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Teaching Mathematics and Computer Science

The shift of contents in prototypical tasks used in education reforms

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Abstract. The paper discusses the shift of contents in prototypical tasks provoked by the current educational reform in Austria. The paper starts with the educational backboard of the process of changes in particular with the outfitting of the students' abilities in different taxonomies and its implementation in the competence models of Mathematics. A methodological didactical point of view on the process is given additionally. Examples out of a specific collection of math problems which arise from the educational reform are integrated and analysed in the context of educational principles and methods. The discussion ends with a short evaluation of the role of traditional approaches to tasks in the ongoing reform. A bundle of tasks as proof that they are still alive is presented finally.

Key words and phrases: educational frame, prototypical tasks of mathematical problems.

MSC Subject Classification: 97B50, 97D40, 97D50.

Initial part

The aim of the paper based on the publication of Karl Fuchs, Christian Kraler and Simon Plangg (2017) is to show the importance of task-based learning in school in acquiring general and specific knowledge of our world. By focusing on this, we have had to face a significant shift in task-based learning philosophies. Traditional teaching focusing on calculating has been replaced by other approaches increasingly. Other activities have gained a much higher importance in Mathematics classes.

The process of changes

Hence, one of the underlying ideas of this paper is the participants' attention on these important parameters in ongoing national education reforms in the Austrian school system (Fuchs, 2013). In 2020, we published a survey that focused on the development of teaching styles in Austria's secondary schools over the last 40 years (Fuchs & Kraler, 2020, p. 482ff). In particular, cognitive psychology allows a new understanding in learning mathematics which transcends the traditional abilities to a new approach. An approach addressing cognitive competences was already worded as ANALYSIS, SYNTHESIS, EVALUATION by Benjamin Bloom et al. (1956) (see Figure 1).

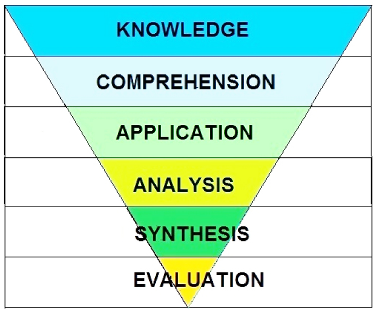


Figure 1. Taxonomy of Bloom et al., 1956

The model was renewed in the revision of Bloom's Taxonomy by Lorin Anderson and David Krathwohl et al. (2000), wording the higher competences as ANALYZE, EVALUATE and CREATE (see Figure 2).

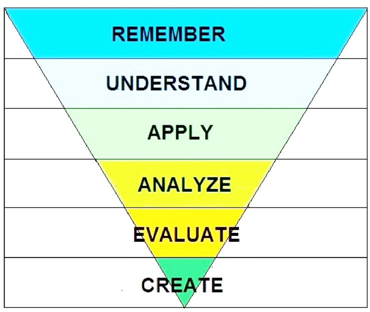


Figure 2. Taxonomy of Anderson and Krathwohl et al., 2000

The SAMR (Substitution/Augmentation/Modification/Redefinition) Model, a technology affine model based upon Anderson and Krathwohl, by Ruben Puentedura (2013) was published recently (see Figure 3).

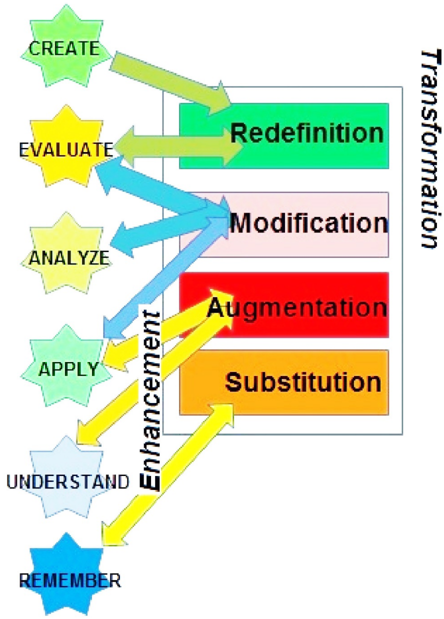


Figure 3. SAMR-Model of Puentedura at http://www.hippasus.com/rrpweblog/archives/000095.html, last retrieved: 4 January, 2022

The different levels of cognitive competences have found their way in the acting dimension of the competence models of Mathematics Secondary Level.

