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Teaching Mathematics and Computer Science

Artworks as illustrations in Hungarian high school Mathematics textbooks

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Abstract. Three different series of Hungarian Mathematics textbooks used in grade 9-12 education for the past 30 years have been analysed in this research. Our aim is to show and evaluate how the visual arts have been connected to mathematical ideas in these textbooks. We have applied the six dimensions of evaluation, which have recently been introduced in (Diego-Mantecón, Blanco, Búa Ares, & González Sequeiros, 2019) to categorise the illustrations of the three different series. We show examples for each dimension from the textbooks, and we find that even if the number of artistic illustrations in these coursebooks have significantly increased, in most cases these sporadic examples are not closely related to the mathematical context, mainly used for ornamental purposes to decorate the core text. Based on this classification we conclude that the number of artistic illustrations with underlying math concepts making students' participation more active could and should be significantly increased.

Key words and phrases: STEAM education, illustrations, artworks, mathematical textbooks, creative thinking.

MSC Subject Classification: 97U20.

Introduction

As it has been demonstrated through the well-known dual coding theory (Clark & Paivio, 1991), the acquisition of knowledge is done through the combined use of our imaginative, pictorial, and verbal, linguistic thinking skills. The visual, imaginary (non-verbal) effect results in a more direct, the linguistic (verbal) decoding results in a more abstract acquisition of knowledge due to the mutually reinforcing effect of the two. Illustrations are therefore important tools for acquiring knowledge, complementing what is written, and illuminating the path to understanding.

The word 'illustration' comes from the Latin verb illustro. Meaning: illuminates, brings to light, explains. The illustration can be a drawing, photograph of a painting, sculpture or building to illustrate and supplement printed text. To illustrate something is to make a text visible, to explain, to interpret a principle or task.

What do we expect from an illustration? To be illustrative, authentic, clear and interesting, not least to encourage thinking. By combining the image and the concept, offer an opportunity for further thinking and stimulate the recipient to associative thinking. The highest level of cognition is thinking, including mathematical thinking and reasoning. We connect what we see with what we know, so that by interpreting the sight we can come to further insights.

The relationship between the text or computation and the image can fundamentally be of two kinds: on the one hand, when the image displays the content of the text, and on the other hand, when it translates not the content and the text of the work, but its mood elements into its own formal language.

Previous work

The importance of visuality and images in various levels of education and science has been widely studied, as well as the artistic value of scientific images. These educational-philosophical aspects of images and the importance of their aesthetic value are well summarised e.g. in (Benedek & Nyíri, 2012; Elkins, 2010; Cazeaux, 2015; Reinhuber, 2017).

But how these principles and aspects are manifested in textbooks is a more empirical question. A comparative analysis of Mathematics textbooks, including Hungarian textbooks has been conducted by (Sutherland, Winter, & Harries, 2001), but this comparison is very much focused on one specific mathematical aspect (multiplication). Hungarian textbooks has been analyised from the viewpoint of teachers' choice in (Czeglédy & Kovács, 2008). The requirements for a differentiated textbook, and the extent to which the textbook in question meets these requirements are studied in (Czeglédy & Szász, 2005). A retrospective study of Hungarian mathematics textbooks in the light of the notion of function and function oriented thinking has been presented in (Ambrus, Filler, & Vancsó, 2018).

A comprehensive review of Hungarian textbooks have been conducted in (Dárdai & Kojanitz, 2007), including studying the illustrations. They examined, inter alia, the changes in graphics and images in terms of the number of illustrations, the pedagogical functionality of the illustrations, the existence of maps, explanatory figures, diagrams, etc. complexity, comprehensibility, and the relationship between visual appearance and learnability.

Romanian mathematics textbooks written in Hungarian have been evaluated in (Baranyai & Stark, 2011) studying the question if elementary school mathematics textbooks meet the (mathematical) requirements imposed on them. In this latter study aesthetic requirements have also been evaluated, including the appropriateness of illustrations. Teachers themselves assessed this issue, and they rated the quality below averaged from this point of view. A very important note in this paper is the fact that school children pay often as much attention to the aesthetic design as to the content of the textbooks (Baranyai & Stark, 2011). This inspired us to evaluate the illustrations of textbook not purely from the viewpoint of functionality, but also from artistic point of view.

