18/3 (2020), 101-110
DOI: 10.5485/TMCS.2020.0480
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Teaching Mathematics and Computer Science

A computational thinking problem-thread for grade 7 students and above from the Pósa method

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Abstract. Lajos Pósa has been developing his "learning through discovery" (Győri & Juhász, 2018) method since 1988. His weekend math camps are focused on fostering problem-solving skills and high-level mathematical-thinking skills in gifted students from grades 7 to 11. One of the core aspects of the method is the structure of the problems, all problems are part of a complex, intertwined, and rich network. In this article we analyze a computational thinking problem-thread and its role in the camps's network of problems (Gosztonyi, 2019), and show some aspects of the method. The insights gained using this method can be useful in other contexts. The possible adaptation of the method to secondary and high schools is briefly discussed as well.

Key words and phrases: Pósa method, network of problems, problem posing, talent-nurturing, inquiry-based learning.

MSC Subject Classification: 97D40.

Introduction

Lajos Pósa started to organize math camps in 1988, he wanted to create a special environment where gifted students could enjoy the joy of problem solving through a mix of discovery and guided learning (Győri & Juhász, 2018). The program is focusing on teaching problem-solving skills (Pólya, 1988). The program is for highly gifted children from grade 6 to grade 11. Students from the same grade have two or three Pósa weekend-camps each year. There are two parallel groups for each grade, in one group there are typically 26-28 students. Pósa in these past 30 years has developed a 5-years-long quasi-curriculum. In the end, each group has a carefully constructed curriculum based on the same network of well-tested problems. While the final set of problems is slightly different for each group.

The problems are heavily structured with several connections to other problems creating different problem-threads, and these problem-threads are creating a network of problems. These connections are mostly not topical connections; problems are rather connected because of their similar approach. In the camps reflecting on the nature of mathematics, reflecting on the structures of problems, and connections between them is an essential part of the learning. Being mindful about these connections is not only a tool for the teacher to design activities but also an important and integral part for the students to learn, reflect on and to discuss it with each other. We are going to describe a problem-thread with the help of the newly developed theoretical concepts: networks of problems and series of problems (Gosztonyi, 2019). These two concepts are used in recent research of the Hungarian tradition of teaching mathematics: past and present traditions as well (Katona & Szűcs, 2018).

The origin of present-day Hungarian mathematics education culture can be traced back to the late 19th century. Several mathematicians, philosophers, psychologists, and mathematics educators shaped the way this tradition thinks about the teaching of mathematics (Gosztonyi, 2016). Varga's work and the Pósa method grew from this tradition. There are numerous similarities between the two methods, most notably their inquiry-based teaching methods were inspired by the heuristic approach of the Hungarian tradition. Notably, Varga and Pósa were both influenced by Rózsa Péter's work (a Hungarian mathematician and educator). Although Pósa independently worked out his method, his work was presumably influenced by Varga. From 1982 to 1989 he was a member of the Research Group on Mathematics Education led by János Surányi and Mária Halmos. This research group was influenced by the reform movement of Varga.

The topic of the chosen problem-thread is computational thinking. In the camps, there are several threads related to computational thinking. Discrete mathematics in general has a strong tradition in Hungary (Erdős, Lovász), they are widely used in talent nurturing programs as well (preparing for International Mathematical Olympiad). For the term computational thinking, Alfred V. Aho's definition is used (2012): "We consider computational thinking to be the thought processes involved in formulating problems so their solutions can be represented as computational steps and algorithms." The reason for choosing this problem-thread is to demonstrate several aspects of the method. Problems of the thread are present throughout the camps, they have several connections inside and outside of this thread, the thread is leading to open problems in mathematics, the problems are used as a starting point to encourage students to pose additional problems.

Long-term planning is central in the Pósa method: with the ideas and methods learned from the earlier problems, students are enabled to solve difficult problems years later, thus developing a competent problem-solving skill. Another important aspect of the method is to learn how to pose new problems. Posing new and meaningful problems is important in mathematics (Silver, 1994) or rather in any scientific research. "It is far more difficult to create a problem than to solve it" (Engel, 1987). Students in the Hungarian school system rarely prompted to ask further questions. The problems in this thread were generated from the initial problem by the students. Several ideas for new problems and for developing of existing problem-threads in the program is based on these student-driven, meaningful, yet achievable questions.