In General Higher Education Schools (Allgemeinbildende Höhere Schulen AHS) they are labelled by

- *Illustrating*,
- *Modelling*,
- *Operating*,
- *Interpreting*,
- *Arguing* and
- *Reasoning*. (Fuchs & Kraler, 2020, p. 487)

In Higher Vocational Schools (Berufsbildende Höhere Schulen BHS) cognitive competences are labelled by

- *Modelling* and *Transferring*,
- *Operating* and *Use of Technology*,
- *Interpreting* and *Documenting*,
- *Arguing* and *Communicating*. (Fuchs, 2013, p. 3)

From the methodological didactical point of view, the **Genetic Principle** in its methodological characteristics (Führer, 1991; Wittmann, 1974) and the **Concept of Fundamental** or **Universal Ideas** (Bruner, 1960) have forwarded the intended changes fundamentally.

The characteristics of the genetic approach are:

- The consideration of the stage of cognitive development of the learners when presenting mathematical contents.
- Acquiring mathematical content should always be accompanied by communication.
- Heuristic considerations of the learners lead to strict formulations.
- The setting of a task should allow the learners to bring in their own experiences.
- The contents should always question the relations to Real World.
- The use of technology is commended. (Fuchs & Landerer, 2021, p. 70)

Fundamental or **Universal Ideas** appoint techniques or strategies which should concerning Andreas Schwill (1985) own the following properties:

- They own a comprehensive applicability in many areas, and they arrange and integrate plurality of phenomena (**horizontal criterion**, see Figure 4).

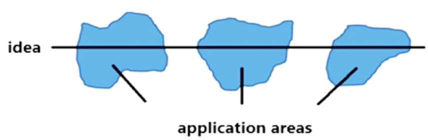


Figure 4. Illustration horizontal criterion (Schwill, 1993)

- They structure the content within an area vertically. Namely an idea can be taught at nearly any arbitrary mental level (from primary students similarly up to High School students) (**vertical criterion**, see Figure 5).

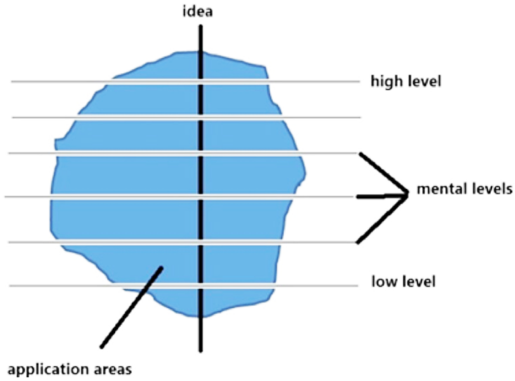


Figure 5. Illustration vertical criterion (Schwill, 1993)

- They must be applicable in manifold variations additionally. Namely they must own an anchorage in everyday life as well as social relations (**criterion of sense**).
- They must result from the observations of the historical development of concepts and structures. Therewith the long-term character of Fundamental or Universal Ideas is secured (**criterion of time**).

Several examples taken out of the collection of math problems (SRDP Aufgabenpool) of the ongoing major educational reform in the Austrian school system, namely *Standardized Maturity and Diploma Exam* (in written form) (*Standardisierte* (schriftliche) *Reife und Diplomprüfung* (sRDP)), will be integrated in the discussion. They should give evidence of the intended educational problem-solving goals and students' competences and will be rated in the context of educational principles and methods.

A first teaching goal is *Prototyping.* For example, it activates *basic understanding* of *prototypical characteristics of functions* (Dörfler, 1991) as an educational problem-solving strategy.

