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Teaching Mathematics and Computer Science

Rational errors in learning fractions among 5th-grade students

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Abstract. Our paper focuses on empirical research in which we map out the errors in learning fractions. Errors are often logically consistent and rule-based rather than being random. When people face solving an unfamiliar problem, they usually construct rules or strategies in order to solve it (Van Lehn, 1983). These strategies tend to be systematic, often make 'sense' to the people who created them but often lead to incorrect solutions (Ben-Zeev, 1996). These mistakes were named rational errors by Ben-Zeev (1996). The research aims to show that when learning fractions, students produce such errors, identified in the literature, and that students who make these kinds of mistakes achieve low results in mathematics tests. The research was done among 5th-grade students.

Key words and phrases: rational errors, learning fractions, operations with fractions, interpretation of fractions.

MSC Subject Classification: 97C10; 97C30; 97C70; 97D60; 97D70; 97F50.

Introduction

All over the world, researchers are interested in what the cause of low mathematics results may be. We present some results that deal with rational number learning, errors in learning and performing operations, and exploring the causes of errors. In the Hungarian-speaking area, error research has been unduly ignored. Rational numbers are the set of numbers that students get entirely acquainted with by the end of public education. We assume the problem lies at the beginning of learning fractions. Many students struggle with the understanding of the fraction concept and what qualities they have. This is one of the sources of future mistakes. We undertake to present only some errors; these are the ‘rational errors’, which were named by Ben-Zeev (1996). We make an attempt to uncover these mistakes among Hungarian fifth-graders at the beginning of learning fractions. No such study has been performed before.

Empirical results on rational errors with fractions

According to McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen (2015), for a complete mathematical understanding of rational numbers, two sub-concepts are necessary:

a) the representations of the magnitudes of rational numbers and
b) the density of rational numbers.

They measured 10-12-year-old students' conceptual knowledge of rational numbers. In their opinion, the knowledge of magnitude representations is necessary, but not sufficient, for the knowledge of density concepts.

They found that there are many qualities of natural numbers that cannot be extended to rational numbers. For example, we do not know what the next number in a sequence of rational numbers is because there is always another fraction between two fractions. Rational numbers are infinitely dense. Furthermore, if someone sees a bigger number in the denominator, they may think it is bigger. However, a rational number with a larger denominator can represent a smaller magnitude.

Research in cognitive psychology and mathematics education has repeatedly shown that students and some adults have difficulties with understanding different aspects of rational numbers. One explanation for these difficulties relates to the natural number bias, when someone uses the properties of natural numbers in rational number tasks in the wrong way.

When rational numbers are introduced, the properties of natural numbers are no longer working. The research literature distinguishes four main areas where such systematic errors can be found (Van Dooren, Lehtinen &Verschaffel, 2015).

- Counting sequence does not work when determining the magnitude of rational numbers (1, 2, 3, ...).
- Arithmetic operations can give unexpected results (Multiplication does not always increase, division does not always decrease).
- Infinitely many symbolic figures.
- We don't know what the next number is - not discreet (Rational numbers are infinitely dense and there is always another fraction between two fractions).

Other studies confirm that learning and understanding rational numbers is a major challenge for students, even though many properties of natural numbers can be generalized, not all properties of natural numbers can be extended. For instance, natural number magnitudes can only be represented by one term, but operations with rational numbers obey different rules (DeWolf & Vosniadou, 2015, Merenluoto & Lehtinen, 2004, Ni & Zhou, 2005). Rational numbers can be represented by an infinite number of terms. e.g. $0{,}5 = 0{,}50 = 0{,}500 = \ldots = \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots$

This is a difficulty for students; the expansion of numerical representation options represents a fundamental change in the number concept (McMullen, Laakkonen, Hannula-Sormunen & Lehtinen, 2015).

Knowledge of fractions is important in mathematics achievement. Studies by TIMMS - TIMMS measures curriculum knowledge - have confirmed that students from lower-performing nations (USA) have a poorer understanding of rational numbers, but especially when it comes to identifying relationships between representations. At the same time, students from outstandingly performing nations (Singapore, Japan) were more flexible in interpreting rational numbers (Mullis et al., 1997).

