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A constructive and metacognitive teaching path at university level on the Principle of Mathematical Induction: focus on the students' behaviours, productions and awareness

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Abstract. We present the main results about a teaching/learning path for engineering university students devoted to the Principle of Mathematical Induction (PMI). The path, of constructive and metacognitive type, is aimed at fostering an aware and meaningful learning of PMI and it is based on providing students with a range of explorations and conjecturing activities, after which the formulation of the statement of the PMI is devolved to the students themselves, organized in working groups. A specific focus is put on the quantification in the statement of PMI to bring students to a deep understanding and a mature view of PMI as a convincing method of proof. The results show the effectiveness of the metacognitive reflections on each phase of the path for what concerns a) students' handling of structural complexity of the PMI, b) students' conceptualization of quantification as a key element for the reification of the proving process by PMI; c) students' perception of the PMI as a convincing method of proof.

Key words and phrases: mathematical induction; quantification; constructive teaching/learning; metacognition; proofs/refutation of conjectures; university level.

MSC Subject Classification: 97B40, 97C70 .

Introduction

The Principle of Mathematical Induction (PMI) is a topic that generally students meet at the university as a tool for proving statements $\mathrm{P}(n)$ depending on a variable n in the set of natural numbers. It is well known that its learning is very problematic: very often the students feel the PMI as untied from their previous mathematical experience and do not understand the reasons of the validity of the PMI as a proof method, so they learn to apply it as a recipe without any awareness about its meaning.

As we show in our theoretical frame, many studies: a) highlighted the overlapping of different types of difficulties met by students in their PMI learning; b) gave indications about how to foster the students learning of the PMI; c) analysed and classified students behaviours and conceptions in proving statements by PMI. But, although the big number of papers on the PMI, there is a lack of studies about the role of the teacher and the effects of his/her behaviours on the students' understanding and perception of the PMI as a method of proof. This paper is aimed to highlight this correlation.

We believe that the most common difficulties that inhibit an aware understanding of PMI are due to two main factors: first, a problematic relation of the students with the matter, that displays in little attention to the interpretation of the formulae and their logical connections; second, the traditional approach to the PMI, often characterized by a concise and rushed teaching, mainly focused to its application to prove closed-form statements.

We are convinced that is possible to minimize the detected difficulties by a reflective teaching which pays specific attention to make explicit all the logical and semantic 'knots' hidden in the statement of the PMI. This should favour the students' understanding of the deep sense of the proving process.

To verify this hypothesis, we designed and implemented a constructive and metacognitive teaching path on the PMI, articulated in various phases and based on different types of activities where the students have to work in different ways (individually, in small group work, in collective discussion with teacher). All these activities are aimed at engaging students as *active learners*, allowing them to conceptualize, apply, explain the PMI and embed this mathematical content into a meaningful net of connections with previous knowledge and experiences. The main idea is to involve at the beginning the students in the exploration of connected questions that should make them explicit the reasoning underpinning the PMI, so that the formulation of this principle can be devolved to them. After the collective validation of the statements of the PMI formulated by the students, our educational path envisages the exposure of the students to fallacious induction arguments, in order to stress the essentiality and independence of the base step and of the induction step in the PMI. Yet, the students are called to autonomously prove some not simple sentences by PMI and then – by a metacognitive questionnaire – to reflect on the experiences made and to express their understanding and perception of PMI. Finally the students are involved in a role-playing game where they act also as a teacher, either in posing problems involving the PMI to classmates or in assessing solutions of those problems and proofs by PMI done by classmates.

In such an educational path, clearly the role of the teacher is essential, since he/she should foster the students' understanding of the contents and focus their attention on the several aspects involved in the processes in play.

Our specific research foci are related to both students and teacher. From the students' side we focus: i) on their handling of the structure of the PMI when they use it to prove statements, and ii) on their perception of the PMI as a method of proof. From the teacher's side, our main aim is to highlight iii) the different roles played by the teacher in the various phases of the teaching path and iv) the incidence of her didactical choices on the students' learning in terms of abilities in proving by PMI and awareness on the effectiveness of PMI as proving method.

Our Theoretical Background

In this section, we describe the theoretical elements which we used in the *a priori* cognitive and epistemological analysis that guided the design of the educational path. Moreover, we recall the theoretical lenses we used for the *a posteriori* analysis of the steps of the educational path.

To fix a formulation of the PMI, we refer to the following statement:

Let $P(n)$ a proposition depending on a natural number n. If

(1) $P(n_0)$ is true for a certain n_0 in $\mathbb{N}$;

(2) for all $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true;

then $P(n)$ is true for all n in $\mathbb{N}$, $n \geq n_0$.

Obviously in point 2) we could use the same letter n used before, since the index is saturated by the quantifier, but we choose to differentiate the letters n and k to avoid inducing the students' common misconception concerning the circularity of the PMI (i.e. we are assuming what we would like to prove). As usually, condition 1) is said 'Base Step' (BS) and condition 2) is said 'Induction Step', (IS); moreover, in IS the premise of the implication is said 'Inductive Hypothesis' (IH). From now we shall use these abbreviations.

Difficulties and non-traditional approaches for teaching/learning the PMI

Based on the literature (Avital & Libenskind, 1978; Dubinsky, 1986, 1989; Ernest, 1984; Harel, 2001), we distinguish some main categories of difficulties in learning and understanding the PMI: *logical and interpretative difficulties, conceptual difficulties, technical difficulties,* and also *psychological difficulties.* The *logical and interpretative difficulties* regard the structure of PMI, for the intrinsic complexity of its logical form. Its statement is a conditional proposition *"if..., then..."*, where the premise is the conjunction of two conditions, the second of which is a quantified conditional proposition itself. The difficulties regard the two settled implications and their different value, the double occurrence of the universal quantification, and the IS, which contains infinitely many inferences. Very often the students do not grasp this chain because they see the variable n simply as a generic fixed number and do not reflect on the role of quantification. The *conceptual difficulties* are related, on the one side, to the essentiality of both the BS and the IS, often not grasped by the students, and the link between them for the validity of the PMI as proof method. On the other side there is the apparent circularity between what we assume (the IH) and what we need to prove: the students do not control the changes of role of the proposition $\mathrm{P}(k)$ in the inferential chain (from thesis in a certain step to hypothesis in the subsequent step) and above all they do not understand why this process assures the general validity of the proposition. The *technical difficulties* concern the algebraic manipulation, i.e., the capability to face the specific form of the proposition to be proved. Indeed, sometimes it is necessary a strategic way of thinking and the capability to orient in an original way the transformations of the IH towards the desired aim, activating forms of anticipatory thought (Bell, 1976; Boero, 2001). Finally, we call *psychological difficulties* those that depend on a sense of discontinuity between the PMI and the mathematics previously learned, and those arising by the expression "principle of induction", which in the common language and for empirical sciences means the generalization by few specific cases.