Rudolf von Hofe terms the following two constitutive elements of basic understanding:

- The composition of appropriate (visual) representations, respectively internalisations, allow functional action at the level of imagination as first element.
- The opportunity applying an idea (here real functions) to reality by recognising the appropriate structure in factual connections or modelling work-related

problems with the help of the mathematical structure constitutes the second element. (von Hofe, 1995)

We will discuss the different elements of basic understanding (of appropriate (visual) representations respectively, and the opportunity applying an idea to reality) in the following task.

Half-Value Period of Knowledge

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Addressed Competences: Abilities to identify the parameters of the exponential and linear function given by graphs and equations including the abilities to see the relations and to interpret these relations in a context. Augmentation and Modification in tenor of Puentedura.

Fundamental/Universal Idea: functional dependance.

Genetic Principle: heuristic approach, problem related to real world, technology cited as **familiar resources possible**.

Use of Technology: familiar resources possible.

Problem: The knowledge earned in a certain time loses up-to-dateness and relevance due to social changes, technological innovations etc. in the course of time. The subsequent image describes the decline of relevance of knowledge in the different special fields. The percentage of primal knowledge which is still relevant is shown for each year.

(a) One may assume that the relevance of vocational knowledge declines exponentially and the Half-Value Period accounts for five years. Sketch into the given image the course of relevance of vocational knowledge in the interval [0,15]!

(b) The relevance of knowledge about technology declines exponentially with a Half-Value Period of three years. Formulate the exponential function which describes the relevance of knowledge about technology in dependence of time!

(c) The relevance of knowledge gained at university can be described by the following function $N(t) : N(t) = 100 \cdot e^{-0.0693 \cdot t}$, where t specifies the time in years, $N(t)$ the relevance of knowledge gained at university at time t expressed in percent of the relevant initial knowledge gained at university. Estimate the decline of percentage of knowledge gained at university after seven years (from the beginning)!

(d) Finally, the relevance of knowledge gained at school can be described as a linear function. Pick the slope of the linear function out of the given image above!

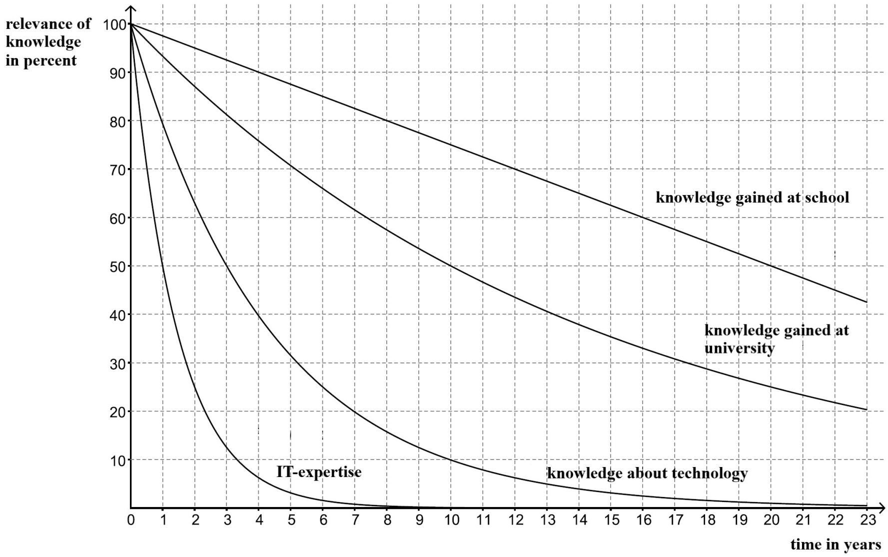


Figure 6. Prototyping as educational problem-solving strategy (using prototypical characteristics of real functions)

Andreas Filler recapitulates the experiences with students solving tasks such as **Half-Value Period of Knowledge** calling for Prototyping as follows:

In a first step, the students investigate the meaning of the function values of the given functions by investigating the graphs. Step by step, the students loosen the concrete values and make generalisations (symbolic representation of functions with parameters). In this process of Prototyping the students must go through Processes of variations. For this step, the author recommends the use of technology. The teaching unit ends in summarizing the results. (Filler, 2012/13)

Illustrating, the second teaching goal, addresses the transfer of a strange new problem to an image out of an area the students are familiar with. Inversely, they should be able to solve the strange new problem by abstraction.

