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Categorising question-question relationships in the Pósa method

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Abstract. The doctoral research of the author – with a reverse didactic engineering (RDE) methodology – aims at reconstructing the theoretical background of the 'intuitively developed' Pósa method for inquiry-based learning mathematics (IBME) in Hungarian talent education. Preliminary results of the second step of this theorization is presented, which applies tools of the Anthropological Theory of the Didactic (ATD). A model is proposed for categorizing question-question relationship with 3 categories: helping question, follow-up question and question of a kernel. The first two of them are claimed to represent two types (relevant or not) of generating-derived questions relationship. The model is also a prospective tool for connected task- and curriculum design and analysis within IBME development.

Key words and phrases: reverse didactic engineering, question-question relationship, connected task-design, helping question, follow-up question, question of a kernel, web of problem threads, generating and derived questions, relevant generation, ATD.

MSC Subject Classification: 97D20, 97D40, 97D50, 97E50, 97K30.

RDE research on theorizing the Pósa method – Research problem & goals, methodology and previous results

Lajos Pósa, a Hungarian mathematician and mathematics teacher has been organizing extracurricular weekend mathematics camps for 12-18 years old talented students since 1988 (Juhász & Katona, 2019), based on his method called (by him) ‘discovery learning mathematics’. Pósa’s method was developed without an explicit, recorded theoretical background. The present PhD study of the author at the Eötvös Loránd University aim at subsequently *reconstruct the theoretical frame of the task- and curriculum-design of the Pósa method*; for understanding and describing the method, potentially redesigning and extending its curriculum for public education use, as well for contributing to the international discourse on conceptualizing IBME (Artigue & Blomhøj, 2013).

The research was started by observing sessions of the Pósa camps, the many times validated, continuously re-developed (through an in-vivo process of teaching and (self-) analysis), and implemented design of Pósa’s method, and analysing samples from the used problem set (not publicly available), also supplemented by communication with teachers of the camps, first with as ‘untheorized researcher’s glasses’ as possible. It has then been leading towards reconstructing the preliminary ideas, epistemological views and mathematical content that may lay behind this practice, with the application and (re)creation of more and more, continuously revised theoretical scaffoldings. Didactic engineering is a research methodology that has similar phases (Artigue, 2014; Barquero & Bosch, 2015), but somewhat the other way round. Therefore, the term *’reverse didactic engineering*’ was (personally) suggested by Angelika Bikner-Ahsbahs at CERME11, for this research methodology.

As a first step, independently of other theoretical frames and research results, the model ‘*web of problem threads*’ (WPT) has been theorized to describe one decisive feature of the Pósa problem set, which is based on the connections between the posed problems. A ‘problem thread’, which term has been consciously used by teachers of the camps as a task- and curriculum design tool, is a “set of connected problems, in a not completely fixed order”, to “foster the development of specific ways of mathematical thinking, or methods”, such as “‘recursive thinking’, ‘(proof of) impossibility’ and ‘movement’”, or ‘working with remainders’ (modular arithmetic) (Juhász & Katona, 2019). However, on the one hand, different threads may be built to foster the development of the same such ‘method’, and on the other hand, a part of a thread may foster the development of several such ‘methods’. Therefore, later, the concept ‘*kernel of a problem thread’* was theorized and introduced in (Katona & Szűcs, 2017) by the author, denoting these ‘methods’ that ‘creates’ the threads. For a sample of the Pósa WPT, see (Katona & Szűcs, 2017).

From ‘praxeology’ to ‘generating and derived questions’ – Theoretical background on ATD

The research on theorizing the Pósa method, in its second step, applies tools of ATD (Chevallard, 2005), an already established framework within IBME. The present paper focuses on the part of this step that attempts to categorise relationships between questions (for students to answer) emerged in the Pósa camps, to reveal the nature of the connectedness of the task- and curriculum design of the Pósa method, by applying the following ATD tools.

Within ATD, the central concept *praxeology* was introduced to reinterpret and generalize the concept (body of) ‘knowledge’, being percolated within institutions and individuals, and to account for the complexion of human ’action’ and ’thinking’, regarding primarily ’learning and teaching’, i.e. didactic phenomena, by reconnecting these three concepts: human action, thinking and knowledge. A praxeology – as (something like) a unit of human knowledge, action and thinking – consists of a more pragmatic *praxis block* – composed of a *type of tasks* (”what people do”) and a *technique* (”how they do it”) – and a rather epistemic *logos block*, serving for explanation and justification of the technique – formed of a corresponding *technology* (“what they think” why they are using this technique to accomplish this task) and a *theory* (why and how they think so) (Barquero & Bosch, 2015, pp. 260-272; Chevallard, 2005, pp. 21-23; Chevallard & Bosch, 2019, pp. xxxii-xxxiii). *Praxeological analysis* is a complex study of selected praxeologies, by identifying and characterizing their elements and their relations to each other, to the (e.g.) individuals and institutions involved, regarding also corresponding conditions and constraints.

