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Prime building blocks in the mathematics classroom

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Abstract. This theoretical paper is devoted to the presentation of the manifold opportunities in using a little-known but powerful mathematical manipulative, the so-called prime building blocks, originally invented by two close followers of Tamás Varga, to support discovery of various concepts in arithmetic in middle school, including the Fundamental Theorem of Arithmetic or as it is widely taught, prime factorization. The study focuses on a teaching proposal to show how students can learn about greatest common divisor (GCD) and least common multiple (LCM) with understanding, and meanwhile addresses internal connections and levels of abstractness within elementary number theory. The mathematical and methodological background to understanding different aspects of the concept prime property are discussed and the benefits of using prime building blocks to scaffold students' discovery are highlighted. Although the proposal was designed to be suitable for Hungarian sixth graders, mathematical context and indications for the use of the manipulative in both primary and high school are given.

Key words and phrases: number theory, concept formation, manipulative, IBE, Varga method.

MSC Subject Classification: F60, C30, E40, U60.

Number theory in school mathematics

Researchers agree that prime numbers, prime factorization, greatest common divisor (GCD) and least common multiple (LCM) are unavoidable in number theory. However, in many countries, partly including Hungary, these are procedurally taught, in other countries these are marginally taught or simply not taught in compulsory education. In the following I attempt to find a possible reason to this discrepancy and offer a feasible resolution by introducing these topics in an inquiry-based way.

"A prime number is a whole number greater than 1 whose only factors are 1 and itself." School definitions of a prime number are similar to this in math books worldwide. Slight differences a little closer to the general definition with "other than" instead of "greater than" also occur. In some countries including Hungary the most used definition is "A prime number is a whole number with *exactly* two factors, one and itself." [1] Similarly to the "other than" definition, it does not use orders in the definition, moreover, the exclusion of the unit is more integrated into the definition. Yet, all three are simplified variations of the *irreducible property* in rings.

Several very good ways have been reported to help students discover primes, for example in Varga (1969) and Zazkis et al. (2009), all of them leading to the definition based on the irreducible property.

In number theory another property, the so-called *prime property* (also known as Euclid's lemma) is defined as well: if a prime p divides a product ab of two integers a and b, then p must divide at least one of those integers a and b. *This* is the key in the proof of the Fundamental Theorem of Arithmetic (FTA), also known as unique-prime-factorization theorem. Then why do educators choose to teach irreducible property to find primes? Because prime property is out of reach for school-age students. To decide whether a number is a prime or not, a student would have to examine *every* product ab of two integers a and b, check if p divides the product and if so, if it divides at least a or b. It is obviously impossible. If educators wanted students to work with primes, they had to rely on the irreducible property in the school definition. The FTA, however, *does not follow* from solely the irreducible property. This is the gap educators in different countries tried to bridge in different ways – or they just eliminated number theory from their education.

An example to the latter case is the Common Core State Standards for Mathematics (CCSSM) of the USA. Schmidt and Houang (2012) report that primes and factorization (number theory) was not consistent on average across the states in coverage even before the CCSSM. One of the chief writers of the CCSSM, number theorist William McCallum on an internet forum for teachers explained, that greatest common factors and least common multiples are treated with a “very light touch” in the standards, while prime factorization is not a topic in the CCSSM. The attribute he attached to using prime factorization to find the least common multiple (LCM) of numbers in school education was “mysterious”. [2]

Does McCallum’s description fit the process of teaching LCM in school mathematics by prime factorization well? How is prime factorization taught and what is it used for at school in countries which kept number theory in their curriculum?

In Hungary, a procedural way of prime decomposition has long been in practice. Traces of this tradition can still be seen in one of the most widely spread mathematics coursebooks. If one looks up prime factorization a similar process is described on the internet. “Start by dividing the number by the first prime number 2 and continue dividing by 2 until you get a decimal or remainder. Then divide by 3, 5, 7, etc. until the only numbers left are prime numbers.” [3] Instead of reasoning and problem solving, students are directed to carry out procedures, which is likely to lead to rote learning according to Jäder et al. (2019). There would be some minor differences in the process in Hungary, for example, divisibility rules are taught, therefore division would only be carried out if the number was actually divisible by the given prime.

Brown, Thomas and Tolias (2002) show that there are three main strategies to find LCM: listing and choosing, creating a multiple and dividing, and by the “higher exponent rule” (including prime factorization). While the first two strategies are more or less straightforward, problems arose when preservice teachers were asked to explain why and how the third method worked.