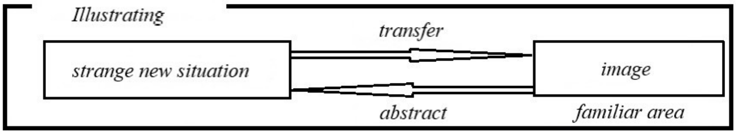


Figure 7. Process of Illustration (Profke, 1994)

Following Lothar Profke, transfer processes can happen in several different ways:

- *Iconic Illustrating by Geometry*,
- *Illustration by Everyday Experiences and Actions*,
- *Illustration by Prototypical Tasks*,
- *Symbolic Illustration by Proofs*,
- *Illustration by Analogy*,
- *Illustration by Stereotyping* (namely searching for related tasks),
- *Illustration by Schemata* (*algorithms*) and *Symbols* (namely by change of representation, cf. Jerome S. Bruner et al.'s *EIS* (*Enactive Iconic Symbolic*) *Principle* (1988),
- *Illustration by Habituation* (namely permanent handling with terms lends them independent existence).

We will address the aspects *Iconic Illustration by Geometry* and *Illustration by Schemata* (*algorithms*) and *Symbols* (combining different types of representation) in the following task.

Curve Progression

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Addressed Competences: Abilities to describe the character of functions with the help of the derivative (function): monotonicity, local extrema, bending to the left and right, turning positions.

Fundamental/Universal Ideas: functional dependance, approximation (linearisation) and iconic representation.

Genetic Principle: active participation/bringing in own experiences, technology cited as **familiar resources possible**.

Use of Technology: familiar resources possible.

Problem: The following image shows the tangent t in a point $P = (x_p, f(x_p))$ of the graph of the polynomial function f. Though, P is the only common point of the graph of f and the tangent t. Statements about $f'(x_p)$ and $f''(x_p)$ are stated in the table following the graphical representations (see Figure 8).

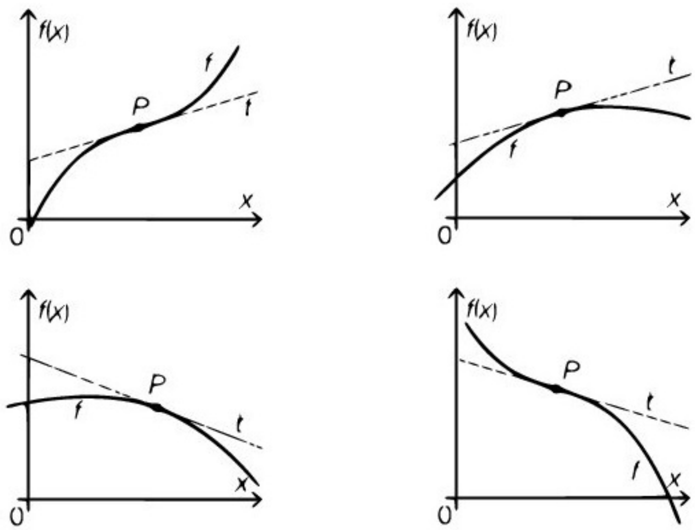


Figure 8. Arguing based on iconic representation as problem-solving strategy

Assign the four images of Figure 8 in each case to the correct statement (out of A till F)!

A	$f'(x_p) > 0$	and	$f''(x_p) > 0$,
B	$f'(x_p) > 0$	and	$f''(x_p) < 0$,
C	$f'(x_p) < 0$	and	$f''(x_p) > 0$,
D	$f'(x_p) < 0$	and	$f''(x_p) < 0$,
E	$f'(x_p) > 0$	and	$f''(x_p) = 0$,
F	$f'(x_p) < 0$	and	$f''(x_p) = 0$.