“In ATD, a *condition* is anything purported to have influence over at least something. . . . A *constraint* is any condition which appears to be unmodifiable” (Chevallard & Bosch, 2019, pp. xx-xxi), and a *dialectic* is “any praxeology that enables one to overcome two opposed types of constraints by turning them into a new kind of conditions that supersede them" (p. xxii). The *dialectic of questions and answers* (or Q-A dialectic) (Bosch & Winslow, 2016) plays a central role in inquiry according to ATD. Questions and answers have an influence on learning – that is, they are conditions of it – and some of them, or at least the existence of them, appears to be unmodifiable for the learners and teachers involved; so they may be constraints, moreover, in some way, they are opposed types (unknown/ known). Within ATD's model for inquiry, called *study and research paths* (SRP) (Bosch & Winslow, 2016, pp. 18-37), questions and answers are connected in a way that they are turned into a new type of condition: *inquiry*. The SRP model is a (didactic) praxeology that connects questions and answers and enables us to overcome them. It is therefore a dialectic of questions and answers. "Questions are the starting point and the main incentive of didactic life: to ask oneself—or to ask someone—a question is the basic act that will ultimately cause praxeologies as yet unknown to be met." (Chevallard & Bosch, 2019, p. xxxiii). In an SRP, a question Q_0 is the starting point of the inquiry (into Q_0) to find the answer A_0, during which other questions, e.g. Q_1 may be posed (by the inquirers), induced by the inquiry into Q_0, *generated* by Q_0, so that A_1, the answer to Q_1 may help inquirers to establish A_0. Q_1 is *derived* from Q_0. Considering the concept derived question, we may detect (and separate) two characteristics. One is, that a derived question of a generating question Q is "induced by the study of Q" (Chevallard & Bosch, 2019, p. xxix), it emerges from the particular inquiry into this Q. The second one is, that derived questions (of Q_0) play the role of producing "partial answers from which the answer $A^♥$ [to Q_0] will be produced" (p. xxiv), or more precisely, "*relevant" derived questions* are "capable of leading temporary answers A_k that can be helpful in elaborating a final answer $A^♥$ (Barquero & Bosch, 2015, p. 261). In the followings, the term 'derived question' is used solely on the basis of the first characteristic, that is, any question is to be regarded derived that was induced by the study of its generating question, whether it is 'relevant' or not from the perspective of the particular inquiry into its generating question. Any answer may be of two basic 'types'. $A^♥$ (A heart) is a "more or less personal" result of a (part of) inquiry (Bosch & Winslow, 2016, p.31), "the inquirers' proper answer" (p.21), that satisfies conditions specific to the (part of) the inquiry. $A^\lozenge$ (A diamond) is an already established answer, not resulting from the particular inquiry the inquirers might use to elaborate the answer to Q_0.

Question generation and the development of kernels: helping question, follow-up question and question of a kernel – Preliminary results, conclusion and issues for further research

Preliminary results are presented on the basis of a praxeological analysis of selected Pósa problems, supported by some examples and explanations. Questions are denoted as posed problems (Problem 1, Problem 2,…), numbered as they appear sequentially in the camps, but at the same time, also using a different notation to express the results of the analysis, whether they are generating (Q_G) or derived (Q_D), or belong to the same kernel of a thread (Q_K), in their relations to each other. Problems 1, 3, 4 and 6 were planned to be posed and were actually posed in the Pósa camps by teachers as problems to be solved by the students. Problem 2 emerged during solving Problem 1 (it is to be discussed how); it has never been posed 'alone', without discussing Problem 1. Problem 5 was not, but may and could be posed in the camps; it is being suggested by the author as a possible extension. The following question-relationship categories are proposed (may be overlapping).

(1) Helping question: For answering question Q_G, question Q_D is posed, so that the inquiry into Q_D, and mostly the answer $A^{\heartsuit}_D$ or $A^{\Diamond}_D$ helps find answer $A^{\heartsuit}_G$. The praxeology, mainly *the praxis block connected to Q_D helps to elaborate the praxeology, mainly the praxis block connected to Q_G, assisting to work out A_G*, for instance as the technique of Q_D forms part of the technique of Q_G. In that case, *the derived question Q_D is definitely 'relevant'*. It is the type of question generation that corresponds exactly to ATD's original concept of the 'generating-derived question relationship'.