Several scholars proposed interesting suggestions aimed at fostering a better understanding of the PMI. In the following we review the approaches that mainly informed our teaching/learning path. Avital and Libenskind (1978) introduced a naïve approach to mathematical induction, based on the verification of the BS and of some local implications $\mathrm{P}(k) \to \mathrm{P}(k+1)$ for successive values of k. Precisely, students should be invited to check the validity of the statement $\mathrm{P}(n)$ to be proved for the first value of n, say $n = 1$, verified by the BS, and for a few other consecutive values. They should show how P(2) follows from P(1), P(3) from P(2), and so on. In agreement with Avital and Libenskind (1978), we think that, through this approach, the common structure of each specific inference should suggest the structure of the proof of the IS. Moreover, the sharp connection between the BS and the IS should arise, making the proof via PMI clear and explaining.

Analogously Cuoco and Goldenberg (1992) proposed that students should be induced in recognizing *self-similarity* of the stream of syntactical transformations. This focus on an invariant structure arose also in the study by Harel (2001), where the author talked about *generalization of result-pattern* and *generalization of process-pattern*. According to these notions, a statement turns out from the regularity of a result or a process respectively. In the quoted paper Harel proposed that the conceptual learning of the PMI springs from the progressive assimilation of the generalization of process-pattern ("quasi-induction"). Namely, first students should become able to apply the PMI autonomously and spontaneously and then they should conceptually recognize it as a method of proof. According to Harel (2001), in "quasi-induction" students recognize the implication $P(n) \rightarrow P(n+1)$ as the last *inference step* needed for assuring the truth of the proposition $P(n+1)$, starting from the BS; instead in mathematical induction $P(n) \rightarrow P(n+1)$ is a variable *inference form* which represents the generic ring of the entire sequence of inferences.

In our path the notions of generalization of result- and process-pattern are used to foster the students' deepening of the meaning of the IS; the concepts of inference steps and inference form help us to analyse the progress of students' understanding of the PMI. If 1 is the first natural number for which $P(n)$ holds, we go through the verification of the local inferences $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, etc., which should suggest how to prove the generic implication $P(k) \rightarrow P(k+1)$ through the generalization of a process. This step should be made explicit also to promote the shift of meaning of k from generic number to a variable in $\mathbb{N}$. Moreover, to bring the students to conceive the IS as the synthesis of the infinite chain of the successive implications starting from $P(1) \rightarrow P(2)$, the introduction of the universal quantifier is essential. Surprisingly, this focus on quantification does not appear in most studies on the PMI.

We believe that to induce the transition from the inference steps view to the inference form view, and, most importantly, the view of the entire sequence of inferences, the key point is to make explicit the role of the universal quantification.

We would like it spontaneously emerges from the experiences we have designed for students and from the meanings they attribute to the mathematical induction.

Summarizing, from the literature on PMI the need for constructive approaches arose, taking into account critical issues about the traditional teaching of the PMI and suggestions for remedial: the introduction of the statement of the PMI, often done abruptly, should arise from exploration and conjecturing phases; it would be suitable to make the students feel the need for proofs by PMI, rather than describe the method as a recipe to be applied; students should be required not only to prove statements, but also to produce conjectures; analogies and metaphors should be suggested in order to make clear the connection between the PMI and the structure of the set of natural numbers; students should be exposed to heterogeneous kinds of problems and statements, and not only to finite sums, to be faced via the PMI. All these suggestions have been considered in the design of our educational path.

Roles of the teacher as a model of aware and effective behaviours

The analysis of the various phases of the educational path is realized by referring to the roles played by the teacher who *"poses him/herself as a model of aware and effective attitudes and behaviours for students"* (Cusi & Malara, 2015). The authors identified some behaviours that such a teacher should display during the classroom activities; he/she should:

a) be able to play the role of an *investigating subject*, stimulating the students' attitude to the research, and acting as an *integral part* of the class towards the shared research aim;

b) be able to act as a *practical/strategic guide*, sharing knowledge with students, and as a *reflective guide in identifying practical/strategic models* during the classroom activities;

c) be aware of his/her own responsibility in maintaining a *harmonic balance between semantic and syntactic aspects* in using the algebraic language;

d) *stimulate and provoke* the enactment of fundamental skills for the development of thought processes and play the role of *activator of interpretative processes* and of *anticipating thoughts*;

e) foster *meta-level attitudes*, being an *activator of reflective attitudes* and *of metacognitive acts.*

Methodology

The context of the study

The project has been carried out in the second term of the academic year 2018/19, within an optional course of 48 hours for Computer Engineering sophomores at Polytechnic University of Marche. The class consisted of about 30 attending students. The PMI was a major topic of the course, and the other subjects treated were propositional logic, sets, relations and functions, natural numbers and divisibility, algebraic structures. Each specific content was taught with the aim to give an overview on the mathematical proof, a theme which was an underlying *fil rouge* along the course (Lolli, 2005).

The educational methodology

From the first lectures, the course was dialogue-based, to favour the interactions of the students with the teacher and the peers, and laboratory-based, to foster the students' work in groups, communication about mathematics and argumentation. Both these features are non-traditional at the tertiary level of instruction, at least in Italy, and allowed us to involve students in subsequent cycles of 1) *exploration*; 2) *formulation* of questions and their solution supported by argumentations; 3) *refinement* of the productions and meta-reflections. In these cycles the individual or small group work are intertwined with collective discussions. These choices are in tune with the recent trends of research studies at university level (Jaworski & Matthews, 2011).

The PMI was the focus of four lectures for an amount of ten hours (excluded the role-play, see below). In the lessons preceding the treatment of the PMI, the concepts of logical implication and *modus ponens* were treated. The students had already met the PMI in the preliminary course, before their entrance at the university. According to the hypotheses discussed above, we aimed to (re-)introduce the PMI with a non-traditional approach, in order to bring the students – after some opportune mathematical experiences – to formulate themselves the principle. We especially wanted the students to rediscover the PMI, without previous conditioning. This is why we abstained from mentioning words and symbols traditionally linked to the PMI, such as "recurrence", "induction", "P(n)", "step", etc.

During the educational path, we collected many oral and written productions by the students, to follow their progress. In particular, we collected a) the students' interventions during the audio-recorded discussions made in the lectures or workshops; b) the students' proofs or refutations of conjectures; c) the formulations of the PMI provided by the students working in small groups; d) the students' analysis of fallacious proofs by induction, in tune with Brumfield (1974), and the formulation provided by the students of false proofs by induction, together with the reasons of their fallacy; e) the students' answers of a cognitive formative test; f) the students' answers to a metacognitive questionnaire.

The last phase of the project – excluded from the ten hours devoted to the PMI – , has been organized as a role-play. During the role-play students played first the role of teachers who assign problems about the PMI, then their role of students who solve problems (assigned by a peer), and finally the role of teachers again, since they evaluated the solutions of the problems (produced by a peer) on the basis of some provided criteria of correctness, completeness, and proficiency. All the steps of the role-play also envisaged an argumentative phase, sometimes consisting in a metacognitive analysis – like the discussion of the difficulties encountered in learning the PMI –, sometimes consisting in the justification of given solutions or responses – like the motivations for which a specific problem is considered simple or difficult.

