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Teaching Mathematics and Computer Science

Some logical issues in discrete mathematics and algorithmic thinking

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Abstract. The role of logic in mathematics education has been widely discussed from the seventies and eighties during the "modern maths period" till now, and remains still a rather controversial issue in the international community. Nevertheless, the relevance of discrete mathematics and algorithmic thinking for the development of heuristic and logical competences is both one of the main points of the program of Tamás Varga, and of some didactic teams in France. In this paper, we first present the semantic perspective in mathematics education and the role of logic in the Hungarian tradition. Then, we present insights on the role of research problems in the French tradition. Finely, we raise some didactical issues in algorithmic thinking at the interface of mathematics and computer science.

Key words and phrases: didactics of mathematics – logic – semantics – discrete mathematics – algorithmic thinking.

MSC Subject Classification: 97E30.

Introduction

The role of logic in mathematics education has been widely discussed from the seventies and eighties during "modern maths period" till now, and remains still a rather controversial issue in the international community (Durand-Guerrier, Boero, Douek, Epp, Tanguay, 2012).

Nevertheless, the relevance of discrete mathematics and algorithmic thinking for the development of heuristic and logical competences is one of the main points of the program of Tamás Varga. This relevance has been also underlined for long by didactic teams in France, in particular in Lyon and Grenoble. Nowadays, the reintroduction of computer science in curricula in France raises new questions and new didactic research perspectives at the interface between mathematics and computer science.

In a first section, we present the main features of a semantic perspective in mathematics and computer science education. In the second section, we give some insights on the place and role of logic in Hungary in the modern maths period. Then, we exemplify the role and place of research problems in the teaching and learning of mathematics in didactic research in France. Finally, we raise some issues of algorithmic thinking at the interface mathematics – computer science.

Semantic perspectives in mathematics and computer science

In our didactic research, semantics, syntax and pragmatics are considered in a logical perspective consistent with the definitions given by Morris (1938). Semantics concerns "the relation of signs to the objects which they may or do denote" (op. cit. p.21). Syntax concerns the "relations of signs to one another in abstraction from the relations of signs to objects and interpreters" (op. cit. p.13). Pragmatics refers to "the relation of signs to their users" (op. cit. p.29). Morris (1938) argues "Syntactic, semantics and pragmatics are components of the single science of semiotic but mutually irreducible components" (op. cit. p.54).

The semantic perspective in logic appears in Aristotle's theory of formal syllogism, and has been developed in the late nineteenth and early twentieth centuries, mainly by Frege, Wittgenstein, Tarski and Quine. In particular, Tarski (1933, 1943) provides a semantic definition of truth, which he describes as formally correct and materially adequate through the crucial notion of "satisfaction of an open sentence by an object". Relying on this definition, he has developed a model-theoretic point of view, of which semantics is at the very core. Tarski argues that his only intent is to grasp the intuitions formulated by the so-called "classical" theory of truth, i.e. the conception that "truly" has the same meaning as “in agreement with reality” (contrary to a conception that “true” means "useful in such or such regard" (Tarski, 1933). He developed a “Methodology of deductive sciences” (Tarski, 1936) providing a clarification of the relationships between truth in an interpretation and logical validity (Durand-Guerrier 2008). This distinction has been later popularised by Quine (1950). The model-theoretic point of view emerged in Tarski (1954, 1955), but the main ideas were already present in his previous papers. It relies on a simple and very fruitful idea. At first, Tarski considers the notion of model of a formula. Given a formalized language L, a syntax providing recursively well-formed statements (formulae): F, G, H..., an interpretative structure (a domain of reality, a piece of discourse, a mathematical theory, a computation model) is a model of a formula F of the formalized language L *if and only if* the interpretation of F in this structure is a true statement. Some formulae are true for every interpretation of their letters in every non-empty domain. They are said to be universally valid (Quine 1950). This is a generalisation of the notion of tautology in propositional calculus. A classical example is the logical equivalence between a universal implication and its contrapositive that provides a logical basis to proofs by contraposition. From the concept of model of a formula, Tarski defines the key concept of logical consequence in a semantic perspective: "The sentence X follows logically from the sentences of the class K *if and only if* every model of the class K is also a model of the sentence X" (Tarski, 1983, p. 417). This provides an extension of the Modus Ponens.