The aim of this study is not to analyze the core mathematical content of mathematics textbooks, but to review, examine and classify the pictorial and visual elements of these books. We have found that the vast majority of current textbook illustrations appear only as a decorative element, as an illustration of the text of a task in the broad sense, or rarely as a presentation of a concept. We hardly find pictures, especially works of art on the levels that require more active participation, concrete, constructive activities and analytical thinking.

Methods to classify illustrations

The vast majority of these figures evidently belongs to to the category of technical drawing, directly supporting the understanding of (mostly geometry-related) problems. But now we are focusing on artworks. To evaluate the figures from this viewpoint, one has to clarify an important initial question: what do we consider as artwork among the illustrations.

In some cases it is quite easy to answer this question. A famous painting or sculpture evidently belongs to the class of artworks, while it is trivially not the case in terms of geometric drawings, such as, e.g., a triangle with its Euler's line passing through the orthocenter, center of the nine-point circle, centroid, and circumcenter. This latter figure has an important role in visualising the geometric relationship of these elements, but it has no real aesthetic value in the classical sense (the study of potential inner beauty of these geometric technical drawings, which may exists, is beyond the scope of this paper).

So what is an artwork? There is no unique scientific definition of this notion. In his seminal work, Danto declares that a work of art is always defined by two essential criteria: meaning and embodiment, as well as one additional criterion contributed by the viewer: interpretation (Danto, 2013)). Meaning and embodiement are also relevant in our specific case: (artistic) illustrations needs to be meaningful and needs to be embodied in the surrounding mathematical context. For our practical classification goals it is sufficient to consider the general description of artwork as a physical item or creation with aesthetic value. In our context, which is consonant to the approach followed by (Diego-Mantecón et al., 2019) this can be

- a piece of fine art, such as painting, etching or sculpture
- a (non-documentary) photograph
- an architectural structure, including interior design

Even this way one may sometimes have doubts as to whether an illustration in the maths textbook can be considered a work of art. In Figure 1 one can see two portraits of famous mathematicians. While one of the (depicting Paul Erdős) is a simple, documentary photo, therefore not considered as artwork, the other one (depicting René Descartes) is a reprint of a painting, thus falls into our artwork classification.

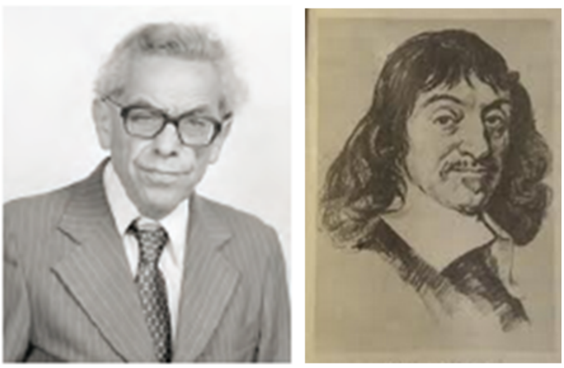


Figure 1. Portrait of Paul Erdős from (Barcza, Basa, Tamásné Kollár, Bálint, & Kelemenné Kiss, 2018) (New generation series, p.58) and René Descartes from (Hajnal & Némethy, 1991a) (it is from Hajnal-series p.147, but the same portrait appears in all of the three textbooks). The right illustration may be considered as an artwork.

In the detailed study of the illustrative artworks we follow the classification recently defined in (Diego-Mantecón et al., 2019). Here we briefly remind the reader to this classification. The authors studied the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks, and they defined six classes (so-called dimensions) according to the depth of the connection between the presented artwork and the mathematical content around that illustration. In their research a textual and pictorial analysis has been conducted in all parts of the textbooks, and the following dimensions have been identified and defined:

(1) artwork for ornamental, decorative purposes (i.e. to make textbooks more appealing);

(2) artwork as a context for calculation and measurement;

(3) artwork as a context for mastering concepts;

(4) artwork as a context for using technological resources in mathematics;

(5) mathematical analysis of works of art; and

(6) artwork created with mathematics.

It is important to note that the six art-mathematics dimensions are not absolutely disjoint subsets of illustrations, they can be overlapped, therefore they are not hermetic compartments, as also declared by the authors. Moreover, learning and teaching activities may involve various dimensions simultaneously. As an example of the authors provided in (Diego-Mantecón et al., 2019), a teaching-learning activity characterized under dimension 5 (mathematical analysis), can normally call for mastering concepts (dimension 3) as well as calculating and measuring (dimension 2). The classification and the dimensions will further be discussed in detail in the next section, parallel to our observations.

