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Some Pythagorean type equations concerning arithmetic functions

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Abstract. We investigate some equations involving the number of divisors $d(n)$; the sum of divisors $\sigma(n)$; Euler's totient function $\varphi(n)$; the number of distinct prime factors $\omega(n)$; and the number of all prime factors (counted with multiplicity) $\Omega(n)$. The first part deals with equation $f(xy) + f(xz) = f(yz)$. In the second part, as an analogy to $x^2 + y^2 = z^2$, we study equation $f(x^2) + f(y^2) = f(z^2)$ and its generalization to higher degrees and more terms. We use just elementary methods and basic facts about the above functions and indicate why and how to discuss this topic in group study sessions or special maths classes of secondary schools in the framework of inquiry based learning.

Key words and phrases: arithmetic functions.

MSC Subject Classification: 97F60, 11A25.

Introduction

In a previous paper (Kézér, 2018) we presented some ideas of how a topic related to the commutativity of the composition of arithmetic functions can be discussed with students in the 9th grade in the special maths program and summarized the experiences of the group study sessions we had with the pupils. The main objective of the present paper is the same: to create opportunities for students to experiment on their own and to do research, to motivate them to propose questions while working on the material. The goal is to investigate problems - related to special equations concerning arithmetic functions - that offer students the framework for experimentation. Number theory, including arithmetic functions, is one of the few topics in high school mathematics that does not require a tremendous amount of prior knowledge. We think the problems studied in this paper do not require higher mathematical knowledge and are accessible for high school students in the special math program. (Unfortunately, the pandemic prevented so far to test our ideas with the students.) The questions involve different grades of difficulty, and most of them can be divided into parts of different depth. They open up some problem solving methods that students know already and can practice, but also give rise to introducing new ones. We think that the questions we investigate create enough space for creativity while also provide the opportunity for pupils to reach different levels in problem solving. We believe that problem solving should be only partly directed: the teacher should be present to support the students' ideas and help them if needed but the main emphasis is on the students' approach and their communication about the problems. These problems are open-ended, and give rise to many more questions than discussed in this paper. We are confident that students will definitely achieve at least partial results in solving these problems.

The approach meets the framework of inquiry based teaching, which is– according to J. Hattie – " ... the art of developing challenging situations in which students are asked to observe and question phenomena; pose explanations of what they observe; devise and conduct experiments in which data are collected to support or contradict their theories; analyze data; draw conclusions from experimental data; design and build models; or any combination of these. Such learning situations are meant to be open-ended in that they do not aim to achieve a single "right" answer for a particular question being addressed, but rather involve students more in the process of observing, posing questions, engaging in experimentation or exploration, and learning to analyze and reason."(Hattie, 2009).

As A. H. Schoenfeld notes: "By definition, problem situations are those in which the individual does not have ready access to a (more or less) prepackaged means of solution." (Schoenfeld, 1985) Of course, not having a prepackaged path of progress, this type of experimentation also includes the possibility of errors, but it is important to develop and improve the sense in the students during their high school years that mistakes are self-evident parts of any discovery, including mathematical discovery, as well. Citing the always appropriate thought of Gy. Pólya: "There is no such thing as a bad idea, something can only go wrong if we accept it without criticism. The only bad thing is not to have thoughts at all." (Pólya, 1971).

We believe that working on these problems serves as a good example how one can develop and extend a question, how the variants of a question can sometimes result in exactly the same answer as the original problem, and how they can lead to a totally different outcome in other cases and how one can make generalizations. Of course, it is not granted that we can solve the analogue problem or the generalization but it is worth an attempt to consider where the reformulation or the generalization of the original problem could lead. It is important to make pupils notice these kinds of opportunities, so they can manage to find similarities, to compare solutions, and to adapt the main ideas of a certain solution in a modified situation. On this journey we can also highlight some connections of arithmetic functions to several other parts in high school math. So studying this topic, besides being interesting in itself, can be beneficial also in solving diverse problems or finding the basic idea behind a solution.

Arithmetic functions form an important area in number theory. Their basic properties and the other number-theoretic notions used in this paper (Fermat primes, twin prime conjecture, etc.) can be found for example in (Freud & Gyarmati, 2000).

We start with determining all solutions of equation $f(xy) + f(xz) = f(yz)$, where $f(n) = \omega(n)$ and $\Omega(n)$, or more generally, f is a positive strongly or completely additive function. Multiplicative functions $d(n)$, $\sigma(n)$, and $\varphi(n)$ show a more diversified behavior even concerning solutions in prime powers. Some results are also related to the twin prime conjecture and other unsolved problems concerning primes.