The research methodology

For each phase, 1) the audio recording of the lectures, 2) the teachers notes about the students' behaviours and her own behaviours and feelings, and 3) the students' productions; have been analysed. This analysis has been done taking into account: a) the indicators offered by the quoted literature; b) the didactical choices and roles assumed by the teacher. A specific attention has been given to the students' proofs and argumentations, classifying them according to the ways of reasoning, the detected difficulties, and the argumentative styles, also highlighting the beliefs and awareness emerged. For each lecture, a study of the teacher's choices and behaviours has been done, also taking into account the teacher's notes about her feelings and actions carried out.

The Experimental Educational Path

Our educational path has been planned with the aims:

a) to engage students at all the levels of learning and to provide students with a set of experiences of exploration and conjecturing activities, bringing them to formulate themselves the PMI by means of a co-construction of a personal comprehension of the principle with their teacher and the peers;

b) to make the IS and the structure of its proof emerge from particular inferences. This should clarify the role of the BS and the IS and their link, by means of a specific focus onto 1) the meaning of the letter involved in the statement as variable; 2) the need and the role of quantification. These aspects should foster the perception of proofs via PMI as clear and convincing.

c) to improve and observe the progressive development of students' mastery of complexity, in connection with the teacher's choices and aware behaviours.

We will only focus on some of the key phases of the educational path, discussing the main results emerged according to the aims of the research.

The exploration and conjecturing phases

The first lecture on the PMI was devoted to: i) highlight the difference between empirical and mathematical laws; ii) propose some exploration and conjecturing activities that should allow students to construct, according to Dubinsky (1986, 1989), the mental schemas needed for deeply understand meanings and processes underlying the method.

First, students were exposed to activities aimed at making them perceive the need for proof of a mathematical proposition (see for example Avital & Libenskind, 1978; Carotenuto *et al.*, 2018; Nardi & Iannone, 2003; Ron & Dreyfus, 2004). The first activity was mediated by a storytelling approach, chosen to engage students and foster their learning at the affective level (see, for example, Albano *et al.*, 2016). The teacher exposed the fictional adventure of the agent Blazkowicz (the main character of a widespread videogame), who fails in his mission as an infiltrator for taking an empirical observation as a law. After a classroom discussion about the differences between empirical and mathematical laws, the exploration of the series of numerical equalities shown in Table 1 – some of which cannot be generalized – was carried out, and the formulation of conjectures about them was required to the students. These problems are in tune with those proposed in Stylianides *et al.* (2016) to encourage a work of exploration and not only the proof of a known result.

On the one hand, this was aimed to accustom the students to formulate conjectures by generalizing a finite number of observations, according to the notion of *cognitive unity* (Boero *et al.*, 1996) and, on the other hand, to make explicit the need for a proof to accept the validity of a proposition and for a counterexample to sanction its falsity. In this phase, the teacher explored the sequences with the students and posed herself as an *investigating subject* (Cusi & Malara, 2015), orchestrating the students' interventions and acting as an element of the class towards the shared objective; at the beginning her interventions aimed at scaffolding attitudes and behaviours in the students, but progressively her support faded.

To give an idea of the classroom interactions during this phase, we report some students' interventions. During the collective analysis of case 1 in Table 1 – the first one faced with this method – students proposed conjectures in the following terms. S1: *"The difference between the sum in a row and the sum in the previous one is the last even number added"*, S2: *"We obtain always odd numbers"* and S3: *"We obtain always prime numbers"*.

Case (i)	1	2	3	4	5
Exploration	17+2=17 17+2+4=23 17+2+4+6=29 17+2+4+6+8=37 17+2+4+6+8+10=47 …	1=1 1+3=4 1+3+5=9 1+3+5+7=16 1+3+5+7+9=25 …	$1-\frac{1}{2}=\frac{1}{2}$ $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\frac{1}{3}$ $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{4}$ …	41+2=43 41+2+4=47 41+2+4+6=53 41+2+4+6+8=61 41+2+4+6+8+10=71 …	1=1 1+2=3 1+2+3=6 1+2+3+4=10 1+2+3+4+5=15 …
Conjecture C_i (by the students)	The sum of 17 with the first n positive even numbers is a prime number.	The sum of the first n odd numbers is n^2.	$\prod_{k=2}^{n}\left(1-\frac{1}{k}\right)=\frac{1}{n}$	The sum of 41 with the first n positive even numbers is a prime number.	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Is the conjecture true?	No Refutation done mainly by the teacher	Yes Proof done in collective interaction	Yes Proof done mainly by the students	No Refutation done mainly by the students	Yes Proof done mainly by the students

Table 1. Exploration and conjecturing activities

All these purposes show that students focused only on one side of the equalities; all the sentences but the first one suggest that students did not look for a general formula through a combined analysis of two or more consecutive rows. Through a collective discussion, the previous statements were clarified and rephrased, giving them the form of conjectures, and they were written on the blackboard. This refinement forced students to find out a general structure of both sides of the equality; they had to use the summation symbol or the dots and write each side of the general formula P(n) by connecting it with the corresponding natural number n.

In this phase, some critical issues arose: the students displayed not being used to have an active role during the lectures, especially to explore and conjecture, and they had many linguistic difficulties. For example, in conjecturing about case 2, S4 said: *"The sum of the first ciphers is the square of the number of the ciphers: the first one is 1, the second one is 2, the third one is 3 and so on"*. In this intervention, we can highlight an attempt to take into account both sides of the equality, but also a poor linguistic capability in explaining what the left-side is: the term "cipher" is improperly used and it is not paid attention to what numbers are added there.

However, the linguistic capabilities and the handling of symbols improved progressively during the teaching/learning path, as we will discuss later.

Refutations and proofs. From the first local implications to their generalization

This step was devoted to confuting or proving the above conjectures. For the conjectures that students felt to be true, the teacher suggested collectively arrange a proof starting from the verification for the smallest value of n, say the first case, and then showing that from the first case the second follows, from the second case the third follows and so on, through *local implications*. Students were organized in small groups of three people and required, for each fixed conjecture, to prove a specific local implication. The productions of the groups were collected and shared with the classroom by the teacher, who highlighted the common structure of the local implications and brought the students to grasp the opportunity of proving a *generic local implication*, that is the implication from a generic value to the subsequent one.

Moreover, in order to state the general validity of the proposition, the insertion of the universal quantifier is a key point, allowing to pass from the *generic local implication* $P(k) \to P(k+1)$ to the *general implication* $P(k) \to P(k+1)$, for all k.