We participate in various international studies. One of them is TIMMS. The achievement of Hungarian students does not differ from the average. According to Torbeyns, Schneider, Xin & Siegler (2015), fraction magnitude understanding and general mathematics achievement are correlated. The correlation remains when controlled for fraction arithmetic. They think the role of fraction understanding in mathematics achievement is important and it indicates that magnitude understanding is central for numerical development. The theory of numerical development mentions, that whole numbers and fractions have many commonalities, for example, the triple code model requires a relationship between the number symbols and the fractional size. And this understanding plays a central role in mathematical competence (Siegler, Thompson & Schneider, 2011).

The triple code theory for fractions asserts there is a similarity between whole number and fraction representations in the brain (Jacob, Vallentin, & Nieder, 2012). Stanislas Dehaene's triple code model (2004) is currently the most accepted theory regarding the brain processing of numbers. In triple code theory, numerical information is represented by three distinct systems, in different ways, as shown in Figure 1. The three systems work in tandem with each other - so any value can be translated from one to the other.

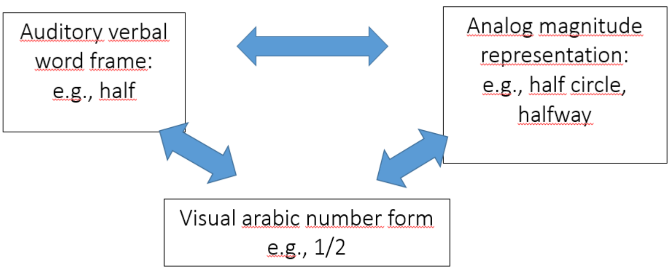


Figure 1. Triple code model for fraction

Methodology

We examined whether rational errors appear in the teaching of fractions among fifth-grade (10-11 years old) Hungarian students. We have written a performance test. When designing the performance test, we took the requirements of the framework curriculum (OFI, 2012) into account and used fifth-grade textbooks and assignments. The tasks were self-developed. The goal was to build a taxonomy system that completely covers the fifth-grade fractions topic. Fraction interpretation, addition, subtraction, comparison, simplifications, expansion with equal and different nominator and denominator, the place of fractions on a numerical line, transcribing a fraction to a mixed number and return, sorting of fractions by size, unit of measurement switch with the fraction, multiplication and division of a fraction by whole numbers, and word problems with fractions.

Sample and quantitative results of the test

118 students participated in the research. The sample included 62 fifth-graders (10-11 years old) from a religious school in a suburban district of Budapest, and 60 fifth–grade students in Kecskemét. The study was aimed at fifth-grade students, trying to find out what level of knowledge they had after becoming familiar with fractions, and what typical mistakes appeared in their work. Children in the capital study mathematics in four lessons a week; one class in the countryside has five lessons a week, and the other class four lessons a week.

The aim of the research is to show that when learning fractions, students produce rational errors. There were two hypotheses:

- The errors that were described in the literature would appear in the test.
- The test results of children who make rational errors are lower than those who do not make this type of mistake.

In the tests, there were other types of mistakes, but the results of those, who made this type of mistake, were low. Obviously, anyone who makes a mistake loses score.

The mean total power is 71 percentage points (standard deviation was 22% p). The achievement is shown in Figure 2. If you plot a curve on the bar chart, it will move to the right. This can be explained by the fact that we measured the students in big cities, the religious school is filled mainly with the children of parents who are motivated to study, and the measurement was carried out at a very early stage in the process of learning fractions, which can be expected to produce good results. The level of knowledge of the students in the capital and in the county capitals is above the national average (Vári, 1997). The key to success in any further fractional operations is to lay the foundations properly.