The situation for equation $f(x^2)+f(y^2) = f(z^2)$ depends heavily on the concrete arithmetic function: for $d(n)$ there is no solution even for any higher degree generalization; for $\sigma(n)$ there is no solution for any even degree generalization; for $\omega(n)$ and $\Omega(n)$, we have infinitely many solutions, and regarding the latter there exist infinitely many solutions even among prime powers; but there are no solutions in primes for any of the five functions, including $\varphi(n)$. We find it really useful for students to experience that though the origin of these questions is the same, the results and methods vary in a wide range.

Solvability of equation $f(xy) + f(xz) = f(yz)$

Additive functions

First we study the solvability if f is a positive strongly or completely additive arithmetic function (by positive we mean that f assumes only positive values except for $f(1) = 0$), and then we apply the results for the number of distinct prime factors $\omega(n)$ and the number of prime factors counted with multiplicity $\Omega(n)$.

Strongly additive arithmetic functions

Definition 1.1
An arithmetic function f is strongly additive, if $f(p^\alpha) = f(p)$ for any prime p and integer $\alpha \geq 1$, and $(a, b) = 1$ implies $f(ab) = f(a) + f(b)$.

The next statement is straightforward:

Proposition 1.2
If $n = \prod_i p_i^{\alpha_i}$, and f is strongly additive, then $f(n) = \sum_{p|n} f(p)$.

Theorem 1.3
If f is a positive strongly additive function, then all solutions of equation $f(xy) + f(xz) = f(yz)$ are $(1; y; z)$, where $(y, z) = 1$.

Proof
As any prime divisor p of y also divides xy and $f(p) > 0$, we have $\sum_{p|xy} f(p) \geq \sum_{p|y} f(p)$ for any $x, y \in \mathbb{Z}^+$. Equality holds if and only if every prime divisor of x is also a divisor of y. The same applies to prime divisors p of z and xz.[1]

This implies a lower bound for the left-hand side of the equation:

$$f(xy) + f(xz) = \sum_{p|xy} f(p) + \sum_{p|xz} f(p) \geq \sum_{p|y} f(p) + \sum_{p|z} f(p) = f(y) + f(z)$$

for any $x, y, z \in \mathbb{Z}^+$, and equality holds if and only if every prime divisor of x is a common divisor of y and z.

Now we establish an upper bound for the right-hand side of the equation:

$$\sum_{q|yz} f(q) = \sum_{q|y} f(q) + \sum_{q|z} f(q) - \sum_{q|y \,\wedge\, q|z} f(q) \quad \text{and} \quad f(q) > 0$$

imply

$$f(yz) = \sum_{q|yz} f(q) \leq \sum_{q|y} f(q) + \sum_{q|z} f(q) = f(y) + f(z)$$

for any $y, z \in \mathbb{Z}^+$. Equality holds if and only if $\sum_{q|y \,\wedge\, q|z} f(q) = 0$, i.e. y and z are coprime.

[1]We denote the set of positive integers by $\mathbb{Z}^+$ and the set of non-negative integers by $\mathbb{N}$.

The lower bound for the left-hand side and the upper bound for the right-hand side coincide, so $f(xy) + f(xz) = f(yz)$ holds if and only if both bounds hold with equality. This means that $(y, z) = 1$ on the one hand, and every prime divisor of x divides (y, z) on the other hand. Therefore, the set of prime divisors of x is empty, i.e. $x = 1$. So all solutions of $f(xy) + f(xz) = f(yz)$ are $(1; y; z)$ where y and z are relatively prime integers. ■

As a special case, all solutions of $\omega(xy) + \omega(xz) = \omega(yz)$ are $(1; y; z)$, where $(y, z) = 1$.

Completely additive functions

Definition 1.4
An arithmetic function f is completely additive, if $f(ab) = f(a) + f(b)$ for any $a, b \in \mathbb{Z}^+$. The next statement is straightforward:

Proposition 1.5
If $n = \prod_i p_i^{\alpha_i}$, and f is completely additive, then $f(n) = \sum_i \alpha_i \cdot f(p_i)$.

Theorem 1.6
If f is a positive completely additive function, then all solutions of equation $f(xy) + f(xz) = f(yz)$ are $(1; y; z)$, where y and z are arbitrary positive integers.