These notions are the basis for the Methodology of deductive sciences with two important results. Given a theory, it is possible to associate a formal axiomatic system (without objects). The initial theory is a model of this system, but also other theories. Two important consequences are established in this frame. 1/ The deduction theorem: all theorems proved on the basis of a given axiomatic system remain valid for any interpretation of the system. 2/ Proof by interpretation: one way to prove that a given statement is not a logical consequence of the axioms of a certain theory is to provide a model of the formal axiomatic system of the initial theory that is not a model of the formula associated with the statement in question. We give an example of such a proof by interpretation given in Durand-Guerrier, Meyer & Modeste (2019) in the Table 1.

Following Sinaceur (1991, 2001), we consider that the model-theoretic point of view offers powerful tools enabling to take into account both form and content and to distinguish between truth and validity, both crucial issues of the teaching and learning of mathematics. In a didactic perspective, this point of view offers fruitful paths to enrich a priori analyses and to analyse students’ activity in mathematics. (e.g. Durand-Guerrier

2008, Barrier, Durand-Guerrier & Mesnil 2019). Regarding computer science, Durand-Guerrier & al. (2019) argue that although one may consider that syntax is at the very core of the discipline, there is evidence that semantic and pragmatic aspects play also an important role.

A research problem: Given a rectangular grid (with integral dimensions), is it possible to tile it with dominoes (1x2 rectangles)? Theorem. A rectangular grid can be tiled by dominoes *if and only if* its area is even. A frequent (incorrect) proof of the above theorem given by students is the following: a grid can be tiled by dominos if and only if its area is $2k$ where k is the number of dominoes, which means that the area of the grid is even. The stated theorem is correct but the proof is not. It is sometimes difficult to invalidate an incorrect proof of a true statement. In order to do so, one can notice that the fact that the grid is rectangular was not used in the proof. So, if this proof was correct, it could be used for any shape consisting of an even number of squares. It is easy to see that a symmetrical quadrimino with three squares on the first line and one square on the second line cannot be tiled by dominoes although is area is even. In other words, the set of grids of arbitrary shapes is a model of the theory used in the proof above. But in this model, the argument becomes false. Hence, the initial proof is invalid (because otherwise it could be transported into the new model).

Table 1. An example of proof by interpretation (Durand-Guerrier & al, 2019)

Logic in Hungary in the math reform period[1]

According to Máté, Andréka, Németi (2012), logicians have plaid a great role in the modernization of mathematical education in Hungary in the sixties and seventies. The authors remind that Tamás Varga was the leading figure in the new mathematics in Hungary, and that he was a student of Rózsa Péter, a world-renowned logician. Rózsa Péter, beyond his famous book *Playing with infinity* addressed to a wide audience, had

[1] In this section, we rely on Andras Máté, Hajnal Andréka, István Németi (2012) The Development of Symbolic Logic in Hungary.
https://old.renyi.hu/pub/algebraic-logic/MateAN12.doc

important contributions in mathematical logic. In particular she founded the field of Recursive function theory, publishing two important books: *Recursive Functions* (Péter 1951), and *Recursive Functions in Computer Theory* (Péter 1976). She shared with Kalmár and Lakatos, the idea that mathematical studies and mathematical research are substantially the same. Katalin Gosztonyi (2013a) points out that for Rózsa Péter the formal language of mathematics appears as a result of solving problems, not an a priori tool. In his paper in Educational Studies in Mathematics in 1972, Tamás Varga argued that logic and probability should be part of the lower grades curriculum, not as an early construction of the discipline, but rather as *"a construction of certain lines and habit of thoughts which can be characterized as logical and probabilistic. This also develops as a by-product, the outfit of ideas and techniques, skills and knowledge which prepares the way to enriching these disciplines. But the logical and probabilistic impregnation of thinking is most important."* (Varga 1972, p. 346)