Results - evaluation of textbooks and illustrations

Looking back over the past thirty years, we have examined the mathematics textbooks of three textbook series used in secondary education in terms of the illustrations used. For a long period these series have been the most frequently used and de facto standard textbooks in Hungarian schools.

The following textbook series have been evaluated in this study:

(1) Textbooks written by Imre Hajnal and Katalin Némethy (cf. Hajnal series) (Hajnal & Némethy, 1991a), (Hajnal & Némethy, 1991b), (Hajnal & Némethy, 1994), (Hajnal, 1995)

(2) Textbooks published by the Mozaik Kiadó, also known as "Colourful Mathematics" (cf. Mozaik series) (Kosztolányi, Kovács, Pintér, Urbán, & Vincze, 2003), (Kosztolányi, Kovács, Pintér, Urbán, & Vincze, 2014a), (Kosztolányi, Kovács, Pintér, Urbán, & Vincze, 2015), (Kosztolányi, Kovács, Pintér, Urbán, & Vincze, 2014b)

(3) Textbooks developed by the Hungarian Institute for Educational Research and Development, also known as "New generation books" (cf. New generation series)(Barcza, Basa, Tamásné Kollár, Bálint, et al., 2019), (Barcza, Basa, & Tamásné Kollár, 2019), (Barcza, Basa, Tamásné Kollár, Bálint, & Kelemenné Kiss, 2018), (Barcza, Basa, Tamásné Kollár, Bálint, Kelemenné Kiss, Gyertyán, & Hankó, 2018).

There have been significant changes in textbooks in terms of image quality of illustrations, but this is mainly due to advances in printing technology. However as one can observe in Fig.2, the number of figures has not been changed significantly during these years. Although a slight increasing is present, there is no change in order of magnitude in the figure-page ratio.

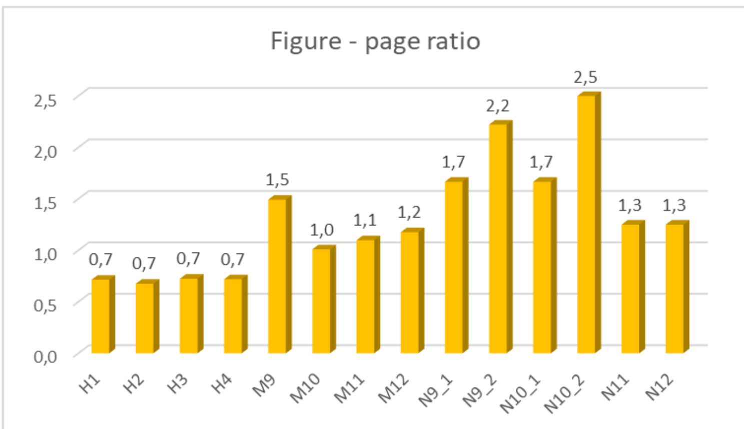


Figure 2. Number of figures in the textbooks (expressed here as figure/page ratio) did not changed significantly (H – Hajnal-series; M – Mozaik-series; N – New generation series)

Even if the overall number of illustrations did not change significantly through the years, the situation is different in terms of artistic illustrations and photos. They is practically negligible in two series (Hajnal-series and Mozaik-series), while it is getting more significant in a New generation series (cf. Figure 3).

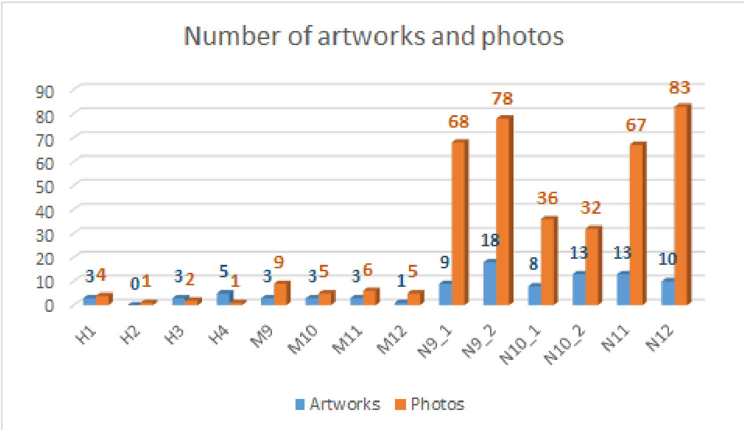


Figure 3. Number of artworks and photos in the textbooks (H – Hajnal-series; M – Mozaik-series; N – New generation series)

