks are designed with the aim to activate the formative assessment strategies: (A) *clarifying and sharing criteria of success*, (B) *engineering learning tasks that elicit evidence of students' understanding*, (C) *providing feedback that moves learners forward*, (D) *activating students as instructional resources for one another*, and (E) *activating students as the owners of their own learning.* Strategy (C) is activated by means of facilitative and specific feedback focused on the task, on the process enacted by students to solve the task and on self-regulation (Hattie & Timperley, 2007). The RTs as a self-assessment tool exploit the functionalities of technology of sending and displaying tasks and solutions, and of providing an interactive environment (Cusi et al., 2017).

Understanding the limitations of a quasi-experimental approach where the EG and the CG are not randomized and where external variables could intervene, the outcomes of the study seem to suggest the effectiveness of the experimental path to promote a meaningful approach to setting up the solution of a problem involving double integrals, with the help of technology. Most of the students in the experimental group (EG) faced the proposed problem in the written exam with more accuracy and a clearer capability of making suitable decisions with respect to the students in the control group (CG). Most of them drew the integration domain, often at different levels of accuracy according to specific aims, which displays awareness; they were able to recognize the integration domain as a normal domain, even in different directions; finally, they displayed strategic competence in choosing the most efficient representation of the integration domain among the different ones available, also in the light of the function to be integrated.

In many cases, the solving processes of the EG students mirrored the structure and the peculiar elements of the RTs. Many students displayed frequent changes of viewpoint and were able to integrate different semiotic registers and representations. Moreover, they displayed having access to more methods and theoretical elements with respect to the CG students, according to the specific needs.

Students in the EG displayed more flexibility of thinking in problem solving processes: they were able to change approach, reverting the order of thoughts and abandoning a chosen strategy when it was not appropriate. In other words, they had *well-connected knowledge*, were able to plan and develop suitable *strategies*, maintained a constant metacognitive *control* on their own solving process and expressed good *practices* (Schoenfeld, 1992).

Further research is needed to understand to what extent these outcomes could be generalized. The specific feedback and the possibility for each student to choose the simplest or preferred problem-solving process within the RTs represents a way to value individual learning needs and habits; hence, finally, individual identity. The opportunity offered by the integration of technology in the education process is essential (Descamps et al., 2006; Albano, 2011).

In this respect, the present study gives a contribution on how to carry out FA purposes through digital tasks with multiple solution strategies, aimed at improving the learning of double integrals and at valuing individuals at the university level. It should be useful for practitioners as a model for teaching/learning at university, within a formative assessment frame. Moreover, it can give insights at the theoretical level, to develop principles for effective task design.

In this respect, in a forthcoming paper we will deepen the students' learning approaches to the RTs on the basis of the analysis of the choices they made within the digital environment.

Despite the effectiveness of the design and implementation of the RTs, a critical issue emerged: not all the students attending the course chose to interact with the digital activities (neither in the computer room, nor independently via the Moodle platform). This fact can be interpreted as an expression of the idea, quite common among engineering freshmen, of mathematics as a service subject: some students considered the interaction with digital tasks as a time-consuming activity, not so necessary to better understand the problem.

The results of the present study, as well as other implementations of the project during the pandemic, suggest the possibility of an enlargement of the educational experiment, engaging more students in future, making them face the digital activities in small groups. Indeed, during the lockdown phase due to the Covid-19 emergency, we proposed the educational path based on the RTs to groups of engineering freshmen (from 4 to 10 students) at distance, via a conference platform. About 80 voluntary students participated, almost all the students who had attended the course, probably also stimulated by the collaboration and interaction with their classmates. Cooperative learning seemed to make students more available to be engaged. From the research perspective, it would be interesting to understand how the interaction in small groups modifies the effects of the routes of digital tasks on the students' learning at different levels.

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