Varga provides examples of activities concerning the relationships between quantification and negation that he points out as a challenge even for educated people, that we have also evidenced in France, Tunisia and Cameroun. We present below some examples of activities (Third grade, 8-9 year olds) (op. cit. p. 350).

"Connect the sentences that mean the same and add the numbers of pictures for which they come true.
The pictures are: a) 3 houses with windows; b) 3 houses without windows; c) 3 houses among them 2 have windows, the third one has not.
Examples of sentences
In no houses there are windows - All houses are without windows- Not all houses are without windows - No house is without windows - In every houses there are windows
Not all houses have windows.

Table 2. Example of activity on negation in Varga (1972, p. 350)

To prove and disprove the association is done by stating the conformity of the pictures with the meaning of the sentence. Deciding for equivalence or non -equivalence relies also to the pictures. This kind of activity is clearly focused on the relation between syntax (the grammatical form of the sentences) and semantics (the interpretation given by the pictures). Varga concluded this section as follow:

"While hoping and wishing that mankind becomes more familiar with the Muses than becomes estranged from them, we have to admit that in the computer age we also need other, more exact, more efficient means of expression, which, to be sure, lack certain human nuances." (op. cit. p. 352).

In our view, this paper of Tamás Varga could be an illustration of the following:

"The specific character of this modernization program against similar efforts in the world was that above the incorporation of new branches of mathematics (logic, set theory, etc.) into the material of public education and the aim that real, demonstrative mathematics should be taught in schools, it laid great stress to the intuitive clarity of mathematics and was not „formalist" in the sense as to teach mathematics as pure manipulation of formulas." (Máté & al., 2012, p.5)

It is worthwhile noticing that, as stressed by the example above out of Varga (1972), in logic, the syntactic form of the sentence plays a crucial role in interpretation. In this respect, we would like distinguishing between "formal logic" which is back to Aristotle (Lukasiewicz 1972) and "symbolic logic" which develops in the late XIX and early XX centuries, with formalized languages. The logic of Aristotle is formal in the precise sense that the validity or the invalidity of a syllogism is depending on the grammatical form of the syllogism. In this respect, as pointed out by Largeaut (1972), Aristotle's brilliant intuitions enabled him to build a formal system based on the fundamental concepts of modern logical semantics: classification and modelling of everyday language utterances by formal statements (singular, universal or specific general statements); definition of propositions; introduction of letters of terms to characterize the form of the statements and construct the figures of the syllogism; notion of interpretation of formal statements; notion of logical validity of syllogism; distinction between de facto truth and necessary truth; setting joint work of syntactic and semantic methods to establish what forms or not a concluding syllogism among those that can be constructed in the four possible figures of his system[2].

[2] This is presented with more details in English in Durand-Guerrier (2008) and in French together with a teaching proposal in Durand-Guerrier (2016).

Research problems in mathematics education

In her PhD dissertation, Katalin Gosztonyi (2015) points out a main epistemological difference on the nature of mathematics underlying the Newmaths reform in France and in Hungary. In France, the movement relies mainly on the reorganisation of mathematics and the development of unifying theories such as abstract algebra or topology while in Hungary, the focus is on logic, computer science, probabilities or discrete mathematics (op. cit. p. 104). She hypothesized that this had an impact also on the development of didactic research, in particular for what concerns theories. This is of course a reasonable hypothesis. Nevertheless, in both countries, the importance of addressing problem solving issues is attested for long. Gosztonyi (2016) emphasizes the crucial role played by the ordering of problems developed in Hungary during the math reform and consequently the relevance of their study as a contribution to characterizing the Hungarian tradition. We will not develop this here but will concentrate on the situation in France where, along with an effort to develop theories by prominent researchers such as Brousseau, Chevallard and Vergnaud, more pragmatic didactic research have been developed in France from the eighties, in relation with the development of the IREM (Research Institute in Mathematics Education). It is in particular the case for what concerns research problems and proof and proving.