Proof
If f is completely additive, then the equation is equivalent to $f(x) + f(y) + f(x) + f(z) = f(y) + f(z)$. This holds if and only if $f(x) = 0$, so $x = 1$. The values y and z can be chosen arbitrarily. ■

As a special case, all solutions of $\Omega(xy) + \Omega(xz) = \Omega(yz)$ are $(1; y; z)$, where y and z are arbitrary positive integers.

Of course, when working with students, we should choose a different approach, the inductive method. We should consider first the question for the specific functions ω and Ω, and move on to formulate generalizations based on the essential properties used in the proofs.

We should start with Ω, as this is (relatively) easy and needs no additional preparation. Then we can ask the students to point out which properties of Ω were used in the proof. After clarifying these, they can be asked to find some other functions satisfying these requirements. They will probably come up with the logarithm and this is a good occasion to review its other properties, too. Then we can inspire them how to construct further such functions by asking e.g.: If (say) $f(2) = 3$, then which other values of f are determined? And if also (say) $f(3) = 5$? Step by step we can conclude that the values assumed at primes determine such a function completely. But we are not yet done, as we must check that choosing arbitrary values at the primes, the induced function satisfies $f(ab) = f(a) + f(b)$ for all positive integers a and b, indeed. Finally, we may add that such functions are called completely additive.

Turning to ω, we should look first for solutions with $x = 1$, and realize that y and z must be relatively prime. This helps to detect that for a fixed $x > 1$ there are no solutions. Finally we can summarize all solutions as $x = 1$ and $(y, z) = 1$. As for the generalization, students are already armed with the experience acquired from Ω, so it will be easier for them to establish the general result for this case, too.

Multiplicative functions

Now we turn to positive multiplicative functions assuming only integer values. Our target functions d, σ, and φ satisfy this condition.

Definition 1.7

An arithmetic function f is multiplicative, if $(a, b) = 1$ implies $f(ab) = f(a)f(b)$ and is completely multiplicative if this applies to any a, b positive integers.

We shall use the following simple observation:

Lemma 1.8

(A) Let f be a positive completely multiplicative function.

(i) Equation $f(xy) + f(xz) = f(yz)$ is equivalent to any of the following three conditions:

(1)

$$f(x)((f(y) + f(z)) = f(y)f(z)\,;$$

(2)

$$f(x) = \frac{f(y)f(z)}{f(y) + f(z)};$$

(3)

$$f^2(x) = (f(y) - f(x))(f(z) - f(x))\,.$$

(ii) The equation has no solution satisfying $x = y$ or $x = z$.

(B) The above are valid also for multiplicative functions concerning the pairwise relatively prime solutions of the equation.

Proof

A/(i) (1) follows from the completely multiplicative property. We obtain (2) by expressing $f(x)$ from (1) (the denominator is not 0 as f is positive). Multiplying (1) by $f(x)$ yields (3).

A/(ii) Substituting (say) $x = y$ into (1), it reduces to $f^2(y) = 0$ which is a contradiction. The same arguments apply to B. ■

We shall need a condition for the right-hand side in (2) to be an integer:

Lemma 1.9

The expression $\dfrac{ab}{a+b}$ defined on $\mathbb{Z}^+ \times \mathbb{Z}^+$ is an integer if and only if $a = r(r+s)k$ and $b = s(r+s)k$, where $(r,s) = 1$ and k are positive integers.

Proof

If a and b satisfy the above conditions, then $\dfrac{ab}{a+b} = \dfrac{rs(r+s)^2k^2}{k(r+s)^2} = rsk$ is an integer, as well.

To prove the converse, let $d = (a,b)$, i.e. $a = rd$ and $b = sd$, where $(r,s) = 1$. So the ratio $\dfrac{ab}{a+b}$ equals $\dfrac{rsd^2}{d(r+s)} = \dfrac{rsd}{r+s}$. As r and s are coprime, $(r+s, rs) = 1$ follows, so the ratio $\dfrac{rsd}{r+s}$ is an integer if and only if $r+s \mid d$. This means that $d = (r+s)k$ for some positive integer k. Substituting back into $a = rd$ and $b = sd$, we get the desired result. ■

Now we turn to the concrete functions d, σ, and φ. We start with d where we exhibit explicitly all solutions in prime powers and describe how to obtain all pairwise coprime solutions in general.

The number of divisors

Theorem 1.10

Consider the equation $d(xy) + d(xz) = d(yz)$.

