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Teaching Mathematics and Computer Science

"How to be well-connected?" An example for instructional process planning with Problem Graphs

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Abstract. Teachers' design capacity at work is in the focus of didactical research worldwide, and fostering this capacity is unarguably a possible turning point in the conveyance of mathematical knowledge. In Hungary, the tradition hallmarked by Tamás Varga is particularly demanding towards teachers as they are supposed to be able to plan their long-term processes very carefully. In this contribution, an extensive teaching material designed in the spirit of this tradition will be presented from the field of Geometry. For exposing its inner structure, a representational tool, the Problem Graph is introduced. The paper aims to demonstrate that this tool has potential for analyzing existing resources, helping teachers to reflect on their own preparatory and classroom work, and supporting the creation of new designs.

Key words and phrases: Hungarian guided discovery approach, teacher design capacity, geometrical concepts, problem posing and solving, problem graph.

MSC Subject Classification: 97D40, 97D50, 97D80, 97G10, 97U30.

Introduction

Despite controversies at the time, the merits of Tamás Varga's reform movement and the guided discovery approach are highly recognized within the Hungarian Mathematics Education community. The basic principles of this approach were elaborated in Varga (1988), Halmos and Varga (1978), and recently revisited by Gosztonyi et al (2018). Five of those principles are of special significance for this paper: *students' autonomy and creativity* – learners rediscover mathematical knowledge through problem solving, *teacher autonomy* – the teacher acts as a facilitator and a guide, *flexibility* – a high degree of responsiveness to students needs and ideas on the spot, *democratic conveyance of knowledge* – great effort to teach 'real' and 'modern' mathematics to a large diversity of students, and *deliberate buildup* – long-term, cross-domain planning processes, concepts being revisited and broadened spirally and gradually.

This paper focuses on the perspective of the teacher. Regarding teachers' design work, Gueudet, Pepin, and Trouche suggested "10 Qs" in their article (2017). We focus on two of them: "What do they design?" and "How do they design?" In the spirit of the guided discovery approach, and also mobilizing the overly problem-oriented Hungarian mathematics education culture and rich problem solving tradition, we have designed a long-term teaching material for regular classrooms. The rationale behind this design is to create a long-term arch of problems that opens up the way towards such basic but complex concepts that are problematic to master for students. Moreover, experience shows that sometimes even teachers and textbooks lack a consistent approach to them.

The design takes the form of a S*eries of Problems*, an instructional object described by Gosztonyi (2018) as indigenous to the Hungarian guided discovery approach. Structurally, it is more flexible than a mere task sequence. To reveal this structural complexity, we created a representational tool, the *Problem Graph*. In the main section of the article, the central elements of the design and the problem graph approach will be discussed with some preliminary results and perspectives on their classroom implementation.

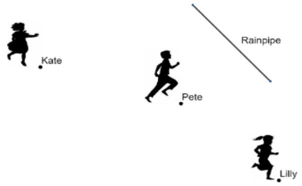
The design

Our design is an extensive problem structure titled 'Distance under the magnifying glass' in the field of Elementary Geometry, a domain that is traditionally discussed in greater depth in Hungarian secondary education than in other European countries. It consists of over 40 individual problems with a number of subtasks and variations in many of them. Regarding the aforementioned question "What do they design?" (Gueudet, Pepin, and Trouche, 2017), this structure is in line with the principles of the Hungarian guided discovery approach for which problem situations are the catalysts of the learning process, and teacher intervention and mediation usually also take the form of new problems and questions. As for the curricular dimensions of this Series of Problems, it overarches 8 years (Grade 5-12), giving a fine example for long-term process planning.