In the first part of the article, we introduce the initial problem of the problem-thread with its solution and insights about students' interaction with it. In the second part of the article, we focus on related problems, giving an overview of the structure of this problem-driven thread, showing where students' questions lead to. Finally, we are going to conclude the article with some remarks and insights about the possible use of the Pósa method in secondary schools.

Initial Problem

The following problem entitled *Rotating Safe* is recommended for grade 7 students and above. To solve the first few problems of this problem-thread, students should be able to understand what a proof is (not in a formal way, of course), have some experiences with how to prove something is impossible. So, this problem-thread has a strong connection to the *impossibility* problem-thread (Artigue et al., 2020). The *impossibility* problem-thread starts early, exploring different topics. For example, an early taste of graph-theory: it is possible to draw a map with seven towns, each town connecting to exactly three other towns? Another problem: how to cover a chessboard missing two “opposite” corners with 1x2 dominoes? This problem is also part of the *coloring* thread. In the *coloring* problem-thread, problems can be solved by coloring. In the case of the chessboard problem, one can use the coloring of the chessboard to solve the problem. These seemingly unrelated problems in the *impossibility* problem-thread have a strong connection to each other and to the following problem-thread. Having experiences with the *impossibility* problem-thread enables students to solve more difficult problems from the present thread.

The *Rotating Safe problem* is a low threshold high ceiling problem. There is no technical knowledge needed to solve or to understand the problem. On the other hand, the generalization of the problem leads to an open problem in mathematics.

The *Rotating Safe problem*. There is a safe with treasure inside. The safe has a lock with special mechanics. The lock has a rotating panel with four holes (see Figure 1 for illustration). Inside each hole, there is a binary switch. The task is to set all switches to the same state. We cannot see from the outside what state each switch is in, we can only tell the state of a switch if we put our hand inside the hole. The initial states of the switches are not known. At one step we can choose two holes, put our hands inside and modify (or not modify at all, independently) the state of the switches. After we get our hands out from the holes, the panel starts rotating at a great speed. At one point the panel stops, and we can once again modify two switches etc. If the switches are in the same position, then the safe opens automatically. Can we open the safe, not relying on “luck”? (There is no guarantee that the rotation of the panel works with a uniform distribution or any other kind of distribution)?

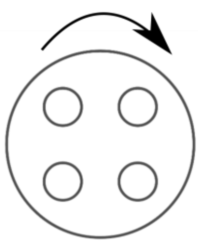

Figure 1. Illustration for the initial Rotating Safe problem. The solution starts here

The safe can be opened. For many students (from children to adults) it is a rather surprising fact. During guiding students, this is a significant hint: "it CAN be opened". This fact alone usually helps students to overcome a mental barrier to solve the problem.

The steps of the solutions are the following (see Figure 2 for a visual aid). First, there are two ways to interact with the panel, a) putting two hands in two opposite holes, labeled as O move, b) putting two hands in neighboring holes, labeled as N move. With the first two steps, using an O move and an N move it can be guaranteed that three switches are in the same states, in our example let us use state 1 (1,1,1,?), independent of the initial state of the switches. This is the point where most students give up, saying: there is no guarantee to be able to put a hand in the "fourth" hole, so if that switch is in a different state, that one cannot achieve to open the safe. Partly, they are right, there is no guarantee to catch the "fourth" hole, but surprisingly it is still possible to open the safe. It can only be done by not wanting to catch the "fourth" hole, instead of focusing on the fact, we are in a better position than in the beginning: we know the safe did not open, so three of the four switches are in the same state (1110). We can take advantage of that, and that way we can solve the problem. For our third step, we use an O move. There are two cases, first, we have found the switch with the different state, we will set into the right position and the safe is open. In the second case, we did not find the switch with the different state, so we modified one of the switches, resulting in two pairs of neighboring switches being in the same state (1100). For the fourth step we use an N move, we modify both switches, resulting in that switches in opposite positions have the same states (0101). Now, we can only have to use an O move to get all switches into the same state (1111).