Werner Wiater postulates the following aspects for a successful instruction through Illustrating:

- Information which should be well received by the students must be clear and impressive for all sensations. For a long time, empirical investigations have proved that visual representations such as graphs bear better students' performances in understanding and keeping of new learning matters than verbal representations.
- Motivation Psychology has proved that the interest of students in a new topic strongly correlates with the kind of representation. (Wiater, 2021)

At last, the third teaching goal, *Modelling*, addresses the transformation of real world problems into the world of mathematics by *Mathematization*. The results gained in the world of mathematics are traced back to reality by *Interpreting* these results in the real world.

Modelling cycles have been published by Werner Blum and Dominik Leiß (2005), Uwe-Peter Tietze, Manfred Klika and Hans Wolpers (1982), Joachim Engel (1998) or Hans-Georg Weigand and Hubert Weller (1997) in the context of technology.

Drug Reduction

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Addressed Competences: Abilities of modelling real world problems.

Fundamental/Universal Ideas: modelling.

Genetic Principle: active participation/solving a problem by using technology (Fuchs, 2007) in the context of relations to reality.

Use of Technology: familiar resources possible.

Problem: The amount of active ingredients after the taking of a pill can be modelled by the following function m:

$$m(t) = 20 \cdot (1 - e^{-0{,}05 \cdot t} - 0,125 \cdot t) \quad \text{with } t \geq 0.$$

t stands for the time elapsed after the taking of the pill in minutes (min). $m(t)$ stands for the mount of active ingredients in the blood circle at time t in milligramme (mg).
Estimate the point in time when the active ingredient will be reduced almost completely!
Identify the point in time when the instantaneous rate of change of the active ingredient in the blood circle is 0,5 mg/min!

Find an argument that m is negative curved using outcomes of the differential calculus!

In fact, numerous iterations of all modelling steps will be necessary to gain satisfactory results (Fuchs & Landerer, 2007). Karl Josef Fuchs and Ulrike Greiner examined modelling tasks like **Drug Reducation** critically and presented the following arguments at the 51st Annual Conference of the German Society for Mathematics Education (Tagung für Didaktik der Mathematik Jahrestagung der Gesellschaft für Didaktik der Mathematik) in 2017, Potsdam:

- Technology used as far as possible for operating. Technology used for concept formations remains unconsidered.
- Modelling process gets to levelling or reduction. The demand of *Mathematization*, namely the transformation of real world problems into the world of mathematics, is rare.
- Very often modelling tasks are underlying *traditional closed tasks* (*eingekleidete Aufgaben*).
- Aspects like the reflections of the role of mathematics, which means the significance of the teaching subject in the cultural and social context, find considerations in no way. (Fuchs & Greiner, 2017)

Filler (2015/16) quotes the following points of view going back to Lutz Führer (1991). Structurally Filler proceeds in the following way: He starts with Führer's hypothesis and cites Führer's answers afterwards.

Hypothesis 23: Teaching mathematics should show and explain applied approaches among other things.
Answer concerning Hypothesis 23: Are we allowed to orientate subject matters application beneficial? For whom beneficial? For all? Certainly, that is what we want. But that what all (students) can require, wish and accomplish is restricted on all hands by the things that are possible and those others require, possess, are able to, know.

Hypothesis 24: Appliance Orientation of Teaching mathematics… can apply two points of view, namely orientation guide inside of mathematics or orientation across things out of mathematics. From a faced approach Appliance Orientation has been only useful where it has been intellectual honest and has had to answer for sociopolitical questions.
Answer concerning Hypothesis 24: Which mathematics is at the bottom of the constraints and barriers of the achievable, the practicable, the understandable?

For filling stations attendants, land surveyors, crystallographs, merchants, insurance agents, ... mature citizen or demagogue as well? Filling stations attendants must be able to calculate but they need not know anything about groups or trigonometry, yet land surveyors and crystallographs do. ... Here we talk about general education. Hence general (aspects) must be detected in the particular.... Maybe this motivates to a closer inspection, creates working hypothesis, endows promising associations, stimulates imagination, wakens career aspirations but it orientates not all necessarily. Usefulness doesn't absolve teachers from the responsibility with respect to content and form. ... The teachers must account for the emerging professional and individual targets very carefully.