Contentwise, as the title suggests, it revolves around the fundamental, yet sometimes deceptive concept of distance, with a number of auxiliary and derivative concepts like perpendicular and angle bisector, line symmetry, inscribed circle, basic components and incidence in geometrical space, etc. These concepts are well embedded in the regular Hungarian curriculum, but experience shows that our problem situations often provoke a deeper classroom conversation touching on extracurricular topics. By no means we aim for setting these for students as requirements, but we observe that broadening the picture leads to their better grasp of both the curricular content in the narrow sense and the surrounding world. Opening up these conversations also accords with Tamás Varga's principle as mentioned in the Introduction: teaching modern and real Mathematics. Geometrical concepts might be rooted more directly in the common human experience than notions from other mathematical domains, but this relationship often lacks harmony: as stated by Mariotti and Fischbein (1997), the double nature of geometrical concepts can result in persistent conflict between the *figural and conceptual aspects*, the *concept image* and *concept definition*.

Problems on the distance concept

First we introduce a selection of problems that delves directly into the concept of distance.



P1. *Kate, Pete and Lilly are caught in a large spring downpour at the playground. They take refuge under a narrow rainpipe as quickly as possible. Draw (in the figure), which route should they run.*
Modify the length/position of the rainpipe. How does it change the route of the children?

b) What could be the position of the rainpipe, if Pete and Lilly run the same distance? (Required: dynamic tools: figures on the board and/or stickers)

P2. *Use navigation apps (e.g. Google Maps) and determine the distance between Budapest and Szeged/Paris/New York. Compare your results. Why is it possible to get different results? (What do you think we should actually call distance?)*

P10. *Our friend lies injured and dehydrated in the sand of the desert, in point B, and he is unable to get to the wadi (temporary stream) marked on this sketch. We ride our camel from point A to bring water for him from the wadi as fast as possible. Construct the shortest possible route to rescue our friend.*

P11. *In March, Sigmund, the stork flies from Windhoek, Namibia to its nest in Budapest in search of its mate. Calculate the estimated distance it has to travel. You can use only the given offline resources (globe, atlas and household items). Present your calculations and results to the whole class.*

Table 1. Problems on the distance concept

P1 is intended for Grade 5-6, but can be revisited later in Grade 9. The text of the main problem is unsurprising as plenty of similar tasks can be found in textbooks. The new element is the dynamic variation of the factors depending on students' responses and ideas. In Grade 5, we count on students' difficulties not only with distance, but also with the notions of segment and line, and the idea of infinity. The tasks intend to scaffold the development of these notions as well. An interesting aspect of task b) is the opportunity to observe whether students abandon the restrictions originating from the actual problem situation and enter the abstract realm. In this case, the perpendicular bisector position occurs as an alternate solution. Note that if the rainpipe is a segment, alternative solutions to the parallel position can also be found. In this case, by moving the child that should run to the endpoint, students can also learn about the circle.

P2 is also a multilevel problem: first treated in Grade 5 and revisited in Grade 9 and/or Grade 12. The possibility of different results originates either from students' different interpretations of the text or from the different settings of the navigation app: distance as the crow flies or shortest route that one can actually travel, optimization for travelling time or cost, different transportation, less walking, etc. Note that all these interpretations are reasonable and have their abstract equivalent in higher mathematics. This problem, and even more definitively **P11** require the consideration of the basics of spherical geometry, with **P2** leading later to other geometries (taxicab), graph theory, optimization problems, and the naturally occurring question of how navigational systems work, with rich mathematical consequences. **P11** goes seemingly the other way round by banning digital devices and resources. This problem is usually implemented in groupwork. A variation of the task is when – after they observed the globe – the students can prepare three questions (not about the actual distance, of course) that help them solve the assignment.

P10, with a slightly different text, is well known in the Hungarian tradition. Besides the obvious connection to the matter in hand – gradually approaching the concept of distance as a minimal route – what makes it relevant is that a majority of university students and novice teachers report that they "have seen the solution of the problem", not that they were able to solve it themselves. This will be a significant aspect later on, when we investigate the structure of the entire design.