Research problems, proof and proving in France from the eighties

In 1984, Nicolas Balacheff and Jean-Marie Laborde translated in French *Proofs and Refutations* (Lakatos 1976, 1984), popularizing this important essay of Imre Lakatos in the francophone research community. Nicolas Balacheff defended in 1988 his PhD entitled *Une étude des processus de preuve en mathématique chez des élèves de Collège* (A study of proving process in mathematics by middle school students), which is still a reference in the international community. The theoretical framework relies on the theory of didactical situations by Brousseau (1997) and the model of Lakatos of the dialectic between proofs and refutations (Lakatos 1974) (Balacheff 1988a, 1988b). In his dissertation Balacheff explains that the experimental part of his research was made possible thanks to his relations with the team in the IREM of Lyon who was developing the *Innovation probleme ouvert* (Open problem innovation) (Arsac, Germain, Mante, Pichod & Tisseront, 1985). This innovation was mainly a practice (Mante & Arsac 2007) that aimed to engage students in a position analogous to the position of a researcher in mathematics, which is consonant with Rózsa Péter's view (Péter, 1961). A hypothesis was that such situations were likely to foster proof and proving skills. The same authors developed later situations for initiation to deductive reasoning for grades 6 and 7 (Arsac & Mante 1997).

The "*open problem innovation"* is still alive and nourishes many teachers' training programs in France. It has also known new developments by considering not only the proof and proving skills, but also the mathematical objects, properties and relations that are involved during the research process, leading in Lyon to the development of the research group DREAM (Démarches de Recherche pour l'Enseignement et l'Apprentissage des Mathématiques, Research Approach for the Learning and Teaching of Mathematics) publishing on line research problems with their a priori analysis, and description and a posteriori analysis of experimental settings[3]. The project relies on three main issues. 1/The observation that many teachers did not see the mathematical value of research problems, considering that they were time consuming while they had a lot of mathematical concepts to teach. 2/ There were research-based evidence that logical and mathematical issues were closely intertwined in proof and proving (Durand-Guerrier & Arsac 2005). 3/ Research problems are likely to develop the experimental dimension of mathematics in the teaching and learning of mathematics (e.g. Dias et Durand-Guerrier, 2005). The first stage of the project lead to the production of the resource EXPRIME (Aldon & al. 2010) presenting and analysing didactical situations (in the sense of Brousseau, 1997) elaborated around research problem in mathematics such as: Egyptian Fractions - Trapezoidal numbers – The number of zeroes of n! – The greatest product of 5 numbers – Pólya urns. The research group DREAM has been developing for a few years a more ambitious program consisting in the proposal of an annual organisation of the teaching and learning based on problems, either research problems, open problems, problems for the introduction of a concept, or problem for the deepening of the learning of concepts (Front & Gardes 2015). A first experimentation was conducted in two classes of grade 9 in 2016-2017 by a member of the group. It is described in French on the website with a detailed analysis of the problems.