For example, about conjecture C_2 in Table 1, the teacher provided the classical geometrical interpretation involving the gnomons, that suggests the syntactic transformations needed to prove first the passage from a fixed value to the subsequent one (local implication) and then the passage from the generic value k to $k+1$ (generic local implication). Finally, the teacher stimulated the students metacognition, aiming at making the need for quantification arise, in order to prove the conjecture for all the cases. These interventions fit for the teacher's role as an *activator of interpretative processes, metacognitive acts and anticipating thoughts*. Indeed, the teacher favoured the students' insight to the identification of steps needed to construct a proof by PMI. Moreover, she fostered the coordination between different semiotic registers (the arithmetic one and the graphical one) to make the students see first the *generic* and finally the *general* in the particular (Mason, 1996; Mason & Pimm, 1984)

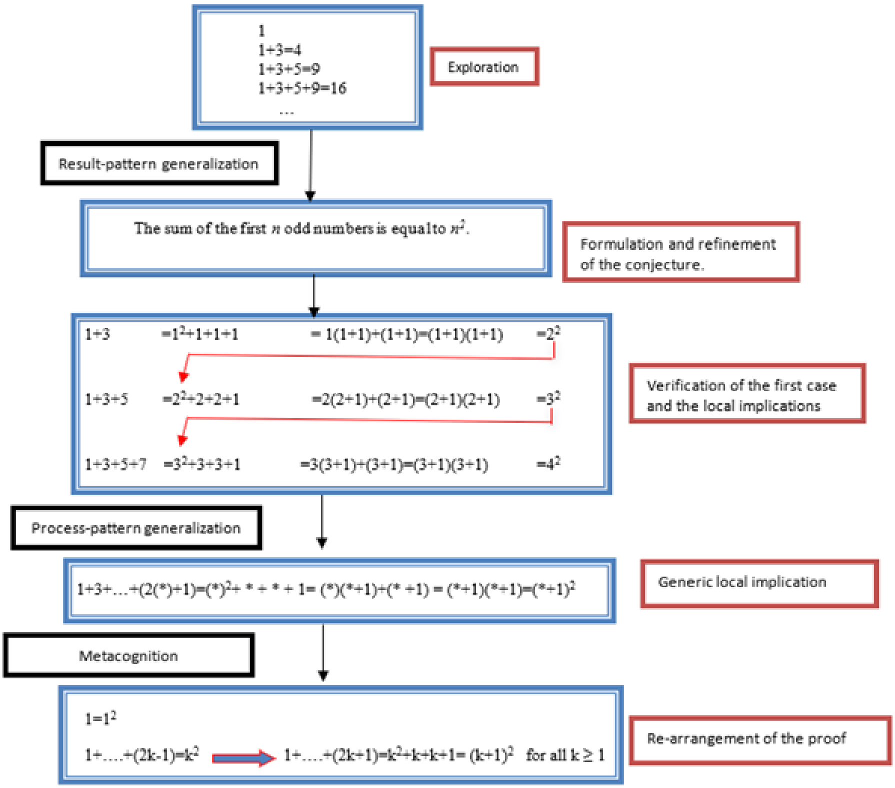


Figure 1. Schema of the activity of the first lesson, from the exploration to the proof of C_2

As an example about false conjectures, let us consider the students' behaviours when facing C_4 (Table 1). Students, generally doubtful about the truth of C_4, used different approaches: some of them looked for a counterexample by the direct analysis of the cases, but they were not able to find it; some other students tried to arrange a proof of its validity. One of them, after a failed attempt, declared *"Maybe the difficulty we have in finding a proof for this conjecture suggests that it does not hold in general"*, referring to the heuristic function of the mathematical reasoning.

In the described activities of conjecturing, refuting and proving, the students should not only recognize the *self-similarity* (Cuoco & Goldenberg, 1992) in the stream of calculation to formulate a conjecture, but also make it visible by highlighting the common structure of the local implications. In other terms, both the *generalization of the result-pattern* and the *generalization of the process-pattern* (Harel, 2001) should be applied, the former when the students found out a conjecture from the exploration, and the latter when the proof of the general implication was obtained as a generalization of the local implications (Figure 1). Both the generalizations are here a remarkable signal of a mature mathematical way of thinking: when students generalized the result-pattern, they were aware that their conjecture could be false, since it was an extension from a finite number of cases. The exposure of the students to fallacious extensions of empirical laws was just aimed to reach this awareness. Just the reflection, fostered by the teacher, on the soundness of the provided reasonings as proofs of the conjectures made the need for the quantification in the IS essential. So, it turned out that there is a chain of local inferences that, from the verification of the statement for the first value, allows to assert the truth of the statement for any value.

The students' formulation of the PMI

After some experiences with propositions of different kinds (equalities, geometrical statements, sentences in verbal language, inequalities), other working groups of three people were formed and required to state the method used to prove the true conjectures, together with the reasons of its validity. In Table 2 we report some prototypes of the groups of students' productions.

The statements have different degrees of formalism, from the very discursive form of G_3 to the formal and symbolic statement of G_2. All the formulations highlight a clear effort to fix the structure of the PMI, which was one of our learning goals. Some elements reveal that the formulations arose from the students' experience during the exploring and conjecturing phase; for example, G_1 used the term *"conjecture"* and focused on the operational steps needed to arrange a proof via PMI, while G_3 showed to understand the link between BS and IS and to conceive IS as a chain starting from the *"verification of some first cases"*.

Some linguistic inaccuracies afflict the protocols and display the students' difficulty in considering the statement to be proved as a propositional form that can be evaluated in the set of natural numbers: for example, G_1 wrote *"assume* *the k-th case is true"*.

Group	Formulation of the PMI	Reasons of validity of the PMI
G_1	Taken a conjecture, in order to verify its truth, I must consider three essential steps: 1) prove that the proposition is true for k_0 (k_0 not necessarily 0); 2) assume that the k-th case is true, $k \geq k_0$, and prove the case $k+1$; 3) from the points 1) and 2) it follows that the proposition is true for every $k \geq k_0$.	The method works well since I can construct a chain for the validity of the proposition: for the first case k_0, it is true (point 1)); by applying point 2), we can say that it is true for k_0+1; by applying point 2) again, we can say that it is true for k_0+2 and so on.
G_2	Let P be a property. P holds for all the natural numbers if: 1) $P(0)$ holds and 2) the implication $P(n)$ holds $\rightarrow P(n+1)$ holds is true for all $n \in \mathbb{N}$.	The method is valid since, for every n, there is a chain of inverse implications in which hypothesis and thesis interchange their positions; the truth for the value n is led back to the truth for the value $n-1$, then to the value $n-2$, up to the value 0, for which the property is true.
G_3	The proof method that we used allows proving a proposition through the verification of some first cases and the possibility to deduce, for every k, from the k-th case the next one.	It is as to go up on a ladder with infinitely many rungs. I have the first rung and I know that I can go up on the second one. When I am there, I know that also the third rung is safe and I can go up. So, if I want to arrive at the n-th rung, I know that I can do it since I know that from the $(n-1)$-th rung I can reach the n-th one; but I can reach the $(n-1)$-th if I can reach the $(n-2)$-th and so on until I say that I can reach the second rung if I can reach the first one. But I verified that I can go on the first rung, hence I can go up to the n-th rung.

Table 2. Some groups' formulations and justifications of the PMI as a method of proof

Moreover, G_2 wrote a) *"inverse implication"*, meaning to go backward in the deductive process, and b) *"hypothesis and thesis interchange their positions"*, which seems to refer to the "change of state" of the proposition P(n), that first is the thesis of the implication $\mathrm{P}(n-1) \rightarrow \mathrm{P}(n)$, and then becomes the hypothesis in the subsequent implication $\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)$. Albeit the linguistic imprecision, this protocol suggests that the students of group G_2 understood the PMI mechanism of transmission of the truth of the proposition P(n) from a natural value to the subsequent one.