These problems demonstrate our aspiration to react to the observation that the unified, consistent concept of distance often eludes students throughout their 12 years of instruction, as they grasp only special cases and measuring techniques instead. In our design we also take into account that the use of everyday language can be both a catalyzing and hindering factor in this aspect.

The Problem Graph

Reverting to the question "How do they (the teachers) design?", we find that expert teachers who follow the guided discovery method usually don't design separate problems, they consider the impact of different arrangements of tasks very carefully in their instructional process planning. Quoting the preface of Polya's famous *Induction and Analogy in Mathematics* (1954): *"In order to provide (or hide) such clues with the greatest benefit to the instruction of the reader, much care has been expended not only on the contents and the form of the proposed problems, but also on their disposition. In fact, much more time and care went into the arrangement of these problems than an outsider could imagine or would think necessary."* No written form can truly reflect the complexity of this thought process, nor can it be observed entirely in classroom enactment, since the structure inevitably becomes fixed and more or less linear at this point. However, it is a crucial aspect of the design process that equips teachers for sensitive, adequate reactions to students' ideas or difficulties. To reveal this hidden structure, we introduced the *Problem Graph* as a representational tool. In the following the problem graph of the series of problems in question will be displayed (Figure 1), and we discuss the potential of the tool through this example.

Problem graphs in general are the representation of a network of problems as a (topological) graph, where the nodes are assigned to the individual problems, and the edges show the connections among them. These connections can be determined by the analysis of a researcher, and in this case, the Problem Graph serves as an analytical tool, but investigating the links can also be part of the teachers' reflection and/or preparatory design work. The connections can reflect similar mathematical content, similar solution strategies, different aspects of the same concept or other didactical considerations. One-way arrows represent corollary relations, in the sense that the antecedent problem supports the solution of the subsequent one. But, as it is discernable in our graph, the arrows do not fully determine the ordering for classroom implementation. Wanderings on the graph give multiple learning trajectories with slightly different outcomes, and forward leaps open the opportunity for differentiation. The forks in the displayed Problem Graph clearly show that the educator, as a guide, has multiple choices.

In the case of this particular design, we also use letter coding for grouping, and the graph consists of three clearly distinguishable parts. In the center of the graph, **C1-4** mark the Core microseries, the kick-off point and mobilizing factor of the whole design. In the upper part, **P** marks the preparatory problems that were built *towards* the Core, and these problems are designed to obtain or mobilize the mathematical knowledge and practice necessary to enter **C1**. These problems are originally designed for Grade 5-8, but many of them can be revisited later. The preparatory problems have thematic labels:

D: Distance G: (Other) Geometries L: Loci of Points

T: (Geometrical) Transformations

In the lower part, the ramifications *from* the Core are displayed, these are interlinked threads of problems that can be opened up as a consequence of the dialogue over the Core. Each of them cover curricular contents from Grade 9-12, but often culminates in problems that exceed the curricular requirements in their complexity. The ramifications are coded thematically:

S: Space Geometry E: Existence Problems R: Radii of Circles

Co: Combinatorics I: Incidence AnG: Analytical Geometry

A: Angles and Proportionality

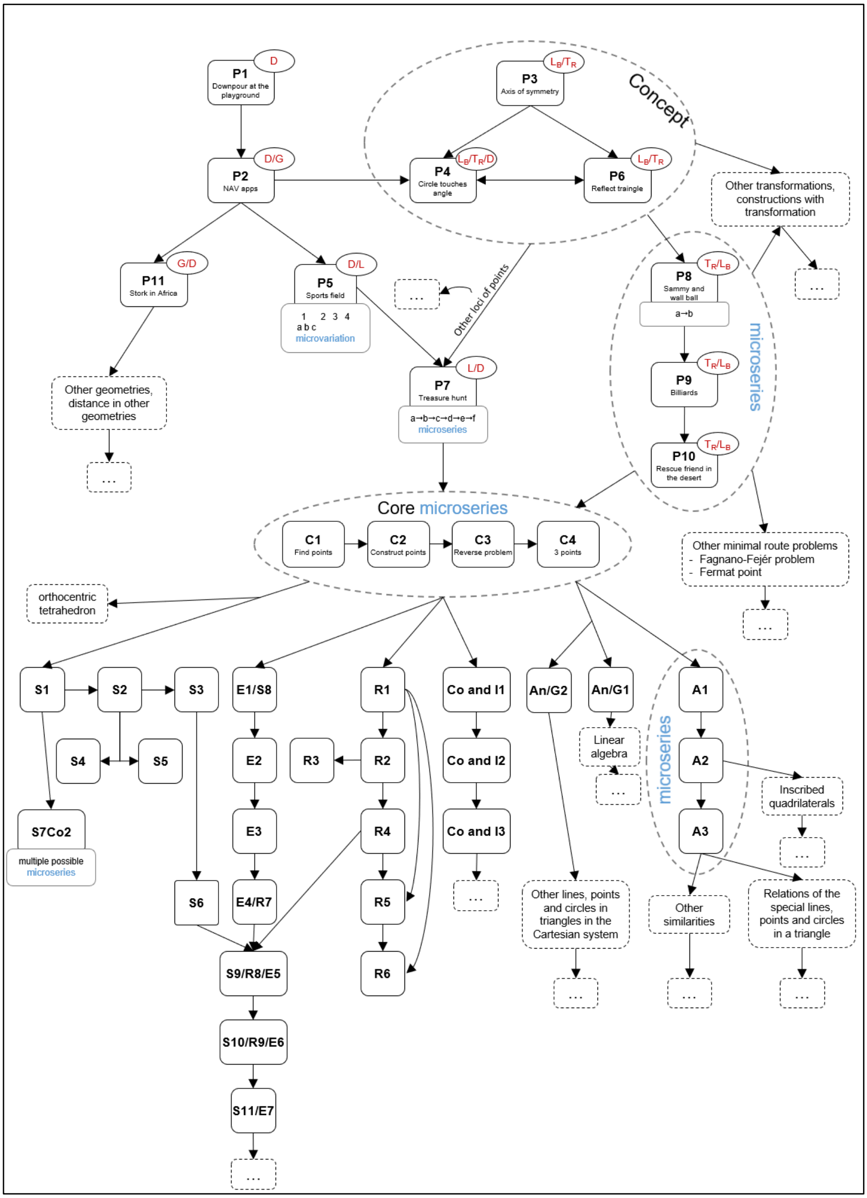


Figure 1. The Problem Graph representation of Series of Problems 'Distance under the Magnifying glass'

The Core Problems

The following four problems are the largest hub of the structure, designed for Grade 9 and intended to be handled in the classroom in close succession:

C1. *Find the equidistant points from three different lines (in the plane).*
C2. *Construct (in Euclidean terms, using only rulers and compasses) the equidistant points from three given lines (on the plane). In case the lines are non-concurrent and mutually non-parallel, rephrase the defining features of the constructed points and lines. What are their relations to the triangle bounded by the given lines?*
C3. P_1, P_2, P_3 *and* P_4 *are four equidistant points from three lines of the plane. Construct these three lines. (The points are given in an illustration, they can't be chosen arbitrarily!)*
C4. *Let* P_1, P_2 *and* P_3 *be three different points that are equidistant from exactly three different lines in the plane. Construct any other points that are equidistant from the same lines. Discuss the number of cases. What does it tell us about the solvability of the previous problem?*

Table 2. The Core Problems

Although **C1** and **C2** might be perceived as roughly equivalent mathematically, from the didactical aspect they differ considerably. **C1** shares its characteristics with other entry point problems of the structure (**P8** or the aforementioned **P1**): it is activating, does not demand extensive and precise knowledge, and it is a good diagnostic problem, meaning that it informs the teacher very well about the students' strengths or difficulties and misconceptions. Note that in **C1**, depending on the level of the class, we consider finding the approximate position of the points a partial solution. The different possible cases add a categorization element to the task, and this leads to the Combinatorics branch in the graph. **C2** necessarily demands the exact mathematical formulation of the solution (the center of the inscribed and ascribed circles derived from the angle bisectors), and leads to further investigation of these circles in branch **R**. **C3** (solvable only if the points are orthocentric), the reverse problem is far more difficult, in fact, together with **C4**, out of reach of a regular classroom as individual problems. When working with this problem structure, the success in solving **C3** mainly depends on the detailed discussion over the constructed figure in **C2**.