After this quantitative survey, now we discuss in detail and provide examples to each dimension in the classification described above. As the authors explain in (Diego-Mantecón et al., 2019), the first dimension is where art is exclusively for ornamental, decorative purposes with no explicit mentioning of mathematics, the artistic element is not necessary even related to the actual text.

This dimension appears more frequently in the recent, technically more advanced series, maybe in order to make the textbook more exciting and more spectacular for the pupils.

An example can be seen in Fig.4 (all specific examples come from the New generation series). It is very typical in this dimension that the text does not refer to the illustration at all.

In dimension 2, artwork may appear in the context for calculation and measurement, the artistic element may provide an overall inspiration, but it is actually not necessary to conduct the task, or understand the mathematical topic.

A typical example can be seen in Figure 5. It is evidently positive that pupils can visually connect the task with the sight of the building, thus understanding the importance of this mathematical topic in real life problems. However, the actual

Figure 4. A typical example of illustration from class 1 (Barcza, Basa, Tamásné Kollár, Bálint, Kelemenné Kiss, Gyertyán, & Hankó, 2018, p. 105) - the photo has a certain artistic, aesthetic value, but it has no clear relationship to the surrounding mathematical topic (actually analyzing graphs)

Figure 5. A typical example of illustration from class 2 (computing a volume of a water tower) - the photo of this beautiful engineering artwork is evidently related to the text, but it is actually not necessary to conduct the task (Barcza, Basa, Tamásné Kollár, Bálint, et al., 2019, Vol I., p.147)

way of illustration still lacks clear support in conducting and understanding the surrounding mathematical text.

In the next dimension (dimension 3) art appears directly as a context for understanding concepts, the artistic illustration is applied to exemplify and improve the understanding of the underlying mathematical notions or theories. However, there is no feedback: the mathematical topic in question is not applied to analyze the artwork.

Figure 6. A typical example of illustration from class 3 (computing the area of the building of Pentagon, using the given scale and photo) - the photo is necessary to the solution (Barcza, Basa, & Tamásné Kollár, 2019, Vol II., p.97)

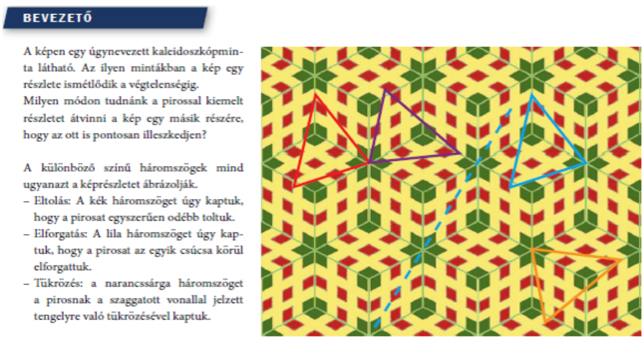
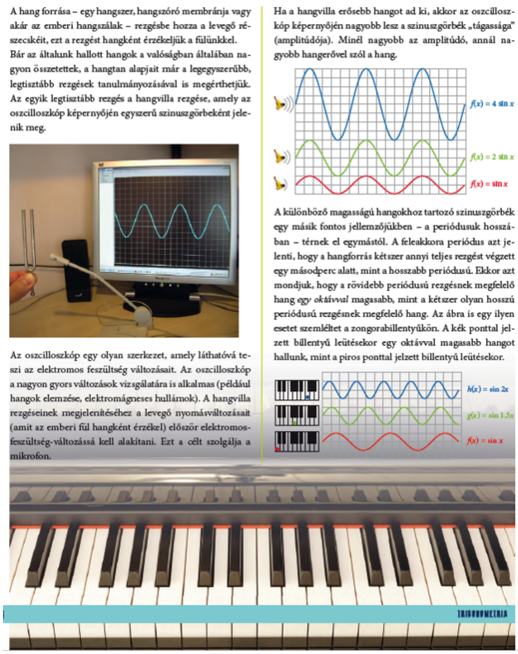