In order to explain the reasons of validity of the method, some groups referred to metaphors or mental images (G_1, *a chain*; G_2, *a ladder with infinitely many* *rungs*), some of which were suggested by the teacher during the previous lessons. These mental images, the utility of which is widely recognized (Ernest, 1984; Harel, 2001; Nardi & Iannone, 2003), seemed to be preferred by the students because they are insightful and facilitate the communication of the mathematical ideas. Moreover, they induce students to reflect on the structure of the set of natural numbers, and on the essentiality of the concepts of "first element" and "successor", that in the PMI method mirror themselves in the "base" and "local generic implication" step. Some groups provided the reasons of validity of the used method by referring to a personal construction and to a dynamic process involving themselves (G_1: *"I can construct a chain"*, G_3: *"I have the first rung and I know that I can go up on the second one. When I am there, I know that also the third rung is safe, and I can go up. So, if I want to arrive at the n-th rung,..."*).

From a general point of view, the overall analysis of the protocols of the ten groups highlights that, except for a group that misunderstood the request and did not provide a formulation of the method:

(1) all the groups identified correctly the logical structure of the PMI;

(2) most groups (6 of 9) did not indicate explicitly the quantification in the IS, while almost all inserted the quantification in the conclusion;

(3) only 5 groups of 9 used a generic first value for which the proposition holds, while the remaining fixed the first value as 0 or 1;

(4) 6 groups showed to guess the reasons of the validity of the PMI as a method of proof (mainly referring to inference steps rather than to inference form) and mentioned suitable mental images, mainly associating the method to a dynamic process of transmission of the validity of the statements;

(5) only 2 groups used an impersonal formulation of the PMI.

The outcomes of this phase suggested some critical points to be fixed and the learning needs of the students, having in mind the learning goals that should be achieved. For example, point 2) induced the teacher to project and enact further activities, like inviting students to come back to previously faced propositions and reflect on them, focusing on the meaning of the IS, from which the need of quantification arises. Also, it turned out to be essential to bring the students to a more mature view of the principle. Indeed, the groups' reconstruction of the PMI as a method of proof was mainly oriented towards the sequence of n inference steps, assuring the truth of P(n) starting from the BS, rather than towards the inference form, that is the view of the generic implication $\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)$ as representative of a ring of the entire sequence of implications (Harel, 2001). In other terms, the students saw mainly the potential infinity, which allows proving the truth of P(n), for each n, by considering the implications linking P(n_0) and P(n), but not yet the actual infinity of the whole sequence of implications. We believe that this view can be reached only by the awareness on the role of the universal quantifier in the IS. This suggests the need to force the students' view of n as a variable in $\mathbb{N}$, and therefore to introduce the quantification in the IS, that enriches the meaning of the sentence, allowing to pass from the vision of potential infinity, intrinsic of the PMI, to the actual infinity of $\mathbb{N}$.

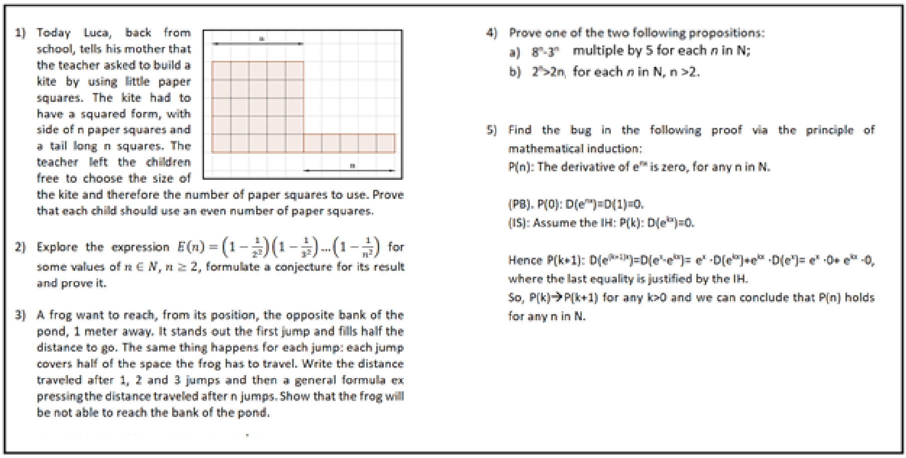
As regards point 3), the teacher opened a collective discussion on the inequality $n^2 < 2^n$ through stimulus questions such as i) for what value it is true? ii) How can we prove the inequality for these values? During the analysis of the proposed inequality, the students observed that $n^2 < 2^n$ holds for $n = 0$ and $n = 1$ and some of them tried to prove the inequality by induction, starting from one of these values, but they are not successful. So, through a direct analysis of the cases for small values of n, it was collectively remarked that the inequality does not hold for $n = 3$ and $n = 4$. *"But for n quite large, the inequality holds, since* 2^n *is an infinity of degree greater than* n^2*!"*, said a student, while other ones justified the same conclusion by referring to the graph of the functions $y = 2^x$ and $y = x^2$. So the students were induced to search the first value for which the inequality definitively holds and finally they found $n = 5$. Hence, they concluded that it is possible to prove the above inequality by induction with BS $n = 5$, and the proof via the PMI was done in the classroom.

All the described activities were aimed to collectively obtain the rigorous formulation of the PMI in quantified form, with a generic first value for the BS.

The role played by the teacher in this phase of the educational path was that of *activator of reflective attitudes*. She was also *responsible for the balance between semantic and syntactic aspects*: she fostered the students' awareness about the value for the BS and she acted so that the quantification in the IS emerged from the students' experience. Moreover, she invited students to adjust by themselves their formulation of the PMI in light of the elaborated reflections.

The formative cognitive test

In order to test the research hypotheses, evaluate the educational path and gain information about the students' level of knowledge about the PMI, a formative cognitive test was submitted to them (see Figure 2). The test includes five problems, having different items (prove that ...; explore ..., formulate a conjecture and prove it; find the bug in the following proof ...; find a model for ...) and referring to statements of different kinds (equalities, inequalities, divisibility statements, verbal language sentences). According to our research foci, the items are aimed not only to evaluate the students' capability to prove statements via the PMI, but also their handling of the structural complexity of the PMI and their use of the PMI as a tool to explore situations and make conjectures.



1) Today Luca, back from school, tells his mother that the teacher asked to build a kite by using little paper squares. The kite had to have a squared form, with side of n paper squares and a tail long n squares. The teacher left the children free to choose the size of the kite and therefore the number of paper squares to use. Prove that each child should use an even number of paper squares.

2) Explore the expression $E(n) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{n^2}\right)$ for some values of $n \in N$, $n \geq 2$, formulate a conjecture for its result and prove it.

3) A frog want to reach, from its position, the opposite bank of the pond, 1 meter away. It stands out the first jump and fills half the distance to go. The same thing happens for each jump: each jump covers half of the space the frog has to travel. Write the distance traveled after 1, 2 and 3 jumps and then a general formula ex pressingthe distance traveled after n jumps. Show that the frog will be not able to reach the bank of the pond.