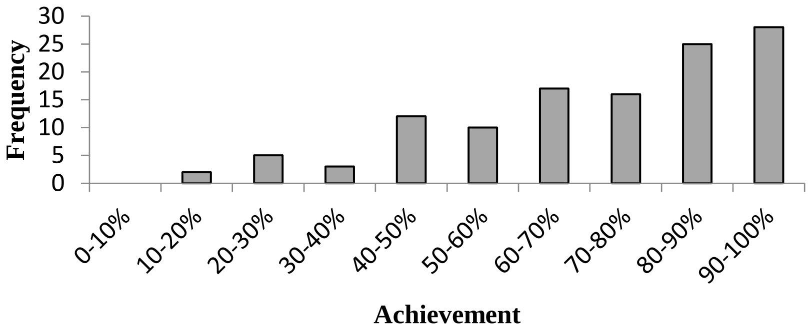


Figure 2. Analysis of knowledge level test

The test results were good, but after a systematic inspection of the tests, we found rational errors. The test results of children who made rational errors were low. Most of these results were between 0 and 60 percent. Those who took the test well but made some mistakes, made another types of mistakes. For example, they did not write anything, or they made a calculation mistake.

A detailed analysis of selected errors

The following errors were found:

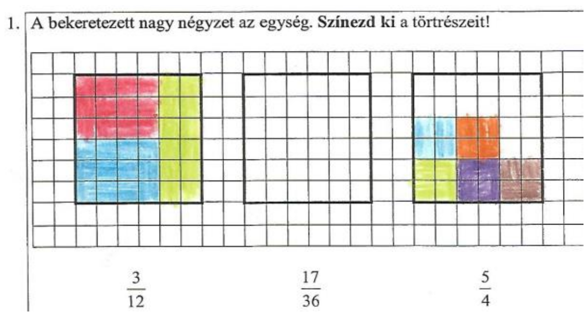


Figure 3. The framed large square is the unit. Colour the fractions of the unit!

In the first exercise, as shown in Figure 3, the magnitude of the fraction had to be plotted. In this case, the concept of a fraction did not develop for him at all. A further four students failed to represent the magnitude of the fraction and made a similar error. In the shown example, $\frac{3}{12}$ is interpreted as 3 pieces of 12 units, and at the $\frac{5}{4}$ similarly. He used different colours. Probably it was what prevented him from portraying $\frac{17}{36}$. He could have done it well without sticking to $17 \cdot 36$ units.

The next task was to compare fractions with equal nominators, and different nominators and denominators, and he should have expanded the fraction as shown in Figure 4. He saw a bigger number in the denominator and he thought it was a bigger number. It is a wrong analogy (Torbeyns, Schneider, Xin & Siegler 2015).

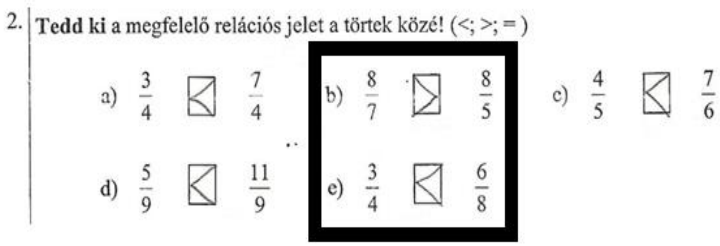


Figure 4. Compare. Write $<$, $=$, or $>$.

According to Nunes & Csapó (2011), it is crucial for the addition and subtraction of fractions that the children understand that the fractions of different forms express the same quantities. Furthermore, international research shows that understanding the equivalence of fractions is not easy for all students. (Behr, Wachsmuth, Post & Lesh, 1984; Kerslake, 1986) As discussed above, it can be assumed that the occurrence of rational errors is associated with a poorer test result.

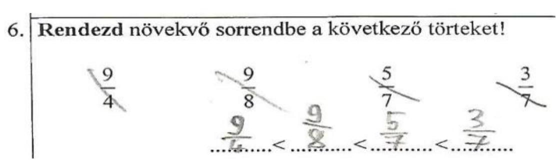


Figure 5. Sort the following fractions in ascending order!

Some students have problems interpreting the magnitude of fractions as shown in Figure 5. Comparisons were made for fractions of equal denominators, equal numerators, and fractions smaller and bigger than one. Nine tests were found, in which the order of fractions is inversely inverted. They have problems interpreting the magnitude of fractions, as Van Dooren, Lehtinen & Verschaffel (2015) mention. The counting sequence is not working.

As shown in Figure 6, they do not understand the distance of the fraction from 0 or the concept of unit.