(i) All prime power solutions are:

(4) $\quad x = p_1^{rsk-1}$, $y = p_2^{r(r+s)k-1}$, $z = p_3^{s(r+s)k-1}$, where p_1, p_2 and p_3 are distinct primes, r, s, k are positive integers, with at least one of them greater than 1, and $(r,s) = 1$, or

(5) $\quad x = p_1^{\frac{k\beta-1}{2}}$, $y = p_2^{\beta}$, $z = p_1^{\frac{k(\beta+2)-1}{2}}$, where p_1 and p_2 are distinct primes, k and β are positive odd integers with at least one of them greater than 1.

(ii) All pairwise coprime solutions can be obtained by fixing x arbitrarily and describing the corresponding infinitely many solutions in y and z.

Remark 1

As a corollary we obtain that the equation has no solutions in primes, but x and y can be primes while z is a 4th power of a prime for $k = 3, \beta = 1$ in (5).

Remark 2

We cannot prescribe the values of two variables arbitrarily. E.g. if both x and y are squares, there are no solutions in z, since the parities of the left-hand side and the right-hand side are different regardless of z being a square or not.

Proof

(i) We distinguish three cases: (A) the three variables are pairwise coprime; (B) exactly two of them are powers of the same prime; (C) they are all powers of the same prime.

A) $x = p_1^\alpha,\ y = p_2^\beta$ and $z = p_3^\gamma$, where primes p_i are distinct and $\alpha, \beta, \gamma \in \mathbb{Z}^+$.

By (2) in Lemma 1.8, we have $\alpha + 1 = \dfrac{(\beta+1)(\gamma+1)}{(\beta+1)+(\gamma+1)}$, since $d(n^\delta) = \delta + 1$, if n is a prime. By Lemma 1.9, this is equivalent to $\alpha + 1 = rsk, \beta + 1 = r(r+s)k$, and $\gamma + 1 = s(r+s)k$, where r, s, k are positive integers and $(r, s) = 1$. As $\alpha > 0$, we have $rsk > 1$, so at least one of r, s, and k is greater than 1. This gives (4).

B) The equation is symmetric in y and z, so we have two subcases: (B/I.) y and z; or (B/II.) x and (say) z are powers of the same prime.

B/I.) $x = p_1^\alpha,\ y = p_2^\beta$ and $z = p_2^\gamma$, where p_i are distinct primes and $\alpha, \beta, \gamma \in \mathbb{Z}^+$. By (1) in Lemma 1.8, we have $(\alpha+1)(\beta+\gamma+2) = \beta+\gamma+1$, a contradiction, since the right-hand side is always less than the left-hand side.

B/II.) $x = p_1^\alpha,\ y = p_2^\beta$ and $z = p_1^\gamma$, where p_i are distinct primes and $\alpha, \beta, \gamma \in \mathbb{Z}^+$. Again, by (1) in Lemma 1.8, we arrive at $\beta(\alpha - \gamma) + 2\alpha + 1 = 0$. This implies $\beta \mid 2\alpha + 1$, so $k\beta = 2\alpha + 1$ for some $k \in \mathbb{Z}^+$, where β and k are both odd, and at least one of them is greater than 1. This gives $\alpha = \dfrac{k\beta - 1}{2}$, hence $\beta\left(\dfrac{k\beta-1}{2} - \gamma + k\right) = 0$. As $\beta \neq 0$, this is equivalent to $\gamma = \dfrac{k(\beta+2)-1}{2}$. This yields (5).

C) $x = p^\alpha,\ y = p^\beta$ and $z = p^\gamma$, where p is a prime and $\alpha, \beta, \gamma \in \mathbb{Z}^+$.

We have $2\alpha + \beta + \gamma + 2 = \beta + \gamma + 1$, a contradiction, since the right-hand side is always less than the left-hand side.

(ii) By (3) in Lemma 1.8, $d^2(x) = (d(y) - d(x))(d(z) - d(x))$. This means that $d(y) - d(x)$ and $d(z) - d(x)$ are complementary divisors u_1 and u_2 of $d^2(x)$, and we can find all these pairs by factoring $d^2(x)$. So $d(y) = u_1 + d(x) = v_1$, $d(z) = u_2 + d(x) = v_2$. Each factorization of v_1 determines a system of exponents

in the standard form of y: if $v_1 = k_1 \cdot \ldots \cdot k_r$ with $r \geq 1$ and every $k_i > 1$, then all possible values of y are $y = p_1^{k_1-1} \cdot \ldots \cdot p_r^{k_r-1}$, where p_i are distinct primes. Similarly, each factorization of v_2 determines all possible values of z. To guarantee the pairwise coprime property, no pair of y and z can contain common prime factors or prime factors of x. ■

Studying the equation specifically for $d(n)$ offers a lot to improve the students' knowledge, skills, and perspective in many aspects. The notion and formula of $d(n)$ are easily accessible for them; the multiplicative property can be detected already from a few examples and then proved from the formula; and forming pairs of divisors is a simple but very efficient method to solve even seemingly hard problems. We can start by rephrasing this particular equation similar to the general formulas (1) and (2) in Lemma 1.8. This induces the natural demand to find the condition for the expression on the right-hand side of (2) to be an integer, so we can prove Lemma 1.9, an interesting question in itself. Armed with this machinery, students will come up soon with some particular solutions of the equation.