Figure 7. Another example of illustration from class 3 (understanding conguence in a pattern) - the photo is necessary to understand the surrounding text (Barcza, Basa, Tamásné Kollár, Bálint, et al., 2019, Vol II., p.134)

An example can be seen in Figure 6, where the photo of the building is absolutely necessary to solve the computational tasks. Another, more theory-oriented example is in Figure 7, where the textual part is in close relationship to the tiling illustration, but this is in fact a one-way relationship. Tilings are typical borderline cases in terms of artistic pieces. While the illustration in Figure 7 may have less artistic value, using, for example, Escher graphical works for the same purpose would provide more relationship and connection to art.

Dimension 4 intends to create connection between art and technology, where the actual task can be the application of technical resources - typically software and apps - to understand mathematics. A very typical case is the creation of tilings through GeoGebra or the use of Wolfram Mathematica to demonstrate that any quadrilateral can be applied in tilings. This dimension is not present visually in Hungarian textbooks, which is fully understandable in older series. In recent issues there are references to these apps and software tools, but its presence is mostly implicit. One rare example is the one presented in Figure 8, using different frequencies of musical tones displayed by the oscilloscope to explain transformation of functions. Although this is not a direct artwork, the music is definitely correlated to this experiment through tones.


A hang forrása – egy hangszer, hangszóró membránja vagy akár az emberi hangszálak – rezgésbe hozza a levegő részecskéit, ezt a rezgést hangként érzékeljük a fülünkkel. Bár az általunk hallott hangok a valóságban általában nagyon összetettek, a hangtan alapjait már a legegyszerűbb, legtisztább rezgések tanulmányozásával is megérthetjük. Az egyik legtisztább rezgés a hangvilla rezgése, amely az oszcilloszkóp képernyőjén egyszerű szinuszgörbeként jelenik meg.

Az oszcilloszkóp egy olyan szerkezet, amely láthatóvá teszi az elektromos feszültség változásait. Az oszcilloszkóp a nagyon gyors változások vizsgálatára is alkalmas (például hangok elemzése, elektromágneses hullámok). A hangvilla rezgéseinek megjelenítéséhez a levegő nyomásváltozásait (amit az emberi fül hangként érzékel) először elektromos-feszültség-változássá kell alakítani. Ezt a célt szolgálja a mikrofon.

Ha a hangvilla erősebb hangot ad ki, akkor az oszcilloszkóp képernyőjén nagyobb lesz a szinuszgörbék „tágassága" (amplitúdója). Minél nagyobb az amplitúdó, annál nagyobb hangerővel szól a hang.

A különböző magasságú hangokhoz tartozó szinuszgörbék egy másik fontos jellemzőjükben – a periódusuk hosszában – térnek el egymástól. A feleakkora periódus azt jelenti, hogy a hangforrás kétszer annyi teljes rezgést végzett egy másodperc alatt, mint a hosszabb periódusú. Ekkor azt mondjuk, hogy a rövidebb periódusú rezgésnek megfelelő hang *egy oktávval* magasabb, mint a kétszer olyan hosszú periódusú rezgésnek megfelelő hang. Az ábra is egy ilyen esetet szemléltet a zongorabillentyűkön. A kék ponttal jelzett billentyű leütésekor egy oktávval magasabb hangot hallunk, mint a piros ponttal jelzett billentyű leütésekor.

Figure 8. Example of class 4 - the application of oscilloscope to explain transformation of functions (Barcza, Basa, Tamásné Kollár, Bálint, & Kelemenné Kiss, 2018, p.58)

In the 5th dimension, artwork appears in the context of analysis and exploration of mathematics. The aim of the illustration is not to support the understanding of the textual source, but it appears as the primary object of understanding. The surrounding mathematical description is about and for this artwork, the experimental work of students is focused on the art piece itself. This work can be the verification of a theoretical statement, or - more frequently - the understanding of mathematical concept the artist applied when the piece was created.