4) Prove one of the two following propositions:
a) 8^n-3^n multiple by 5 for each *n* in N;
b) 2^n>2n, for each *n* in N, n >2.

5) Find the bug in the following proof via the principle of mathematical induction:
P(n): The derivative of e^{nx} is zero, for any n in N.

(PB). P(0): $D(e^{0x})=D(1)=0$.
(IS): Assume the IH: P(k): $D(e^{kx})=0$.

Hence P(k+1): $D(e^{(k+1)x})=D(e^x \cdot e^{kx})= e^x \cdot D(e^{kx})+e^{kx} \cdot D(e^x)= e^x \cdot 0+ e^{kx} \cdot 0$, where the last equality is justified by the IH.
So, P(k)→P(k+1) for any k>0 and we can conclude that P(n) holds for any n in N.

Figure 2. The cognitive test about the PMI

Problems 1 and 3 are the core of the test, since they involve a modeling part, essential for engineering students: a statement should be formulated, proved and finally the result should be interpreted according to the context. Instead, problems 2 and 4 are proposed since also the low achievers should be able to solve them, so avoiding detrimental frustrations. Problem 2 requires the formulation of a simple conjecture that can be proved by straightforward manipulation. Problem 4 does not present specific difficulties in the setting of the solution, but it requires an alignment between *conceptual insight* and *technical handle*, according to the notion of *relational necessity* mentioned in Stylianides *et al.* (2016); indeed, the combination of students' skills and the posed problem requires a deepening of the underlying mathematical relations to find out the solution, and not only blind algebraic transformations. Finally, problem 5 aims at evaluating the students' understanding of the fallacious proofs.

Generally, the students' productions displayed a full comprehension of the PMI structure, the quantifications and the essentiality and independence of the

BS and the IS. Most students indicated clearly all the steps of their proofs via PMI and the corresponding actions, using expressions like *"I will proceed by induction"*, *"I verify the BS"*, *"I will prove the IS"*, *"Let us assume the IH"*, *"I use the IH"* and whatnot (see Figure 3, black boxes).

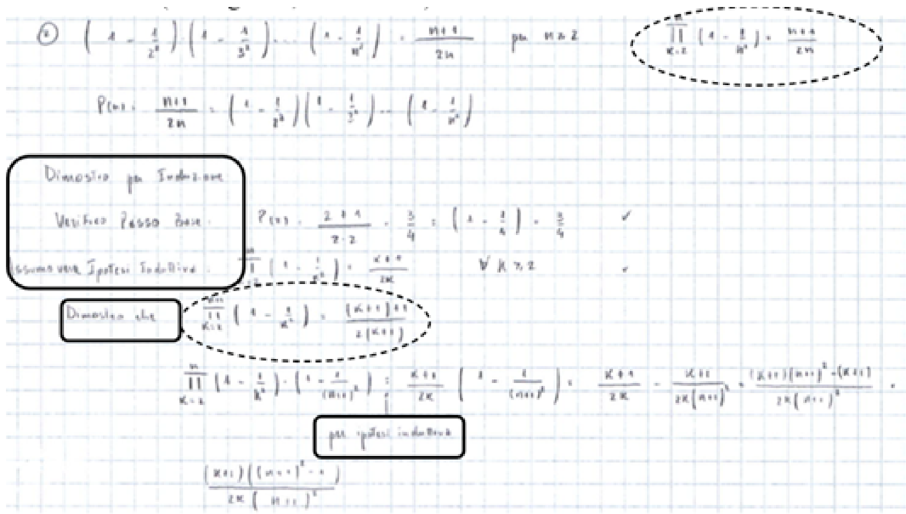


Figure 3. A protocol of the cognitive test; the student sketches out the steps of the proof (black boxes), but does not handle correctly the indices in sums (dotted ellipses)

All but one student recognized and chose correctly the value defining the BS for all the proposed proofs. However, some students were not able to conclude the proof because of troubles in modeling the situation of problem 3, or for algebraic and interpretative difficulties, that emerged mainly in cases of the statements involving inequalities and concerning divisibility of problem 4. Indeed, in these cases generally the proofs are not straightforward and require goal-oriented syntactic transformations. As a positive aspect, we observed that some students showed a good metacognitive effort in recognizing their difficulty in algebraic manipulation, since they wrote declarations like *"I cannot go on"* or *"I am not able to continue to prove the IS"* (see Figure 4).

Moreover, also in problem 2, concerning an equality, some protocols present inaccurate writings; for example, some students used incorrectly indices in sums, as in the expressions highlighted by dotted ellipses in Figure 3. Finally, a group of

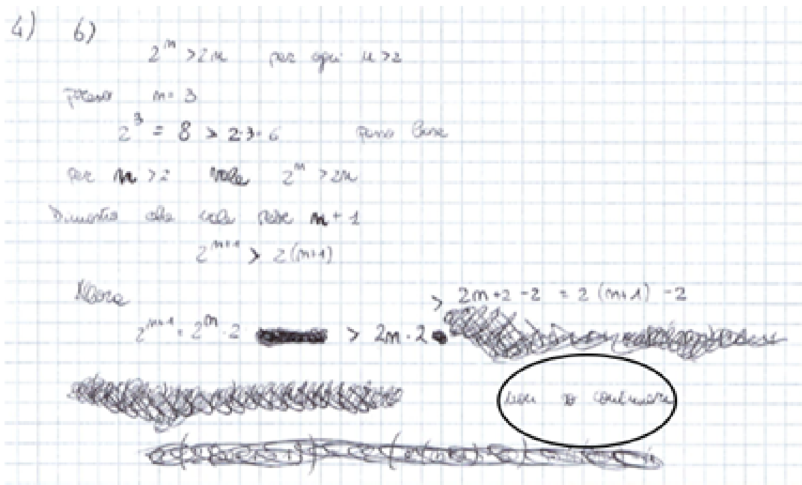


Figure 4. A protocol of the cognitive test; the student writes *"I am not able to continue"*

students did correct algebraic transformations but did not interpret the obtained formulae to conclude the proof, like considering them self-evident.

On the other hand, from a general point of view, the cognitive test displayed that students improved their linguistic, argumentative and reflection capabilities: this also emerged in the metacognitive questionnaire, as we will discuss later. As a further remark, we observe that some students, during their algebraic manipulation, obtained expressions, mainly inequalities, that do not convince them; then, they used again the PMI to prove these inequalities. This can be interpreted as a signal of their perception of the PMI as a proof method that *ascertain*, that is convince themselves, according to the terminology used by Harel (2001).

The metacognitive questionnaire

After the cognitive test, a metacognitive questionnaire, shown in Table 3, was submitted to 28 students. It was aimed at evaluating their perception of the meanings and the processes involved in the PMI, together with its reasons of validity. The design of the questions was informed by the questionnaire presented in Fishbein and Engel, 1989.