After listing these (probably sporadic) examples, we can start a systematic search. The multiplicative property facilitates handling the equation, so it is worth to restrict ourselves to pairwise coprime solutions. The next step can be to fix the value of x starting with $x = 1$ (some of the students' preliminary solutions probably are of this type). This leads to the equation $d(y)+d(z) = d(y)d(z)$ that can be transformed into $(d(y) - 1)(d(z) - 1) = 1$. Hence $d(y) = d(z) = 2$, so y and z are arbitrary distinct primes. For $x = 2$ we get $2(d(y) + d(z)) = d(y)d(z)$ yielding $(d(y) - 2)(d(z) - 2) = 4$, so $d(y) = d(z) = 4$, or one of them is 3 and the other is 6. So this equation gives rise also to solutions where y and/or z are not necessarily prime powers, e.g. $d(y) = d(z) = 4$ holds if y and/or z are products of two distinct (odd) primes. After a few more examples, the students themselves can discover the general pattern and the way to find all solutions for a given x, i.e. to prove part B of Theorem 1.10. In the meantime, they discovered formula (3) in Lemma 1.8, and learned how to solve Diophantine equations of type $uv + au + bv + c = 0$ in u and v where a, b, and c are given integers. Another advantage was to see that characterizing all solutions is not necessarily an explicit formula but can also be some algorithmic description of the above type.

Finally, we can make a search for solutions of special types. Asking the students, they most probably will come up with one or more of the following sets: primes, prime powers, squares, two or three variables assume the same value, and many others. So, we can do part A in Theorem 1.10 and/or A(ii) in

Lemma 1.8 and investigate many other questions depending on the sets suggested by the students. It may turn out that we cannot give a complete or even a partial answer in some cases, but it is good if the students see that it is a natural phenomenon in mathematical research that we are unable to solve a problem, at least temporarily (which may sometimes mean thousands of years, see the famous unsolved problems posed by the antique Greeks). And it is a good occasion for the students to use their creative energies both in raising original mathematical questions autonomously and in working alone or in teams on the solution of these questions proposed by them or their classmates.

Euler's totient function

There are many types of solutions of the equation for φ, and some of them are related to famous unsolved problems concerning primes.

Theorem 1.11
Consider equation $\varphi(xy) + \varphi(xz) = \varphi(yz)$.

(i) There are infinitely many solutions in powers of 2.

(ii) If x and y are powers of 2 in a solution, i.e. $x = 2^\alpha$ and $y = 2^\beta$ (with integers $\alpha, \beta \geq 0$), then $\beta = \alpha + 1$, and there is a simple algorithm to find all solutions for such a pair α, β.

(iii) For any integer $k > 31$, there exist at least 31 solutions where $x = 2^k$ and y is, but $z > 1$ is not a power of 2.

(iv) There is no solution where two variables are powers of the same odd prime.

(v) A triplet $(p_1; p_2; p_3)$ is a solution in primes if and only if p_i are distinct and $(p_1 - 1)^2 = (p_2 - p_1)(p_3 - p_1)$.

(vi) Any prime triplet of the form $(2k + 1; 3k + 1; 6k + 1)$ is a solution.

(vii) Any pair of twin primes induces a solution.

Proof

(i) The triplet $x = 2^\alpha$, $y = z = 2^{\alpha+1}$ is a solution for any integer $\alpha \geq 0$ since $2 \cdot \varphi(2^{2\alpha+1}) = \varphi(2^{2\alpha+2})$.

(ii) Let $x = 2^\alpha$ and $y = 2^\beta$, and $z = 2^s \cdot t$, where $s \in \mathbb{N}$ and $(t, 2) = 1$. After substitution and rearranging the equation we get $\varphi(t) = \dfrac{2^{\alpha+\beta-1}}{2^{\beta+s-1} - 2^{\alpha+s-1}} = \dfrac{2^{\alpha+\beta-1}}{2^{\alpha+s-1}(2^{\beta-\alpha} - 1)}$. Since $(2^{\alpha+\beta-1}, 2^{\beta-\alpha} - 1) = 1$, the ratio is an integer if and only if $2^{\beta-\alpha} - 1 = 1$, i.e. $\beta = \alpha + 1$ and $0 \leq s \leq \alpha + 1$.