Hubs and leaves

The graph structure reveals a number of other hubs, rich, complex problems that require the synthesis of wide-ranging mathematical knowledge, and therefore can be identified as critical points of the instructional process. **P7**, a microseries itself contains treasure hunting tasks similar to certain textbook problems. What sets them apart is the gradual buildup, and, similarly to **P1**, the capitalization on the opportunities provided by dynamic tools and the inaccuracy of everyday language.

In the lower part of the graph, problem **S9/R8/E6** deals with inscribed spheres of solids, strongly relying on the plane-space analogies and mobilizing the knowledge discovered in threads **R** and **E**.

In the leaf elements, as culmination points, we find some problems that are often deemed inaccessible for regular classrooms, although they have no extracurricular content. **C3, C4** and **A3** are on the level of national competitions. Nevertheless, it is possible to approach them through the designed problem structure with motivated students. By showcasing to students the creative power of their own, even lesser successes evoke the spirit of Tamás Varga’s Complex Experiment.

P10 has already been discussed briefly in the second chapter of the main section. This series of problems design embeds it into a culminating thread, with subgroup **P3, 4** and **6** familiarizing the students with both the process and the properties of line symmetry, and **P8** and **9** serving as immediate guiding problems. **P9** is a different problem with the same solution as **P10** (hit the billiard balls). As an alternative to the solution with reflection, students sometimes come up with a solution to **P8** that rather provokes a solution with proportionality. That can lead to a much less elegant, but sufficient solution with calculations for **P9** and **P10**.

Concluding remarks and perspectives

In this article we attempted to demonstrate with some examples how the principles of Tamás Varga’s Complex Mathematics are manifest in the design of a teaching material. This particular design is part of a collection of series of problems with detailed commentary that our research group plans to publish online in 2020. This collection intends to provide generic examples of working with the guided discovery approach for

Hungarian teachers. Regarding the question ‘what the practitioners of the guided discovery method design’, the examples show they primarily prepare problem situations. Dealing with the question ‘how do they design’, we build on Gosztonyi’s research (2018) and demonstrate the tendency to use problem structures rather than individual problems. The particular example detailed in this paper, as a generic example for long-term process planning, shows that this structure can be far more complex than a linear task sequence. Finding and analyzing other extensive examples in the literature, and investigating practicing teachers’ capacity in designing such complex structures are important perspectives of our ongoing research.

In this paper, we propose the problem graph representation for both analyzing and designing problem structures. We find this representational tool promising, and intend to investigate its potential in our ongoing research. Some pilot experiments have already been conducted in this regard. As a first step, the presented material was used in three PD programs with prospective and in-service teachers. In ensuing sessions, teachers used the Problem Graph approach to reflect and develop their own teaching materials. The detailed analysis of the experiments and the refinement of the PD sessions with the associated questionnaires are in progress. The pilot experiments suggest that both quantitative and qualitative differences could be detected between the graphs of expert and novice educators, and the complexity of the recognized relations may improve with the use of the graph approach, but these claims need further investigation. The participants’ self-evaluation on their professional growth were enthusiastic, and that inspires us to move our focus in this direction. As a closing remark, we would like to quote a colleague’s post-session questionnaire: *“...(it improves) the structural reasoning to such a depth…I analyzed both what I want to teach and what can be taught through certain problems, and this was new...”*.

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