Brown (2002) discusses how LCM is generally taught. The reference “higher exponent rule” and the verbalization of the article both show that *explanation* is at stake. An excerpt from a similar *explanation* is recorded in Wagner and Herbel-Eisenmann (2014) and Tatsis and Wagner (2016).

Country	FTA	Factorization	GCD	LCM
Australia	Not stated	Procedural (outcome of a factor tree)	Procedural	Procedural
England	Stated as a fact	Procedural	Listing and choosing	Listing and choosing
Germany (Hessen)[1]	Not in the curriculum	Through some examples	Systematic trying	Systematic trying
South Africa	Not in the curriculum	Factor pairs and then procedural	Grade 7 by inspection Gr. 9 by factorization Gr. 8 either	Grade 7 by inspection Gr. 9 by factorization Gr. 8 either
Sweden	Stated as a fact	Procedural (outcome of a factor tree)	Not in compulsory education[2]	Not in compulsory education

Table 1. Number theory in some example states

Table 1 shows by some examples that in countries with number theory in their curriculum the trend is to state the FTA as a *fact*, according to the curriculum, current coursebook(s) and personal reference. [4, 5, 6, 7, 8] Teaching of GCD and LCM varies. Zazkis and Campbell (1996) believe that some pedagogical alternative is needed to compensate for the lack of proof of the FTA. Although they consider the proper understanding of prime factorization to be indispensable in the proper understanding of the structure of whole numbers, they found that after being convinced by deductive arguments, the subject of their study needed more empirical verification. FTA, especially the uniqueness of the factorization has not been sufficiently grasped by preservice teacher participants of the study.

1 There’s some variance among states

2 It is included in teacher training

An inquiry-based teaching proposal

In 2000 a new series of Hungarian mathematics books emerged aiming to encourage inquiry-based education (IBE, in particular in mathematics education IBME) on the basis of the new national curriculum of Hungary [9], along with the European trend. Artigue and Blomhøj (2013) give an excellent survey about IBE and point out the importance of education being for all, stimulating students' interest for learning, cultivating students' autonomy, preparing students for an active role in society, and opposing teaching practices based on instruction and drill. The authors also give a brief overview of how all these ideas have roots in earlier texts and realizations.

Hungary's acknowledged innovator in mathematics teaching Tamás Varga worked out his methodology in the complex mathematics teaching experiment from 1963 based on such similar grounds that it could well be introduced as another branch of IBL in Artigue and Blomhøj's article. Its main characteristics include respect for the autonomy, zest and activity of both teachers and students, creating a positive attitude towards thinking and cognitive problems yet unsolved (it focuses on thinking rather than knowledge), considering age characteristics and individual differences of children, and regarding the capacity of the individual to form abstract ideas an ability that builds on concrete experience and that can be developed – as Klein summarizes the method in the foreword to the Mathematical Lexicon by Varga (2001). An important distinctive feature compared to the other branches of IBE, is the special emphasis placed on the autonomy of teachers and cultivating their interest in teaching well which creates ground for teachers' original ideas in the spirit of Varga up to our days.

All these values shared by Varga's innovative mathematics teaching methodology and IBE play an important role in the proposed implementation of the number theory chapter of the Hungarian mathematics books mentioned above, in which the authors introduced a new way of teaching prime factorization with the verbalised purpose of "interpreting prime numbers as the building blocks of numbers". [10] The idea, originally worked out by Éva Szeredi and Csongorné Kovács, two former co-workers of Varga, was to lead students in their discovery about the FTA using appropriate manipulative.

The manipulative is called *prime building blocks*. These are primes written on 3x4,5 cm cardboard pieces, ten of each single-digit and six of each double-digit prime under 20 for each student. The teacher needs plenty of at least business-card-size pieces of primes under 20 magnetized on both sides.

In the following the rough process of inquiry of the FTA is presented in ten steps. The process is tested and is suitable for at-least-11-year-old students the findings of which are to be elaborated in a future separate paper.

Step 1. Students prepare their prime building blocks based on the worldwide-used definition and process called Eratosthenes' sieving method described by Freudenthal (2012), for instance. Students discover primes under 100.

Step 2. Students are encouraged to build different numbers ("houses") from their building blocks using multiplication as mortar.