[3] https://dreamaths.univ-lyon1.fr/

Research situations for classroom based on discrete mathematics

In a paper of 1998, Denise Grenier and Charles Payan argue the relevance of discrete mathematics as scholar knowledge for the development of proof, proving and modelling skills in mathematics education. The context of the research is the French educational system in which the approach of proof is mainly developed in geometry, with emphasise on syntactic aspects, and modelling is absent. As a matter of fact, although the official instructions insist on the importance of a progressive development and recommend to avoid any premature requirement of formalization, their analyse of textbooks from 1979 to 1993 shows the lack of activity around issues of truth, formulation and study of conjecture and modelling (Grenier & Payan, 1998, pp. 76-81). In the fourth section of the paper, the authors give several examples highlighting the potential of discrete mathematics as an alternative approach to proof. They consider that the issue of truth is enhanced by 1) the nature of objects; 2) the proximity with lived knowledge and the possibility to propose students rich problems that are still open in the research field. They illustrate on examples the role of modelling in proof and proving using tools such as pigeonhole principle or graphs. They point out some mode of reasoning such as: decomposition/re-composition; induction; structuration of objects. Finally they propose the tiling situation evocated in the first section of this paper as a fundamental situation for developing the targeted skills. In the abstract in English of the 1998 paper they wrote:

"The paper presents bases of a research aimed to the long run toward an ambient environment for discrete mathematics, a construction which would not only facilitate the first acquaintance to this mathematical field, but would also provide an alternative approach to some transversal concepts, such as proof and modelling. The study which has been achieved here only involves an a priori analysis, prior to any attempt of elaboration of didactical engineering." (Grenier & Payan 1998, p.59). The research program has been developed since this period with many experimental settings and several PhD, leading to a model for research situations for class (SiRC) (Grenier 2013) and to the Federation of Research "Maths à Modeler"[4].

[4] http://mathsamodeler.ujf-grenoble.fr/

1. A SiRC is similar to an actual question in mathematical research or in a non « didactified » one. This condition, very restrictive a priori, aims to avoid that the question or the answer may seem obvious or familiar. The objective is to give relevance to the research activity. This condition can be artificially recreated by the « staging » of the problem.
2. The initial question is easy to understand at various levels of knowledge. Our intention is to break with the usual didactic practice that tends to attribute any problem to a specific grade level. To fulfil this requirement, the statements must in general be not as heavily mathematized. However, we try to avoid random real life « noises », which complicate the task of students in non-mathematical « concrete » problems, and sometimes prevent them from entering into actual mathematics.
3. Strategies to start with are available, but they won't solve the problem completely – usual techniques or properties are not sufficient. In other words, one must ensure the devolution of the problem, by leaving space to some uncertainty that cannot be reduced just by applying known techniques or usual properties (i.e., what Brousseau described in his theory as a « good » situation). The theoretical framework of resolution is neither given nor obvious.
4. One can use several strategies, such as «trials and errors », study particular cases, etc; relevant conjectures are not obviously true, counter-examples are attainable. These points are meant to encourage the construction of conjectures by students, based on an exploration of the question investigated. These conjectures can be examined by the students, through accessible examples and counterexamples.
5. Hypotheses, or the initial question, can be changed. One can change the assumptions of the original question, and grab a new problem. The initial question can lead to related questions: closing a problem through the choice of certain parameter values, or starting a new research activity.

Table 3. Characterization of the model "SIRC" (Grenier, 2013. pp. 177-178)

Beyond the development of heuristic skills, this model has been proved to offer the possibility of enhancing logical competencies linked with implication, quantification, and various mode of reasoning, including generalisation. It provides large autonomy to students and offer the opportunity of creation of new problems by changing some variables during the exploration phase, or for generalisation. For example, in the tiling situation : *the form of the grid to tiled and its area (number of elementary squares)–the form of the* *piece to tile with (domino, trimino, pentamino, other polymino) – the position of holes if any etc.*

Algorithmic thinking at the interface Mathematics – Informatics

Algorithmic thinking has a long story, but it is quite clear that the development of computer science has renewed the methods and challenges of its study in a didactic perspective. In this respect, Modeste (2012) proposed a model of advanced algorithmic thinking from an epistemological point of view, based in particular on Knuth (1996) and Chabert (1999). This model enables study how algorithms are transposed in different institutions (curricula of mathematics, curricula of computer science, textbooks...). One fundamental element of this model is the notion of problem. It links many other aspects of algorithms such as proof, effectivity, or complexity. It is also very important in the theoretical models of algorithms (to study decidability questions, for example). The definition of problem is adapted from the theory of algorithmic complexity.