In the '$A^{\heartsuit}_D$ case' (*helping question – heart)*, Q_D was not posed as a problem (or question) before posing Q_G, the inquiry into Q_D is supposedly new to the students. It emerges during the inquiry, and it is to be inquired into as part of the overall (original) inquiry into Q_G, like a sub-inquiry. An example (posed usually for 7th grade students) is the following (Q_G_1 and Q_D_1).

Problem 1: Q_G_1 – ***Map drawing****:* Can you draw maps containing cities and roads between some cities (bidirectional roads go directly from one city to another, no dead end, they can cross each other), so that exactly 3 roads 'start' from each city. We need four maps with 6, 7, 8 and 9 cities.

Problem 2: Q_D_1 – ***Number of roads***: How many roads are/ shall be there in these maps?

$A_D^{\heartsuit}$_1 and 1st part of A_G_1: If you have e.g. 7 cities, and 3 roads start from each city, you can add it up, city by city, which results in 7×3 roads altogether. However, you counted every road twice, as every road has exactly two ends, so there must be (7×3) /2 roads altogether.

2nd part of A_G_1: It is not a whole number, so it's impossible to do it.

In the '$A^{\Diamond}_D$ case' (*helping question - diamond*) A_D is an already established answer (from a previous inquiry by the students) at the moment of posing Q_G. However, Q_D is still a 'relevant' derived question, having the (main) role of facilitating the inquiry to find A_G. An example is the following (Q_G_2 and Q_D_2).

Problem 3: Q_D_2 – ***Shortest ways of astronauts***[1]: An astronaut lives in a space station consisting of 27 modules, which are at the vertices of the little unit cubes that make a 2×2×2 bigger cube. There are passages between each neighbouring modules along the edges of the unit cubes. Our astronaut uses these passages to move between the modules. (See Figure 1.) She is now at module F_2, and would like to go to the opposite one, M_2. How many edges does her shortest way contain?

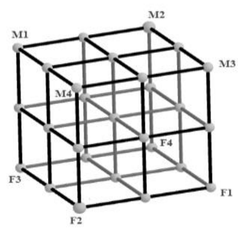

Figure 1. Space station with modules (vertices) and passages (edges)

A_D_2: The astronaut needs at least 6 moves (edges), as she needs to get to the rightmost set of (9) cabins from the leftmost set, and also to the topmost set from the bottommost set, and to the backmost set from the frontmost set. Each needs at least 2 distinct moves, which results in 3×2 = 6 edges (7 modules) for any shortest way.

Problem 4: Q_G_2 – ***Astronauts in love***[2]**:** 7 new astronauts have arrived at the space station, so now the crew consists of 4 man and 4 women, living in the space modules illustrated by the tagged vertices in Figure 1. The passages are all closed due to security

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[1] Basically the same problem is in (Katona & Szűcs, 2017, pp. 22-23)

[2] also in (Katona & Szűcs, 2017, pp. 29-30) in a different formulation

reasons. However, during the spacewalks, love springs among two astronauts, F1 and M1, living in 'opposite modules'. Therefore, by opening some passages (edges), a corridor is created between the two sweeties. After some months, love springs again, now among other two 'opposite astronauts', and again, and again. Can it be done with all the 4 couples if no two corridors shall contain a same module?

A_G_2: There is no solution with all the 4 couples. Suppose, indirectly, that there exist 4 proper corridors. Based on A_D_2, each proper corridor goes through at least 7 modules. As no two corridors can go through the same module, we need at least $4 \times 7 = 28$ modules altogether, one more than we have.

(2) Follow-up question: Based on the study of a question Q_G, a connected question Q_D is posed, initiating a new, connected inquiry, but *not necessarily for facilitating the inquiry to find A_G*. Moreover, *Q_D may (or rather usually) be posed on the basis of the study of the already established answer A_G*. Therefore, the derived question Q_D is not necessarily (and is usually not) 'relevant' in that case. Q_D is to be called the follow-up question of Q_G. Q_D is generated on the basis of the study of usually at least the whole praxis block, rather also the technology part of Q_G. This type of Q_G–Q_D relationship is also regarded as a generating-derived question relationship, though not 'relevant' to the particular inquiry into Q_G, which seems to be an extended interpretation of the original concept of ATD's 'question generation'. An example is the following (Q_G_3 and Q_D_3).