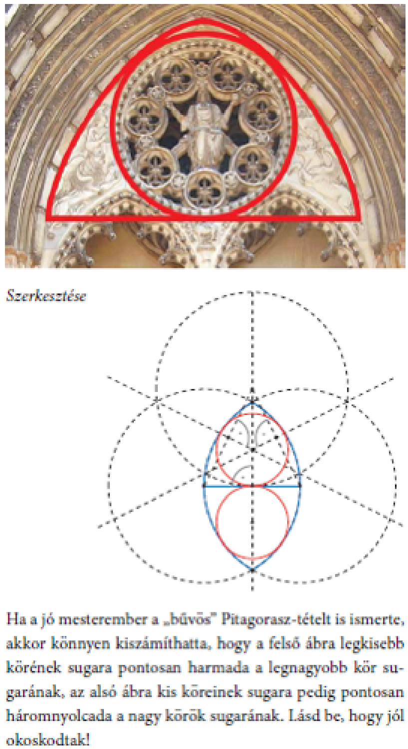


Figure 9. Example of class 5 - the focus of study is on the artwork itself, understanding the concept the architect applied (Barcza, Basa, Tamásné Kollár, Bálint, et al., 2019, Vol II., p.171)

An excellent field of this type of involvement of art is architecture, where the example of Figure 9 also comes from. This is also the dimension where, e.g., golden section may appear in some textbooks, discovering it in classical paintings and sculptures.

Finally in the 6th dimension, mathematics is directly applied for creating artistic images and three dimensional shapes. This would be the essence of the interdisciplinary approach and connection between arts and mathematics. The already mentioned tilings can be shifted to this dimension, too, but three-dimensional geometric shapes and graphs of functions (perhaps of two variables) are also among the potential mathematical field to create artistic values.

Two examples of this dimension, the Klein bottle and a fractal image are seen in Figure 10. Both images are excellent examples for artistic creations that can inspire teachers and pupils to discuss the (not so deep) mathematical concept behind these phenomena.

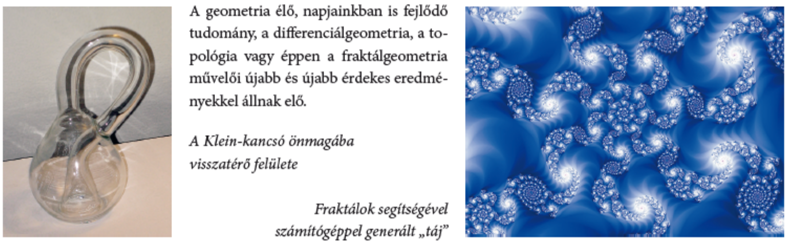


Figure 10. Examples of class 6 - the transparent Klein bottle and the fractal image have clear aesthetic value (Barcza, Basa, Tamásné Kollár, Bálint, Kelemenné Kiss, Gyertyán, & Hankó, 2018, p.172)

Conclusion

Figures and illustrations must be inherent parts of any mathematics textbook, and, as we have seen in the graph of Figure 2, the overall number of illustrations did not change significantly through the years. The situation is different with figures that could bring math students closer to real life and the arts. The number of artistic illustrations and photos is practically negligible in two series (Hajnal-series and Mozaik-series), while it is getting more significant in a New generation series (cf. Figure 3).

This is a very positive change and is only partly due to the development of printing technology, all the more so due to the concept of the series and the commitment of the series authors. Of course, this does not mean that higher number of artistic figures will automatically improve the quality and make the books better in terms of mathematics or education. But it does mean that teachers will be given another opportunity and potential tool to bring mathematics closer to students and possibly show the connection between mathematics and the arts. This is a welcome process and could be completed even more in the upcoming series.

Having that said, there is still room for improvement. Considering the New generation series, where the overall number of artistic illustrations and photos are relatively high, the above mentioned dimensional classification of these illustrations shows strong bias towards lower (less valuable, less inspiring) dimensions. $39,3\%$ of these illustrations belong to the first dimension, and 38.9% belongs to the second dimension.

Even if the classification cannot be absolute precise (there can be few illustrations on the borderline of two dimensions), it is evident that the majority, more than $3/4$ of these illustrations are only for decorative purposes or for inspiration without direct connection to specific mathematical principles or tasks. On the other end of the classification, we have found only one pair of examples for dimension 6 (the one seen Figure 10).

All this means that, although there are vauable positive changes in mathematics textbook design, we could use and exploit the available resources, the technical possibilities and the existing connections in a more thorough way in the upcoming series. Our further aim is to develop a series of tasks and find appropriate artworks for specific fields of mathematics to help to increase the dimensionality of these connections.