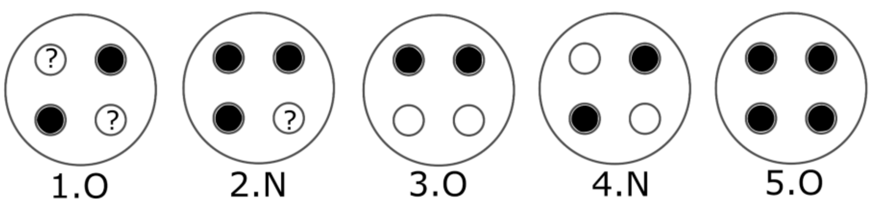

Figure 2. Solution for the rotating safe problem. State 1 is denoted by black, state 2 by white and an unknown state by a question mark

Further problems

The initial problem is used as a starting point for students to pose additional problems, it is easy to generate engaging but not too difficult related questions. Exploring further problems is one of the most important aspects of the Pósa method. After solving the initial problem, there is a plenary discussion with the students to propose new problems, to brainstorm. All questions are welcome. The easy questions are solved on the spot with the students, and usually, there is a discussion about what makes a question an interesting, a "good" one. Is the question well-defined? Is it too easy/too hard? Does it bring new, original elements to the problem? After collecting the questions, the teacher decides in which direction they are going to explore the topic further, based on the students' experiences. The exploration of this topic takes years, from the beginning of the program to the end. This is one of the longest threads in the program. Many questions are revisited when the students are more prepared to solve them.

Here are five additional problems related to the initial problem. There are many more related questions, these five have been chosen because most of the students are going to encounter these problems in the camps. These problems, besides their educational value, demonstrate the surprise elements of a mathematical discovery: based on one question the solver can think the next question is impossible or vice versa only to find out that the solutions differ from their expectations. Observe the following problems in order: initial problem, Q3, Q4, and Q5. Each problem has a surprising result, especially after solving the previous problem, the intuition is working against finding a solution. Only the solution to Q2 is discussed. The purpose of the next section is to give an overview of the thread, their connection to each other, and to other threads.

Q2 Rotating Safe with ten states. Imagine the same problem, only the switches have more states, namely ten distinguishable states (1 to 10). To solve the problem, one only needs to "iterate" the previous solution. First, with an O move and an N move we set three switches to state 1, we do not know the state of the fourth switch, let us denote with X. Then, if the door did not open, we set three switches to state 2 with an O move and an N move. If during these two moves we detect a state bigger than 2, we set it to 2. Now, we only have two states and we can solve the problem with the solution to the *initial* problem. If we could not detect a state bigger than 2 with the two moves and the door did not open, then we did not encounter switch X and the state of X is greater than 2. After that, with an O move and an N move we set three switches to state 3 and so on. In this way, there are two cases. First case: we find X and revert to the previous problems with only two states. Second case: the door will open at one of the stages. This problem-thread connects to a different problem-thread: “recursive/iterative thinking”.

Q3 Rotating Safe with buttons. Instead of switches now we have buttons, meaning, we cannot tell which states the buttons are in. Reflecting on the solution of the initial problem, we realize that we relied heavily on the information about the states of switches we gained throughout the steps. Throughout the years, there is an ongoing discussion about how mathematics search for the line between possible and impossible. After the solution of the *initial* problem it comes as a surprise for students, that the problem can still be solved. The origin of this problem is also interesting. This problem was created by accident, a student misunderstood the original problem and solved this instead.