"A problem (e.g. finding the gcd) is: I a set of instances (e.g. N^2, all the pairs of two integers); Q a question about these instances (e.g. what is the gcd of the 2 integers?). This definition is perfectly suitable for algorithms: an algorithm is a systematic method that must give an answer to a question, for all instances of the problem, and after a finite number of steps (e.g. Euclid's algorithm solves the problem of gcd for any couple of integers). (…). This model is also useful to characterise problems with potential for the learning of algorithmic thinking, in order to design and study teaching situations. (Modeste 2013, pp. 266-267)

We will illustrate this by the example of "Exponentiation by squaring"[5]

A classic algorithmic problem is that of computing for any natural n the n-th power a^n of a real number a. A naïve solution is, starting with value 1, to multiply n times this value by a. The final value one obtains is indeed the expected result, which is not very difficult to establish. The fact that this algorithm terminates is also trivially true since it contains a single bounded repetition. Finally, the complexity of this computation is asymptotically bounded above and below by n, counting for instance the number of multiplications performed, and assuming that multiplication by a is an elementary operation.

[5] We retrieve this example from Ouvrier-Buffet & al. (2018).

A more efficient technique relies on the observation that $a^n=(a^2)^{n/2}$ if n is even, otherwise $a^n = a\ .\ (a^2)^{(n-1)/2}$. This allow to write the recursive algorithm below • If $n = 0$, power (a, 0) = 1 • If n is congruent to 0 modulo 2, return power (a^2, n/2) • Else return a . power (a^2, (n-1)/2)

Table 4. An efficient algorithm for exponentiation by squaring (Ouvrier-Buffet, Meyer, Modeste 2018, p.261)

This example is used in Ouvrier-Buffet & al. 2018 (pp.261-262) to address some main issues in algorithmic thinking, linked to the general question about a given algorithm “Does it works?”. Given a problem P and an algorithm A supposed to solve the problem P and P(a) an instance of the problem, the following aspects are crucial: 1/ Termination: on any instance of P, A performs at most a finite number of computation steps. In the example above, this can be establish by a proof by reduction ad absurdum relying on the property that the relation *less than* is a well ordering on the set of natural numbers. 2/ Partial correctness: on any instance P(a) of P, the value computed by A is P(a). In the example above, this is established by a proof by complete induction. The conjunction of both properties might be seen as expressing correctness. 3/ Complexity - worst-case upper bound: a function f is a worst-case upper bound for the complexity of A if there exists *a* positive constant c such that, on any instance of size n of P, the number of computation steps performed by A is at most $c\ .\ f\ (n)$ for n large enough. In the example above, considering the operation of dividing a number by two and rounding down and choosing the binary representation of integers for which dividing a number by two and rounding down correspond to erasing its rightmost digit allow to prove that the total number of multiplications performed by the algorithm power (a, n) is asymptotically bounded above by $\log_2(n)$, against n with the naïve solution.

Reasoning about programs or algorithms, in particular about their termination, correctness and complexity, draws upon a rather wide assortment of proof techniques and mathematical concepts, including notions usually attributed to discrete mathematics. In this respect, some knowledge of formal logic is of course useful. The notion of worst-case complexity is a relevant illustration of this claim: it involves several objects and concepts, and a non-trivial use of quantification. It is however a very important and classical notion which is a part of many computer science courses. (Ouvrier-Buffet & al. 2018).