The possible values for $z \neq y$ are determined by the odd solutions t of the equation $\varphi(t) = 2^{\alpha-s+1}$. For instance, if $\alpha = 2$, i.e. $x = 4$, $y = 8$, while $0 \leq s \leq 3$ holds, we have to check whether there exist odd solutions t of equations $\varphi(t) = 1$, $\varphi(t) = 2$, $\varphi(t) = 4$, and $\varphi(t) = 8$. These are: $t = 1$, $t = 3$, $t = 5$, and $t = 15$, respectively. So the solutions in z of equation $\varphi(xy) + \varphi(xz) = \varphi(yz)$ with $x = 4$ and $y = 8$ are $z_1 = 2^3 \cdot 1 = 8$, $z_2 = 2^2 \cdot 3 = 12$, $z_3 = 2^1 \cdot 5 = 10$, and $z_4 = 2^0 \cdot 15 = 15$, i.e. (4; 8; 8), (4; 8; 10), (4; 8; 12), and (4; 8; 15) in this special case (the first one being listed in (i), too).

(iii) If $x = 2^k$, then the assumptions and (ii) imply $y = 2^{k+1}$ and $z = 2^s \cdot t$, where $s \in \mathbb{N}$, $t > 1$ and $(t, 2) = 1$. We saw in the proof of (ii) that all solutions are characterized by $0 \leq s < k + 1$ and $\varphi(t) = 2^{k+1-s}$.

Since $\varphi(t)$ is a power of 2 and t is odd, t must be a product of distinct Fermat primes, i.e. primes of the form $2^m + 1$. The presently known five Fermat primes belong to the exponents $m \in \{1; 2; 4; 8; 16\}$. This implies that $\varphi(t) = 2^n$ has exactly one odd solution in $t > 1$ for any integer $1 \leq n \leq 31$ as n has a unique binary representation $n = \sum_{i=0}^{4} \varepsilon_i \cdot 2^i$, $\varepsilon_i \in \{0, 1\}$, $0 \leq i \leq 4$. For any fixed $k > 31$, we have at least 31 values of s satisfying $1 \leq k+1-s \leq 31$, and for every possible pair $(k; s)$, we have exactly one odd t with $\varphi(t) = 2^{k+1-s}$, so we have at least 31 solutions $z = 2^s \cdot t$.

(iv) If $x = p^\alpha$, $y = p^\beta$, and $z = p^s t$, where $s \in \mathbb{N}$, $p > 2$ and $(t, p) = 1$, then the equation is equivalent to $\varphi(t) = \dfrac{p^{\alpha+\beta-1}}{p^{\beta+s-1} - p^{\alpha+s-1}}$. This cannot hold for a prime $p > 2$ since the denominator is even, but the numerator is odd, so the fraction is not an integer.

If $x = p^s t$, $y = p^\alpha$, and $z = p^\beta$, where $s \in \mathbb{N}$, $p > 2$ and $(t, p) = 1$, then the equation is equivalent to $\varphi(t) = \dfrac{p^{\alpha+\beta-1}}{p^{\beta+s-1} + p^{\alpha+s-1}}$. This cannot hold for a prime $p > 2$ since the denominator is even, but the numerator is odd, so the fraction is not an integer.

Since the equation is symmetric in y and z, we have no more cases to check.

(v) We know from (ii) in Lemma 1.8 that there are no solutions for $x = y$ or $x = z$. Also $x = p$ and $y = z = q$ is impossible with primes $p \neq q$ since $2(p-1)(q-1) = q(q-1)$ holds only for $p = q = 2$. So x, y, and z have to be three distinct primes $(p_1; p_2; p_3)$. Since $\varphi(n) = n - 1$ if n is a prime, we obtain the result by (3) in Lemma 1.8.

Such prime triplets are e.g. $x = 5$, $y = 7$, $z = 13$, or more generally, if x and y are twin primes $x = p, y = p + 2$ and also $z = \dfrac{p^2 + 1}{2}$ is a prime.

(vi) This follows immediately from (v).

We obtain such prime triplets e.g. for $k = 2, 6, 26, 90$ or 200. For $k > 2$, only $k \equiv 0, 6, 20,$ or $26 \pmod{30}$ may work but it is unsolved whether there exist infinitely many such prime triplets.