Hypothesis 25: Normally Application Orientated teaching phase require, due to reasons of complexity, competences and responsibility teacher comments as corrective.
Answer concerning Hypothesis 25: This hypothesis as a methodological consequence of Hypotheses 23 and 24 ought to be clear actually, but it must be mentioned in the light of the very common fetishization of Learning by Discovering on the one hand, and qualified instruction on the other hand. Teachers should not hide that they are grownups and that they have studied their profession generally. Following my experiences, this fact depresses students little similarly to a geography, music teacher or trainer who knows the real world (of which he/she is speaking constantly) from his/her own experiences. A teacher who is at home in his/her subject is allowed to play students these experiences every now and then but he/she mustn't do this continuously... (Filler, 2015/16: Führer, 1991)

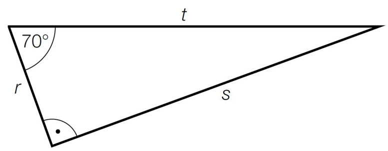
Final part: The role of old-fashioned tasks

Wading through the tasks in the collection of math problems shows oneself that the pattern changes needed in the design of the individual tasks in the desired direction have not yet been a widely realized area due to the constructive criticism of many math teachers and experts in mathematics. Single tasks still employ old-fashioned approaches (focusing on one basic competence, namely calculating). Most of the tasks treat innermathematical problems, some of them like **Coin Flipping** problems where the context is only adornment. These tasks are without deeper purpose. A reform concerning technology is planned actually. The students will have to work out tasks especially of the old-fashioned type without technology.

Traditional Task 1: Triangle

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Given a right-angled triangle with side length r, s and t.



Problem: Calculate the ratio $\frac{r}{t}$ for this triangle!

Traditional Task 2: Orthogonal Vectors

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Given the subsequent vectors:

$$\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} x \\ 0 \end{pmatrix}, x \in \mathbb{R}, \qquad \vec{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \vec{d} = \vec{a} - \vec{b}.$$

Problem: Calculate x so that the vectors $\vec{c}$ and $\vec{d}$ are normal!

Traditional Task 3: Identify a Coefficient!

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

Given function $f : \mathbb{R} \mapsto \mathbb{R}$ with $f(x) = a \cdot x^2 + 2$ and $a \in \mathbb{R}$.

Problem: Calculate the value of the coefficient a so that $\int_0^1 f(x)dx = 1$!

Traditional Task 4: Linear Function

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

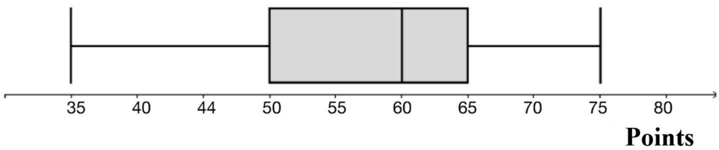
Given a linear function $f : \mathbb{R} \mapsto \mathbb{R}$ with $k, d \in \mathbb{R}, k \neq 0$ and $f(x) = k \cdot x + d$. It is imperative: $\frac{f(5)-f(a)}{2} = k$ for $a \in \mathbb{R}$.

Problem: Calculate a!

Traditional Task 5: Mathematics Competition

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

The allocation of points is pictured in the subsequent Boxplot.



Problem: Calculate the width of the given points!

Traditional Task 6: Coin Flipping

Subject/Level: **Mathematics**/Secondary Level – Age of the students: 18

A coin shows *pitch-and-toss* after each throw. The probability that the coin shows *pitch* in each throw is as high as the probability that it shows *toss*. The outcomes are independent from one another.

The coin is thrown 20-times.

Problem: Calculate the probability that the coin shows 12-times *pitch* when it is thrown 20-times!

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