Conclusion

In this paper, we argue that logic and problem solving have long been considered as essential aspects of mathematics and computer science education, from Tamás Varga paper in 1972 to now-a-day researches linked to the re-introduction of computer science in French curricula. Discrete mathematics in general, and algorithms in particular, is a field at the interface between mathematics and computer science and a relevant domain for developing proof and proving skills, including logical proficiency. The importance of computer science in a digital area needs the development of research aiming to better understand the similarity and the difference of the epistemological foundation of both domains in order to support the development of didactical engineering. This is developed in the research project DEMaIn (Didactic and Epistemology of Interactions between Mathematics and Informatics) led by Simon Modeste in Montpellier (France). Logical issues in a semantic perspective linked with proof and proving play an important role in this respect (Durand-Guerrier & al. 2019).

References

Aldon, G., Cahuet, P., Durand-Guerrier, V., Front, M., Krieger, D., Mizony, M., & Tardy, C. (2010). *EXpérimenter des PRoblèmes Innovants en Mathématiques à l'Ecole*. Cérédom INRP, IREM de Lyon.

Arsac, G., Germain G., Mante M., Pichod D., & Tisseron C. (1985). *Varions notre enseignement avec les problèmes ouverts. Chercher, essayer, conjecturer, découvrir, prouver. Bande Dessinée*. IREM de Lyon, Villeurbanne (France).

Arsac G., & Mante M. (1997). Situations d'initiation au raisonnement déductif. *Educational Studies in Mathematics* 33, 21-43.

Balacheff, N. (1988a). *Une étude des processus de preuve en mathématique chez des élèves de collège* (Doctoral dissertation, Institut National Polytechnique de Grenoble-INPG; Université Joseph-Fourier-Grenoble I).

Balacheff, N. (1988b). *A study of students' proving processes at the junior high school level.* Second UCSMP international conference on mathematics education, 1988, Chicago, United States.

Barrier, T., Durand-Guerrier, V., & Mesnil, Z. (2019). *L'analyse logique comme outil pour les études didactiques en mathématiques*. Éducation & Didactique, Presses Universitaires de Rennes.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Kluwer Academic Publishers.

Chabert, JL. (1999). *A history of algorithms - From the Pebble to the Microchip*. Springer.

Dias, T., & Durand-Guerrier, V. (2005), *Expérimenter pour apprendre en mathématiques*. Repères IREM, 60, 61/78.

Durand-Guerrier, V. (2016). Initier les étudiants à la distinction entre vérité dans une interprétation et validité logique en s'appuyant sur la théorie du syllogisme formel d'Aristote. *Electronic proceedings of History and Pedagog of Mathematics*, July 2016, Montpellier, France.

Durand-Guerrier V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, 60/2, 149-172.

Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof. *ZDM The International Journal on Mathematics Education*, 40/3, 373-384.

Durand-Guerrier, V., Boero, P., Douek, N., Epp, S., & Tanguay, D. (2012). Examining the Role of Logic in Teaching Proof. In G. Hanna & M. de Villiers (éds), *ICMI Study 19 Book: Proof and Proving in Mathematics Education*, chap. 16, pp. 369-389. Springer, New-York.

Durand-Guerrier, V., Meyer, A., & Modeste, S. (2019). Didactical issues at the interface of mathematics and computer science. In G. Hanna, D. Reid, & M. de Villiers (Eds.) *Proof Technology in Mathematics Research and Teaching*, pp. 115-138. Springer.

Front, M., & Gardes, M.-L. (2015). Un projet d'enseignement fondé sur les situations de recherche. In Aldon, G. (Ed.) *Actes de la 66ème CIEAEM Mathématiques et réalités*. 21-25 Juillet 2014 Lyon (132-138).

Grenier, D., & Payan, C. (1998). Spécificités de la preuve et de la modélisation en mathématiques discrètes. *Recherches en Didactique des Mathématiques*, 18/1. 59-100.

Grenier, D. (2013). Research Situations to Learn Logic and Various Types of Mathematical Reasonings and Proofs. *Proceedings of the eighth congress of the European Society for Research in Mathematics Education*. Middle East Technical University, Ankara, Turkey.