References

Ambrus, G., Filler, A., & Vancsó, Ö. (2018). Functional reasoning and working with functions: Functions/mappings in mathematics teaching tradition in hungary and germany. *The Mathematics Enthusiast*, *15*(3), 429–454.

Baranyai, T., & Stark, G. (2011). Examination of mathematics textbooks in use in hungarian primary schools in romania. *Acta Didactica Napocensia*, *4*, 47–58.

Barcza, I., Basa, I., & Tamásné Kollár, M. (2019). *Matematika 10. I-II.* Hungarian Institute for Educational Research and Development, Budapest.

Barcza, I., Basa, I., Tamásné Kollár, M., Bálint, Z., & Kelemenné Kiss, I. (2018). *Matematika 11.* Hungarian Institute for Educational Research and Development, Budapest.

Barcza, I., Basa, I., Tamásné Kollár, M., Bálint, Z., Kelemenné Kiss, I., Gyertyán, A., & Hankó, L. (2018). *Matematika 12.* Hungarian Institute for Educational Research and Development, Budapest.

Barcza, I., Basa, I., Tamásné Kollár, M., Bálint, Z., Kelemenné Kiss, I., Gyertyán, A., & Hankó, L. (2019). *Matematika 9. I-II.* Hungarian Institute for Educational Research and Development, Budapest.

Benedek, A., & Nyíri, K. (2012). *The iconic turn in education.* Peter Lang GmbH, Internationaler Verlag der Wissenschaften.

Cazeaux, C. (2015). The aesthetics of the scientific image. *Journal of Aesthetics and Phenomenology*, *2*(2), 187–209.

Clark, J. M., & Paivio, A. (1991). Dual coding theory and education. *Educational psychology review*, *3*(3), 149–210.

Czeglédy, I., & Kovács, A. (2008). How to choose a textbook on mathematics? *Acta Didactica Napocensia*, *1*(2), 16–30.

Czeglédy, I., & Szász, R. (2005). The mathematics textbook as an aid to differentiation: A first hungarian example. *Teaching Mathematics and Computer Science*, *3*(1), 35–53.

Danto, A. C. (2013). *What art is.* Yale University Press.

Dárdai, L., & Kojanitz, A. (2007). A tankönyvek változásai az 1970-es évektől napjainkig. *Új pedagógiai szemle*, *1*, 56–69.

Diego-Mantecón, J. M., Blanco, T. F., Búa Ares, J. B., & González Sequeiros, P. (2019). Is the relationship between art and mathematics addressed thoroughly in spanish secondary school textbooks? *Journal of Mathematics and the Arts*, *13*(1-2), 25–47.

Elkins, J. (2010). Visual practices across the university. In *Beyond mimesis and convention* (pp. 169–192).

Hajnal, I. (1995). *Matematika IV.* Nemzeti Tankönyvkiadó, Budapest.

Hajnal, I., & Némethy, K. (1991a). *Matematika I.* Tankönyvkiadó, Budapest.

Hajnal, I., & Némethy, K. (1991b). *Matematika II.* Tankönyvkiadó, Budapest.

Hajnal, I., & Némethy, K. (1994). *Matematika III.* Nemzeti Tankönyvkiadó, Budapest.

Kosztolányi, J., Kovács, I., Pintér, K., Urbán, J., & Vincze, I. (2003). *Matematika*

9. Mozaik Kiadó, Szeged.

Kosztolányi, J., Kovács, I., Pintér, K., Urbán, J., & Vincze, I. (2014a). *Matematika 10*. Mozaik Kiadó, Szeged.

Kosztolányi, J., Kovács, I., Pintér, K., Urbán, J., & Vincze, I. (2014b). *Matematika 12*. Mozaik Kiadó, Szeged.

Kosztolányi, J., Kovács, I., Pintér, K., Urbán, J., & Vincze, I. (2015). *Matematika 11*. Mozaik Kiadó, Szeged.

Reinhuber, E. E. (2017). Beyond visibility–scientific imaging as an artistic tool. *Journal of Media Practice*, *18*(2-3), 229–236.

Sutherland, R., Winter, J., & Harries, T. (2001). A transnational comparison of primary mathematics textbooks: The case of multiplication. *Research in Mathematics Education*, *3*(1), 155–167.

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