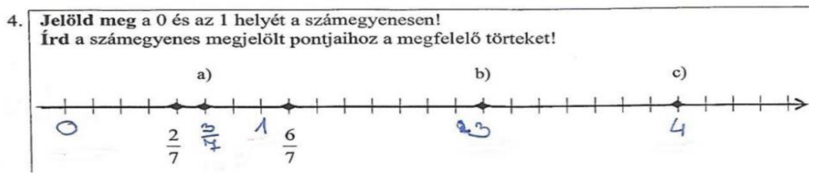
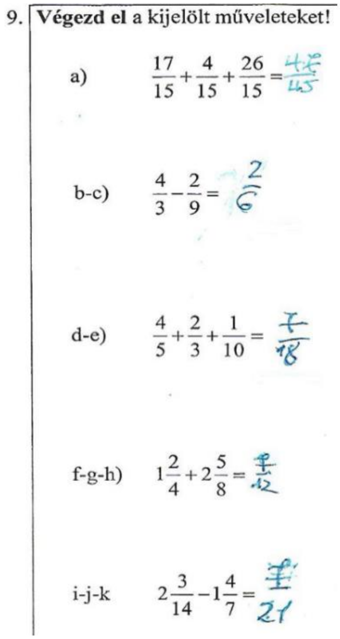


Figure 6. Mark the positions of 0 and 1 on the number line! Write the fractions for the appropriate points on the line!

Marshall (1993) highlights five areas where the fractions are interpreted. One of these is the measurement where the student is expected to understand the distance of the fraction from 0. In the fourth exercise, we examined this aspect. According to Moseley & Okamoto (2008), successful rational number problem-solving is interwoven with the widespread knowledge of rational numbers, in which this area also plays a role.

As shown in Figure 7, the errors are not only in addition but also in subtraction, i.e., fractions are adding the numerator up with the numerator, the denominator with the denominator. It is an indicator of considering fractions componential as two natural numbers rather than holistically as one number (Obersteiner, Van Dooren, Van Hoof & Verschaffel, 2013). Together with what Schoenfeld (1988) mentioned, these students generalize poorly from previous knowledge, subtract the smaller from the larger number.



9. **Végezd el** a kijelölt műveleteket!

a) $\frac{17}{15}+\frac{4}{15}+\frac{26}{15}=\frac{47}{45}$

b-c) $\frac{4}{3}-\frac{2}{9}=\frac{2}{6}$

d-e) $\frac{4}{5}+\frac{2}{3}+\frac{1}{10}=\frac{7}{18}$

f-g-h) $1\frac{2}{4}+2\frac{5}{8}=\frac{7}{12}$

i-j-k $2\frac{3}{14}-1\frac{4}{7}=\frac{7}{21}$

Figure 7. Add or subtract

We found six students who have such an error in one item, and two students who went through with this rational error throughout the ninth task. Their achievement was very poor.

Discussion

This research has revealed a special type of rational errors in mathematics education. We focused exclusively on mistakes committed with fractions by fifth-grade students. During the systematic examination of the tests, we did find several occurrences of rational errors. On the whole, students, who committed rational errors, has an overall poor achievement in the test. Those, who had a good performance on the test and made some mistakes, committed entirely different types of errors than those with poor test results. E.g., lack of any answer or errors in the calculation process. The students, who committed several rational errors, achieved poor or medium overall test results.

The research can be pursued in several directions. We can collect typical errors in other grades (e.g. from third grade to seventh grade), and perform a longitudinal survey to track changes in some errors. We can map what representations in which age-groups appear in children's minds. How does the concept of fraction develop spontaneously and then under the influence of education? What is the impact of fraction representations on the mathematical performance of tests? In addition to the paper-pencil method, we can use think-aloud protocols to reveal what they thought/were thinking when they committed a rational error.

Our results suggest some educational implications. It seems to be worth putting more emphasis on the relationship between different fraction representations. How and to what extent do different educational practices help develop the concept of fractions? Do textbooks used in public education help or hinder the formation of fractional representations? These questions set new directions for research.

In the Hungarian-speaking area, error research has been unduly ignored. If we strive to achieve the best possible results for our children in mathematics, we need to look at the source of the mistakes they are making and pay more attention to develop appropriate mental representations in formal education settings. If we are aware of the wrong conclusions our children can draw, we will be able to spot mistakes and correct them more effectively, so that they can achieve better results in the future.

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