The responses to the questions highlighted that most students understood the deep meanings involved in the PMI and the reasons of its validity. In this respect, 26 over 28 students were able to explain the connection of the BS and

1) Giacomo and Luca talk on the PMI. Giacomo says: *"Right now I proved a theorem using the method of mathematical induction, but I don't know if the theorem proved is really true, because I relied on the inductive hypothesis (the truth of the statement for a certain k) and I don't know if this statement is true for this k."* Luca says: *"But it is the base step that ensures the truth of the statement for a certain k: to understand the mechanism of proof, you have to connect the inductive step to the base step. This connection guarantees the truth of the statement for that certain k"*. Do you agree with Giacomo or Luca? Explain your viewpoint and justify your choice.
2) Carlo is doubtful by the PMI proof method. He says: *"In a proof based on the PMI, there is a defect contained in the inductive step. At this stage we suppose that the statement is correct for a generic k; relying on this, we say that it is correct for $k+1$, finally we conclude that the statement is true for all k. I think there is a leap. How is it done by reasoning on a fixed value to conclude that it is valid for everyone?"* Try to explain to Carlo what he is neglecting to consider and why the conclusion is correct.
3) Ugo states the principle of mathematical induction as follows: *"Let $P(n)$ be a proposition that has meaning for natural numbers. If we prove that $P(0)$ is true and that, for some k, if $P(k)$ is true, then $P(k+1)$ is true, then we can conclude that $P(n)$ is true for every n"*. Do you agree with him? Do you think that the statement is correct or should it be corrected? Justify your answer.
4) We know that every non-zero natural is the successor of a single natural number which is its antecedent. Does this proposition come into play in induction proofs? Justify your answer.
5) In light of what has been done in class and in the collateral activities, do you think a proof by induction is only a procedure that guarantees the truth of the sentence or is it a procedure that makes clear and convincing the reason for the truth of the sentence? Justify your answer.

Table 3. The metacognitive questionnaire about the PMI

IS and to reconstruct the mechanism of the PMI. For example, S8 referred to the *"transmission gear guaranteed by the existence of a successor for each natural number, that allows us to connect the BS to the IS; therefore, this assures the truthfulness of the statement for the k that interests us.* S9 says: *"The strength of the principle of mathematical induction is on the link between the BS $P(k_0)$ and the IS, $P(k) \rightarrow P(k+1)$ for every $k \geq k_0$. [...] $P(k_0)$ is the base of the ladder with rungs, where the rung $P(i)$ is based on the previous one $P(i-1)$ and this is based on the previous one $P(i-2)$ and so on...up to $P(k_0)$"*. 23 over 28 students reacted to incomplete and incorrect formulations of the PMI proposed by fictional colleagues (items 2 and 3) by correctly pointing out the fallacies or inaccuracies. For example, S10 answered question 2 as follows: *"Carlo is missing the quantifier* *giving the condition on the k which is possible to choose; the quantifier* $\forall n \geq n_0$*, where* n_0 *is the smallest natural number such that* $p(n)$ *makes sense, indicates those values of k from which it is possible to reconstruct the chain of implications starting from the BS."*

Moreover, the same student S10 corrected the Ugo's statement proposed in question 3 as follows:

"Ugo's statement has two mistakes that should be corrected:

In the explanation of the BS, Ugo writes that $p(0)$ *needs to be proved, but instead we should prove that* $p(n_0)$ *is true, where* $n_0 \in \mathbb{N}$ *[is] the smallest number for which* $p(n)$ *makes sense; in fact, some propositions do not make sense for* $n = 0$*, but can be treated by induction for* $n > 0$*.*

Then Ugo writes that if for some k $p(k)$ *true, then* $p(k+1)$ *true, we can conclude* $p(n)$ *true. But the allowed values of k are just all those greater than* n_0 *and they are the same values for which we can conclude* $p(n)$ *true."*

This response, albeit some linguistic inaccuracies, displays a good awareness on the meanings involved in the PMI, especially on the value of the BS and on the crucial role of quantification in the IS.

Some students (7 of 28) choose to write directly to the fictional students, referring to mental images to foster their understanding. For example, S11 wrote: *"Dear Carlo, in order to better understand the mechanism founding the principle of induction, you can try to think to a ladder with infinite rungs. With the BS we fix the first rung from which we start to construct the ladder. The IS, in which the proposition* $P(k)$ *(or, in our metaphor, the presence of the k-th rung) is assumed to be true, is considered for every* $k \geq n_0$ *([*n_0*] first rung)".*

The effect of the mental images used by 13 students over 28 seems to be very deep, mainly in realizing the explanatory power of the PMI; for example, in his response to question 5 of the metacognitive questionnaire, S12 writes that the method of proof via PMI *"creates in our mind an image of the infinite implications transmitting the truth of the statement* $P(n)$ *from the first value* n_0 *to any natural number greater than* n_0*".* Most students (19 of 28) thought that the PMI as method of proof is convincing of the truth of the statement because of the link between the BS and the IS; this emerged in many answers, like the following one, in which the student S13 referred to the constructive character of the PMI: *"An induction proof clarifies why a statement is true since it gives a method for verifying the statement by means of the BS and from the proof of the IS it is possible to show the truth of the statement for any* $n > n_0$*, just like climbing a staircase from the first step (base step) to the last one".*

Another student wrote that: *"when for a proposition* $P(n)$ *the BS and the IS have been proved, then it is possible to substitute to* n *each value and the proposition* $P(n)$ *will be true!"*; this response displays that the student was able to coordinate the general and the particular view in the understanding of the PMI (Mason, 1996; Mason & Pimm, 1984) and reached a comprehension of the PMI that overcomes the concept of potential infinity and is in tune with the actual infinity of the set $\mathbb{N}$.

Main General Outcomes

The students' productions collected along the educational path allowed to highlight a progressive acquisition of the structural aspects of the PMI. In the cognitive test, all students correctly addressed both the BS and the IS when proving statements through the PMI. Moreover, most students displayed having interiorized the connection between the BS and the IS grounding the validity of the PMI as method of proof. Another good result regards the students' use and understanding of the quantifications in the statement of the PMI. A direct comparison of the students' productions highlighted that although most groups of students missed the quantifiers in the IS and/or in the conclusion in their formulation of the PMI, this did not happen in the final cognitive test, in which almost all students correctly inserted the universal quantifier when applying the PMI as method of proof. Also, we observed a general improvement in the students' linguistic skills and availability to discuss their solutions, communicate their perceptions and reflect on their own difficulties. This can be interpreted as an effect of the constructive and metacognitive educational path taught in a laboratory- and dialogue-based course, where the collective discussion was a standard methodology. At the beginning of the course, students were loath to interact and report their thoughts on the matters of study; moreover, their argumentation capabilities were very poor, since, as they said, it is not usual to discuss about mathematics at university. Instead, at the end of the teaching/learning path, students were much more available in this respect and their capability to share their reflection about activities, difficulties encountered, progresses done and to explain mathematical ideas (and not only to give definitions or statements) definitively increased. This clearly arose by comparing the students' first argumentative productions and the reflections they reported, sometimes spontaneously, in the final part of the educational path.