Step 3. "Number-houses" are gathered and listed on the board.

Step 4. Students realize that it is worth listing a number built from the same blocks only once. They agree that a number built from the same blocks will be the same regardless of the order of the blocks. If students do not come up with the same number twice on their own, the teacher claims to have one, too, and repeats one of the numbers already listed with a different order of the primes. Students often agree that prime building blocks should be listed in increasing order to help spot repetitions.

Step 5. Existence below 50: Students are encouraged to think whether every number below 50 could be built from such blocks had they got all the primes in their sieve. They are encouraged to find one that cannot be built. If nobody "finds" one, students can challenge each other to show how a number less than 50 of their choice can be built of primes.

Step 6. Students are asked to share their strategy for building a number. The most common strategies use divisibility rules or factor pairs to find possible candidates.

Step 7. Existence in general: Students are asked to guess if all numbers can be built of primes among greater numbers, too. The question remains open at this point. Experience shows that students now have a strong conviction that all numbers can be built. Students can be challenged to try to find one that cannot.

Step 8. Uniqueness below 50: Students (in pairs or groups) are challenged to choose a number from the board and try to build it differently from prime blocks. "Solutions" are discussed as a class.

Step 9. Uniqueness in general: Students are asked if they think uniqueness is only true for numbers up to 50. The question remains open at this point, but experience shows that students to have a strong conviction that it is true in general.

Step 10. Students are prodded to summarize what they have found true at least for numbers below 50. Deficient verbalizations are challenged.

Discussion – in the light of Varga's complex mathematics

The originality of the idea of prime building blocks lies in the fact that it approaches the structure from the opposite direction to prime decomposition. In teaching factor pairs or more formalized, systematic decomposition techniques, whole numbers are *de*composed, while here whole numbers are *composed*. This shift of view creates the space for inquiry, where as Dewey (1938) describes it, the unknown can be approached with what is already known. Due to the simplicity of the first steps and their obvious use in learning about multiplication, the process of inquiry can be spread over a long time, beginning even from lower primary, and so allowing number theoretical concepts to ripen as Varga (1969) recommends at page 51. He argues that communicating these on specific weeks of a schoolyear, even in a form of lead discovery, for most students will still be a sort of *statement* that has been verbalized by one or two kids or even by the active cooperation of the entire class. He believes that arriving at the verbalization phase within the very lesson when experimenting begins is risky.

In the inquiry about FTA students are encouraged to think about and debate whether the general cases could be true or not, but the teacher does not state any part of the theorem as a fact. As in IBE in general, students' natural curiosity is triggered, which according to Gosztonyi (2016) is completely in line with Varga's aims. Students are encouraged to try to find a counter-example for the forming theorem. They are not expected to memorize anything they have not experienced and verbalised for themselves in the first place. Instead of rote learning, students learn with understanding, which is one of the key factors it shares with IBE.

Students are presented with the opportunity to play and discover about school mathematics that is in accordance with abstract mathematics. "Mathematics has the qualities of appealing to pupils – especially if it appeals to their teachers – and this is, we feel, an immense reserve not sufficiently exploited." (Halmos & Varga, 1978)

The role and responsibility of the teacher should be underlined in the process as discussed by Gosztonyi (2020) in this volume. The teacher launches and shepherds conversation, but goes through the discovery process with the students pretending not to know any more than the students to encourage their autonomy. The teacher should not interfere unnecessarily, not even if the students make a mistake. Instead, students are encouraged to challenge their own and their peers' ideas to prove them true or false, a habit which prepares them to play an active role in society. Making a mistake is allowed and sometimes encouraged. Errors are not considered as a fault, but as an opportunity for the whole class to improve. The teacher facilitates collection and systematization of the findings of the class. The teacher is responsible and has the freedom to choose the work forms (individual, group, frontal, mixed) most appropriate for the teaching situation.

Differentiation is carried out by beginning the inquiry process from a level where all students “feel home” - along with IBE’s idea about education being for all. Here it means building numbers from given numbers (small primes) by multiplication and collecting solutions. Students can work with different pace. Students have a choice at several points, which also naturally builds their autonomy. For example, students can choose the size and number of primes they want to multiply to build a whole number (within the range designed to fit the long-term purposes of the class).

Although not for concept-building but concept-reinforcing, Kurz and Garcia (2012) reported the use of the manipulative in this specific case to add to the understanding of the mathematical structure.