Q_G_3 = Q_G_2 (Problem 4)

Problem 5: Q_D_3 – ***Modified space station***: Can we modify the size of the space station in a way that all the 4 pairs of astronauts will be able to arrange their intimate meetings? We keep the cuboid form, but we can change the size.

A_D_3: For instance, if we take a $2 \times 2 \times 3$ cuboid plan of the space station, the number of modules increases by 9, and the number of the shortest paths between the 2 modules of a couple by 1, so the sum of the modules in all the 4 shortest paths is increased by 4. Therefore the contradiction that was the main element of the argumentation for A_G_2 does not apply here. However, we would still need to show (not presented here) that the construction of 4 appropriate paths is possible. (If it is.)

Q_G_4 = Q_G_1 (Problem 1)

Problem 6: Q_D_4 – ***Proper numbers for starting roads***: With what numbers, as numbers of starting roads from the cities (now not necessarily 3 from each) is it possible to construct an appropriate map for a given number of cities?

A_D_4: With odd sums it is impossible. (Further inquiry is needed to find which even sums can be ‘realised’.)

(3) Questions of a kernel: *Q_{K1} and Q_{K2} are connected, as they foster the development of the same kernel of a problem thread*. In a manuscript of the author that is under preparation for the proceedings of the 6th International Congress on the Anthropological Theory of the Didactic (Katona, 2019), it has been shown that an example of *this type of relationship is based on the intersection of the logos blocks of praxeologies connected to the corresponding questions, more precisely, on common elements of their technology parts*. For answering these questions analysed in (Katona, 2019) students need to elaborate such first seemingly different techniques for “counting paths”, “counting tilings”, and arguing on the basis of “number regularity”, that become ‘unhiddenly’ connected (after the inquiries) at the technology level. An example for ‘questions of a kernel’ is the following (Q_{K1}_5 and Q_{K2}_5).

Q_{K1}_5 = Q_G_1 (Problem 1), and Q_{K2}_5 = Q_G_2 (Problem 4)

For solving the problems raised in Q_{K1}_5 and Q_{K2}_5, i.e. also for justifying their solutions, students need to use different series of arguments as the technique parts of the ‘connected’ praxeologies. However, these series of arguments are (also) similar to each other: *both technique parts are explained and justified, at a higher level, by the same corresponding technology of indirect thinking*. During this particular series of inquiry (into Q_{K1}_5, Q_{K2}_5, and other related questions not presented here) students are getting to know and apply indirect proof, for the first times, which is *appearing as justification of the used techniques, therefore being at this time situated at the technology level; and it is just being born as a technique to come (later)*. Indirect thinking/ proof is regarded as one *kernel* within the Pósa WPT, hence it is claimed that *kernels may be interpreted as common technology parts of praxeologies ‘connected’ to questions of the same problem thread, that later may become techniques*. This type of Q–Q connection is named ‘questions of a kernel’.

The graph of the presented problems, highlighting the categories of the detected question-question relationships can be seen in Figure 2.

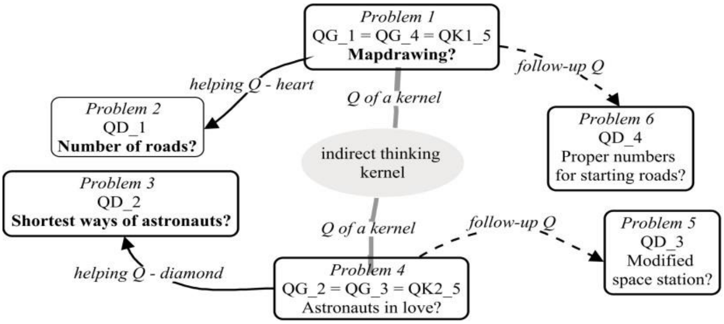

Figure 2. Model for analysing Q-Q relationships during task-design

The presented set of categories, 'helping question', 'follow-up question' and ' questions of a kernels', offer a model for task-design and analysis that, first of all, calls attention to the importance of the focus on the structure of set of tasks designed (connected task-design), and on the roles of types of question-connection and generation (relevant or not) within series of inquiries. However, further research is needed, on one hand, on a deeper understanding of the nature of the presented categories, on detecting more ones (if exist) and on subcategorization. For instance, the two presented example of follow-up questions are both 'generalizations' and we shall research on detecting other subcategories (if exist). On the other hand, continuing research on types of question generation would contribute to a deeper understanding of the presented model and of the applied ATD tools too.

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