In answering the metacognitive questionnaire, most students displayed of perceiving the PMI as a convincing method of proof and discussed its explanatory power by referring to mental images such as a transmission gear, a chain or a ladder with infinite rungs. Students displayed good metacognitive control on the logical knots of the statements of the PMI; they were able to identify the mistakes of the fictional colleagues and to correct their statements. This seems to suggest the effectiveness of the designed educational path to support the overcoming of the logical-structural and conceptual difficulties in learning the PMI.

About the role-playing game, it had a clear impact on the students' evolution and on different levels of their learning. Students displayed to be very engaged within the role-play, generally curious of an unusual activity and progressively more available of getting involved. From the cognitive point of view, students have deepened the meanings involved in the mathematical content at stake by means of *changes of perspective*, either posing problems, solving and assessing them. They displayed improvements in handling the structural complexity of the PMI, and on algebraic skills, on the syntactic and the interpretative levels. From the perspective of linguistic competence, students' behaviours improved in a significant way: they strived of being clear in solving problems and experienced the need of clarity of notation and appropriate use of logic syntax when they acted as assessors. From the metacognitive point of view, they have had the opportunity to reflect on their knowledge, as well as on their own learning needs, also taking the advantage of comparing them with the difficulties of their peers. More details can be found in Telloni (2021).

As to the incidence of the different teacher roles on the students' conceptualization and handling of PMI, we remark how her role of *investigating subject* seemed to have favoured the students' understanding about how to pose themselves for searching regularities and how to formulate conjectures; her roles of *strategic and reflective guide*, played reflecting aloud on the meanings of the two premises of PMI, highlighting the effects of the connection between BS and IS, also using metaphors, and analysing false proofs by PMI to favour the recognition of fallacies, seemed to have fostered students' abilities and behavioural traits that widely appear in their proofs and argumentations; the relapses of her metacognitive acts seemed to have effects in the students' argumentations about the PMI structure and their awareness of the explicative power of PMI as proving method.

Conclusive Discussion

This study is focused on the teaching/learning of the PMI and aims to first and second level objectives, concerning the students handling of the structural complexity of PMI and their perception of it as a method of proof. We hypothesized that a constructive educational path paying suitable attention to all the logical knots of the PMI could help to minimize the obstacles encountered by the students. In such a path, the role of the teacher as a model of effective and aware behaviours (Cusi & Malara, 2015) is crucial, and we wanted to identify and enhance these behaviours and their effects on the students' learning of the PMI. We designed and implemented an experimental teaching path aimed at providing students with a set of experiences allowing them the co-construction of the meanings involved in the PMI with their teacher and peers. Differently by some other paths on the PMI suggested by the literature, which envisage the involvement of students for relatively long periods, our educational path is compatible with the time constraints of a semestral university course.

The outcomes of our study suggested the validity of our hypothesis: the designed constructive approach to the PMI was successful in fostering the students' handling of the logical-structural difficulties in learning the PMI raised by the literature. An important aspect concerns the shift of the students' view of the PMI, from the potential infinity to the actual infinity. Indeed, albeit the inductive process is naturally focused on the potential meaning of the infinity, through a specific focus on the introduction of the universal quantifier in the IS and the saturation of the variable, the objectification of the infinite chain of inferences is obtained. In this way the infinity is reified, and hence grabbed in its actual meaning. In this respect, it was important to bring the students to distinguish between the logical concepts of 'free variable' and 'bound variable' and recognize these different roles respectively in the writings $\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)$ and $\forall k \in \mathbb{N}$ $(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1))$. This specific focus on quantification as a key element to convey the actual view of the infinity in the PMI is a new element with respect to the studies on the PMI available in the literature.

Moreover, our path was also successful in making the explanatory and convincing power of the PMI as method of proof emerge, according to the question set by Hanna (1989, 2000) and Stylianides *et al.* (2016). Generally, our students described a positive learning trajectory about the structural comprehension and the control of meanings involved in the PMI, from the first incomplete protocols to the metacognitive questionnaire. Our students, after the experimental teaching-learning path, were able to control and understand the processes behind the structure of the principle and its elements, together with the quantifications, and displayed to be convinced that the PMI not only proves, but also makes clear why a proposition is true.

We believe that the adopted methodology, aimed at fostering the students' deepening of the meanings of the treated mathematical concepts, contributed to the positive outcomes of the educational path. The different roles assumed by the teacher, which are a peculiarity of our path, were displayed to be important elements for the students' conceptualization. They should have favored the improvement of the students' attitude to learning in a critical way and reflect with a meta-perspective on their own learning. This is a first answer to questions – not yet answered by previous research – concerning the teacher's roles and behaviors that can foster a meaningful understanding of the PMI in the students (Dubinsky, 1989; Cusi & Malara, 2008; Ron & Dreyfus, 2004).

Although the effectiveness of the educational path with respect to our focus on the handling of the structural complexity of the PMI, some critical aspects emerged: first of all, many students were not able to conclude proofs via PMI, displaying difficulties in the algebraic manipulation and in the interpretations of formulae. This happened mainly when they faced statements involving inequalities or concerning divisibility. From this perspective, the collected data suggest that our constructive approach supported them in recognizing their algebraic difficulties. Some students, in fact, after having carried out syntactic transformations, concluded *"I cannot go on"*. Then, this kind of approach, although seems to induce a deep understanding of the principle, is not sufficient itself to bring students to correctly apply it; some collected protocols, indeed, highlight algebraic/ interpretative obstacles that prevent the students to conclude their proofs via PMI.

The positive effects on the understanding of the PMI suggest the possibility to implement the educational path, with suitable modifications, at previous levels of instruction. In fact, the methodology used seems to be particularly adequate for secondary school classrooms, where the interaction of each student with the teacher and the peers is simpler than in the university teaching context. In particular, for secondary school students the envisaged steps could be very suitable to connect and distinguish the concepts of "induction" – as an extrapolation of a law from a finite number of cases – and "mathematical induction" – as the general validity of a statement from its truth for the first case and the transmission of validity from any value to the subsequent one. It is worthwhile to recall that many countries where the PMI teaching is anticipated at the upper secondary school. For instance, in our country, the 'Indicazioni Nazionali' (National Guidelines, (MPI, 2010)) underline the opportunity to bring high school students to reflect from the philosophical point of view on the PMI structure so that they can become aware that it is an effective way of proving sentences related to the natural numbers.

Moreover, the experimental teaching path seems to be useful also for pre- and in-service teachers, because of its specific characteristics of aiming second degree goals, improving argumentative capabilities and going beyond technical aspects, by means of metacognition.

Summarizing, we can say that this study gives an innovative contribution on the teaching of the PMI according to the following aspects:

- Role of the teacher as a learning guide on both the cognitive and metacognitive level.
- Formulation of the PMI devolved to the students after exploring and conjecturing experiences and collective proofs constructed from local implications, to the generic implication, to the general implication.
- Explicit focus on the quantification, as a key point to bring students to acquire the actual infinity view about the PMI.
- Explicit focus on metacognition about the link between the BS and the IS to make the students perceive the PMI as a convincing method of proof,
- The designed learning path is compatible with the usual time constraints of the university instruction.

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