We constructed these solutions using Lemma 1.9 with $r = 1, s = 2$. We can produce further similar types of triplets by choosing other values for r and s. E.g. if $r = 1, s = 3$, then we get $x = 3k + 1, y = 4k + 1$ and $z = 12k + 1$ yielding a solution of our equation when all three values are primes.

So, if there are infinitely many $k \in \mathbb{Z}^+$, for which $2k + 1, 3k + 1$ and $6k + 1$, or $3k + 1, 4k + 1$ and $12k + 1$ are primes – an unsolved problem so far –, then the equation $\varphi(xy) + \varphi(xz) = \varphi(yz)$ has infinitely many solutions in primes.

(vii) If p and $q = p + 2$ are twin primes, then $(2p; pq; 2q)$ is a solution since $p(p - 1)(p + 1) + 2(p - 1)(p + 1) = (p + 2)(p - 1)(p + 1)$ holds. So, if the twin prime conjecture is true, then we have infinitely many solutions, where x, y and z are all products of two distinct primes. ■

We suggest to investigate this problem similarly to the question we studied earlier involving the number of divisors. And while examining this question, students will realize some of the basic differences between the two functions on their own. If we are searching for pairwise coprime solutions and start our examination with $x = 1$, we get $1 = (\varphi(y) - 1)(\varphi(z) - 1)$, hence $\varphi(y) = \varphi(z) = 2$. All solutions of $\varphi(n) = 2$ are $n = 3, 4$ and 6, but only 3 and 4 are coprime, so for $x = 1$ the only solutions are $y = 3$, $z = 4$, or vice versa. As also $\varphi(2) = 1$, we have no coprime solutions for $x = 2$.

This suggests to continue by the values of $\varphi(x)$ instead of x, and study together $x = 3, 4,$ and 6 satisfying $\varphi(x) = 2$. Now our equation is $4 = (\varphi(y) - 2)(\varphi(z) - 2)$. One of the differences and difficulties compared to the number of divisors is that we can only have even factors here, since $\varphi(n)$ is even for every $n > 2$. It means that in this case there is only one possible factorization of 4, namely $4 = 2 \cdot 2$ which leads to $\varphi(n) = 4$, so y and z must be coprime elements of the set $\{5; 8; 10; 12\}$ and coprime to x. So for $x = 3$ the only solutions are $y = 5$, $z = 8$, or vice versa, and we have no pairwise coprime solution if $x = 4$ or $x = 6$.

At this point we can discuss in general that if an odd number n is a solution of $\varphi(n) = k$ for a fixed k, then also $2n$ is a solution. If x is even we cannot be sure that there exist two odd elements in the set we get for y and z. It leads to another question: are there values k in the range of $\varphi(n)$, for which it is guaranteed to have at least one odd solution in n. And this way we can discuss in general the role of Fermat primes in solving these kinds of problems. We can also experiment with other special types of primes, e.g. twin primes, as well.

While searching solutions when $\varphi(x)$ is fixed, we should highlight that opposite to $d(x)$, we have only finitely many possibilities in x for any fixed value of $\varphi(x)$. After not finding pairwise coprime solutions when $x = 2, 4$, or 6, we should make it clear that this does not mean that there are no solutions at all in y and z in these cases. If we do not restrict ourselves to pairwise coprime triplets, students can easily find solutions with $x = 2$, for instance $x = 2$, $y = 4$ and $z = 5$; or $x = 2$, $y = 6$ and $z = 3$; or $x = 2$, $y = 4$ and $z = 4$. For sure some students will come up with one of the last two, and it is a nice and easy task to prove the generalizations: for every positive integer k, triplets $(k; k(k+1); k+1)$ and $(2^k; 2^{k+1}; 2^{k+1})$ are solutions. Upon the latter one we can challenge students to find triplets in which two of the unknowns are powers of the same odd prime. and after a few failed attempts we should prove that there are no such triplets.

Another difference compared to the number of divisors is that there are even integers not in the range of $\varphi(n)$, so called nontotients, e.g. 14, 26, 34, 38... etc. (it is proved that there are infinitely many nontotients), which means that for a fixed value of $\varphi(x)$, not every theoretically possible even factorization is really an option. For instance, for $\varphi(x) = 12$, the equation is $144 = (\varphi(y) - 12)(\varphi(z) - 12)$, but the factorization $144 = 2 \cdot 72$ cannot lead to a solution, since $\varphi(n) = 14$ is not solvable.

The next two theorems examine the case when x and y are powers of two distinct primes. In the first one we determine all solutions where z is divisible by both of these primes.