Gosztonyi, K. (2015). *Traditions et réformes de l'enseignement des mathématiques à l'époque des « mathématiques modernes »: le cas de la Hongrie et de la France*. PhD Dissertation, University Paris Diderot Sorbonne Université Nouvelle & University of Szeged.

Gosztonyi, K. (2016). Playing with infinity of Rózsa Péter. Problem series in a Hungarian tradition of mathematics education. *Electronic proceedings of History and Pedagogy of Mathematics*, July 2016, Montpellier, France.

Knuth, D.E. (1996). *Selected Papers on Computer Science*. Center for the Study of Language and Information - Lecture Notes.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

Lakatos, I. (1984). *Preuves et Réfutations: essai sur la logique de la découverte mathématique*. Traduction Française de N. Balacheff et J.M. Laborde. Paris: Hermann.

Largeault, J. (1972). *Logique mathématique, textes*. Paris: Armand Colin.

Lukasiewicz, J. (1972). *La syllogistique d'Aristote* (translated by F. Zaslaswski). Paris: Armand Colin.

Mante, M., & Arsac, G. (2007). *Les pratiques du problème ouvert*. Canopé-CRDP de Lyon

Modeste, S (2012). *Enseigner l'algorithme pour quoi? Quelles nouvelles questions pour les mathématiques? Quels apports pour l'apprentissage de la preuve?* PhD dissertation, University of Grenoble

Modeste, S (2013). Modelling algorithmic thinking: the fundamental notion of problem. In: *Electronic proceedings of CERME 8*, Antalya (Turkey), pp.267-268

Ouvrier-Buffet, C., Meyer, A., Modeste, S. (2018). Discrete mathematics at university level. Interfacing mathematics, computer science and arithmetic. *Electronic proceedings of INDRUM 2018*, University of Agder, April 2018, Kristiansand, Norway. pp.255-264

Péter, R. (1951). *Rekursive Funktionen*. Budapest: Akadémiai Kiadó. English edition: New York: Academic Press. 1967.

Péter, R. (1961). *Playing with infinity* (trad. by Z. P. Dienes). New York: Dover Publications. (Originally published in 1944 as Játék a végtelennel. Budapest: Dante Könyvkiadó.)

Péter, R. (1976). *Rekursive Funktionen in der Komputertheorie*. Budapest: Akadémiai Kiadó.

Máté, A., Andréka, H., & Németi, I. (2012). *The Development of Symbolic Logic in Hungary*. Published online: https://old.renyi.hu/pub/algebraic-logic/MateAN12.doc

Morris, C.W. (1938). Foundations of the theory of signs. In: *International encyclopedia of unified science*, Chicago University Press, pp 1–59.

Quine, W. V. O. (1950). *Methods of logic*. New York: Holy, Rinehart & Winston.

Sinaceur, H. (1991). *Corps et Modèles*. Paris: Vrin.

Sinaceur, H. (2001). Alfred Tarski, semantic shift, heuristic shift in metamathematics. *Synthese*, 126, 49–65.

Tarski, A. (1933). The concept of truth in the language of deductive sciences. English translation in Tarski (1983), pp. 152–278.

Tarski A (1943). The semantic conception of truth and the foundations of semantics. *Philosophy and Phenomenological Research* 4(3): 341–376.

Tarski, A. (1936). *Introduction to logic and to the methodology of deductive sciences* (4th edition, 1994). New York: Oxford University Press.

Tarski A (1954). Contributions to the theory of models, I–II. *Indagationes Mathematicae* 16:572–588.

Tarski A (1955). Contributions to the theory of models, III. *Indagationes Mathematicae* 17:56–64.

Tarski A (1983). *Logic, Semantics, Metamathematics*: papers from 1923 to 1938. tr. J. H. Woodger. Introduction: J. Corcoran. Hackett Publishing Company.

Varga, T. (1972). Logic and probability in the lower grades. *Educational Studies in Mathematics,* 4, 346-357.

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