Q4 Rotating Safe button version, with three states. Now, the buttons have more states, namely three. It is interesting that while the button version with two states can be solved, it can be proved that it is impossible to open the safe when the buttons have three states. The problem is also part of the *impossibility* problem thread. Additionally, this impossibility proof is in a more specific impossibility problem-subthread: the *relaxation* problem-thread. Here, relaxation means the relaxation of conditions. In this case, we can relax much of the conditions. We only would like to get two opposite switches into the same state. Furthermore, after each rotation, a voice announces about the states of the two switches, only, it will not tell which state belongs to which switch. It is easy to see that one cannot satisfy even these relaxed conditions; thus, one cannot open the safe in the original Q4 problem.

Q5 Rotating Safe button version, with four states. After solving the previous problem, one can argue that surely it is impossible to solve this problem with four states since we could not solve it with three states. The surprising fact is it can be solved. There is an easy way to generalize the solution from Q4 to solve this problem, but that generalization leads to a wrong solution. The Pósa method usually takes advantage of these situations, these ‘thinking traps’ are there to help students understand the underlying ideas better.

Q6 How many hands? There is an out of the box generalization of the original problem, proposed originally by a student during one of the camps. What if we have now six holes in the panel with the original binary switches inside them? It can be easily shown that this version cannot be solved. What if we can have more hands? How many hands do you need at least to be able to open the safe? To prove that three hands are not enough it also falls into the *impossibility* problem-thread.

Beyond Q6. Generalizing to this direction leads to a rather difficult problem: how many hands do you need at least to open the rotating safe with a panel with h-holes with binary switches? Generalizing further we arrive at an open question: how many hands do you need at least to open the rotating safe with a panel with h-holes with s-state switches (a switch with indistinguishable s number of states)? In the Pósa method encountering an open mathematical question would not be the first occasion in the camps, there are earlier examples of it. For example, one of the early open questions is related to $3\times n$ Chomp! (see Gale, 1974). Chomp! is a strategy game for two players on a rectangular grid. The players take turns and choose one block and remove it from the table together with blocks below it and to its right. The one removes the upper left corner is the loser. In the camps, students usually play the $2\times n$, $n\times n$ and 3×5 versions of the Chomp!, because in these cases, the winning strategy can be found. After solving the Chomp! for these cases, students typically ask the following: what is the winning strategy for $3\times n$ in general? This question is still open, despite significant efforts to solve it (Brouwer et al. 2005). For the students, it is surprising to know that the case of $3\times n$ is not solved, especially that they could solve the $2\times n$ and $n\times n$ Chomp! easily.

Final thoughts

In this article, the focus was on the original context of the method, the special camps for gifted children. Only a few aspects of the method were discussed here. Problem posing is widely used to foster problem-solving skills. (Silver, 1994) In Hungary, most students do not encounter problem-posing situations. However, in the Pósa camps, problem posing skills are nurtured throughout the years. The structure of networks of problems is one of the most important aspects of the program. Problems have a rich background, history, and connections to other problems. The connections between problems are thoroughly discussed with the students, which is also a novelty for most students in Hungary. The structure of the program has been developing for several years, new problems are added every year. Several such new problems are originated from the aforementioned problem-posing sessions. In this specific problem-thread, the element of “surprise” was predominant. This element of “surprise”, or in other terms the element of “unexpected results against reasonable expectations”, and “traps” are recurring elements of the camps, fostering critical-thinking and encouraging students to think deeply about problems. Enabling students to encounter open problems in mathematics is also one of the goals of the camps, shaping their beliefs about mathematics (Schoenfeld, 1987). In summary, in the Pósa method students can experience several elements of mathematical research, and develop the habits and dispositions of a mathematician (Schoenfeld, 2016). There are many aspects not discussed here: how students are developing a positive attitude towards mathematics, how group work is structured, how instructors guide students to solve problems, how group discussions are structured.

Many aspects of the Pósa method can be used outside its specific context: carefully designed long-term problem-threads, teaching students how to ask interesting, yet achievable questions. There is ongoing research on the possible adaptation of the method to normal school settings by the MTA-Rényi Research Group on Discovery Learning in Mathematics. The goals of the program are the following: students should develop a positive attitude toward mathematics (the joy of problem-solving), and students should also be able to take a successful final exam (at the end of high school). The first group of students is half-way through the program.

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