Building on: divisors of a number

After finding out about the FTA, students are given *a fix set of blocks* to build numbers. Findings are listed, ordered and examined. After a few examples, students realize that *all* the divisors of a number can be built of its prime blocks and a number that is not a divisor cannot be built. A prime that is not a factor of the original number is not a factor of any of the divisors and vice versa.

This observation is indispensable for finding the GCD and LCM. For example to find the LCM of 9 and 15, students first build the two numbers. Nine is a divisor of any of its multiples, so a multiple must contain all of 9’s building blocks: 3 and 3. Fifteen is a divisor of any of its multiples, so a multiple must contain all of 15’s building blocks: 3 and 5. To find the *least* common multiple, any unnecessary block has to be eliminated. In this case out of 3, 3, 3, 5, one of the threes can be eliminated and the number is still divisible by 9 and 15. More blocks cannot be eliminated. This way the building blocks of the LCM is 3, 3, 5. The method is a blend of “creating a multiple and dividing” and “with prime factorization” in the classification of Brown et al. (2002). Compensation for the lack of the proof of the FTA as suggested by Zazkis and Campbell (1996) is carried out by gently lead experimenting and inquiry.

Further development of the manipulative

In order to shift students' focus from concrete primes to the abstract general idea, primes on the building blocks can be replaced by different colours (an idea of the author's own). This way the manipulative can be used to inquire about problems where the structure is important rather than the concrete prime. In Figure 1 Two example questions are given.

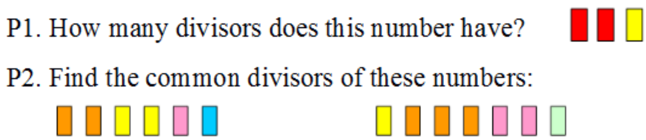

Figure 1. Two questions to be solved by the altered version of the manipulative

P1 is in strong relation with combinatorial games introduced by Varga (1982). The alteration of the manipulative is designed to scaffold discovery in a hands-on way so that students realize that in both problems only the number of each type of prime (colour) matters.

Conclusion

According to Freudenthal (2006), the search for primes and the research on them are largely ones of mathematical content. However, there has long been a trend of teaching basic number theory, including the Fundamental Theorem of Arithmetic (FTA), as a number of processes worldwide, due to the fact that the level of abstractness of prime property surpasses that of schoolchildren's comprehension. Although Maaß and Doorman (2013) report that inquiry-based learning has been increasingly promoted recently, teaching number theory seems to have remained intact in many countries.

On the grounds of Tamás Varga's complex mathematics teaching method, in 2000 a tangible mathematical manipulative was introduced to middle schools in Hungary with the potential to compensate for the lack of proof of the theorem (FTA) and make it accessible for inquiry-based learning. Sfard (1991) argues that tangible mathematical tools promote the construction of corresponding mental objects, consequently, representation helps students in their mental constructions. An appropriate modification of the manipulative is also presented that can serve as a tool for generalisation and abstraction.

Appendix of curricula and coursebooks

[1] AssessOn Maths Quest 7 for the Australian Curriculum 2E (Robert Rowland), John Wiley and Sons

[2] Common Core internet forum https://mathematicalmusings.org/forums/topic/prime-factorization/

[3] Prime factorization method https://www.mesacc.edu/~scotz47781/mat120/notes/radicals/simplify/images/examples/prime_factorization.html

[4] Curriculum of Australia https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

[5] Curriculum of England https://www.gov.uk/national-curriculum

[6] Curriculum of Hessen (Germany) https://kultusministerium.hessen.de/schulsystem/bildungsstandards-kerncurricula-und-lehrplaene/lehrplaene/gymnasium-9

[7] Curriculum of South Africa https://www.education.gov.za/Portals/0/CD/National%20Curriculum%20Statements%20and%20Vocational/CAPS%20SP%20%20MATHEMATICS%20GR%207-9.pdf?ver=2015-01-27-160141-373

[8] Curriculum of Sweden https://www.udir.no/kl06/MAT104?lplang=http://data.udir.no/kl06/eng

[9] Curriculum of Hungary (Core) http://kerettanterv.ofi.hu/02_melleklet_58/index_alt_isk_felso.html

[10] Teachers' book for Mathematics 6. http://ofi.hu/sites/default/files/060834_matematika_6_kk_1.pdf

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