Theorem 1.12

Let $x = p^\alpha$ and $y = q^\beta$, where p and q are distinct primes, $\alpha, \beta \in \mathbb{Z}^+$. Then equation $\varphi(xy) + \varphi(xz) = \varphi(yz)$ has a solution in z divisible by pq if and only if $x = 8$, $y = 9$; or x is a Mersenne prime, $y = x + 1$; or y is a Fermat prime, $x = y - 1$. To each such pair (x, y), we can find the corresponding values of z by a simple algorithm.

Proof

By assumption, $z = p^a q^b t$, where a, $b \in \mathbb{Z}^+$ and $(t, pq) = 1$. After substitution, rearranging and simplifying the equation we have $\varphi(t) = \dfrac{p^{\alpha-1}\, q^{\beta-1}}{p^{a-1}\, q^{b-1}\, (q^\beta - p^\alpha)}$. As

$q^\beta - p^\alpha$ is coprime to the numerator, this equality holds if and only if $q^\beta - p^\alpha = 1$ and $\varphi(t) = p^{\alpha-a}\, q^{\beta-b}$. We distinguish four cases.

I) $\alpha, \beta > 1$

First we prove that $q = 3$, $p = 2, \beta = 2$, $\alpha = 3$ is the only solution to equation $q^\beta - p^\alpha = 1$ in this case.

Since the difference of the powers is odd, p has to be 2 and q an odd prime, or vice versa.

Consider $p = 2$. Then $q^\beta - 1 = (q-1)(q^{\beta-1} + q^{\beta-2} + \cdots + q + 1) = 2^\alpha$. If $\beta > 1$ is odd, the second factor of the product is an odd divisor greater than 1, a contradiction. If $\beta > 1$ is even, $q^\beta - 1 = (q^{\frac{\beta}{2}} - 1)(q^{\frac{\beta}{2}} + 1)$. Both factors can only be powers of 2 if $q^{\frac{\beta}{2}} - 1 = 2$, thus $q = 3$, $\beta = 2$, so $\alpha = 3$.

Consider $q = 2$. Then $2^\beta = p^\alpha + 1$. If $\alpha > 1$ is odd, then $2^\beta = (p+1)(p^{\alpha-1} - -p^{\alpha-2} + \cdots + p^2 - p + 1)$, but the second factor of the product is an odd divisor greater than 1, a contradiction. If $\alpha > 1$ is even, then $p^\alpha + 1 \equiv 2\,(4)$, while $2^\beta \equiv 0\,(4)$ for any $\beta > 1$, a contradiction again.

Thus the only solution is $q = 3$, $p = 2, \beta = 2$, $\alpha = 3$, indeed.

So we get equation $\varphi(t) = 2^{3-a} \cdot 3^{2-b}$, which gives us six options for a and b:

A) If $(a; b) = (3; 2)$, then $\varphi(t) = 1$, so $t = 1$.

B) If $(a; b) = (3; 1)$, then $\varphi(t) = 3$, a contradiction, since $\varphi(t)$ cannot be odd for $t > 2$.

C) If $(a; b) = (2; 2)$, then $\varphi(t) = 2$, a contradiction, since $(t, 6) = 1$.

D) If $(a; b) = (2; 1)$, then $\varphi(t) = 6$, so $t = 7$.

E) If $(a; b) = (1; 2)$, then $\varphi(t) = 4$, so $t = 5$.

F) If $(a; b) = (1; 1)$, then $\varphi(t) = 12$, so $t = 13$. Thus, if $x = 2^3$, $y = 3^2$, then $z_1 = 2^3 \cdot 3^2$, $z_2 = 2^2 \cdot 3 \cdot 7$, $z_3 = 2 \cdot 3^2 \cdot 5$, and $z_4 = 2 \cdot 3 \cdot 13$.

II) $\alpha = 1, \beta > 1$

Then $q = 2$, p is a Mersenne prime, and $a = 1$. For example: if $x = 7$, $y = 2^3$, we need all values of t coprime to 14 satisfying $\varphi(t) = 2^{3-b}$ for some $1 \le b \le 3$. These are $t_1 = 5$, $t_2 = 3$, and $t_3 = 1$. So all solutions for $x = 7$, $y = 2^3$ are $z_1 = 2 \cdot 7 \cdot 5$, $z_2 = 2^2 \cdot 7 \cdot 3$, and $z_3 = 2^3 \cdot 7$.

We always have at least one solution: for $a = 1$ and $b = \beta$ we get $\varphi(t) = 1$, i.e. $t = 1$, so $z = xy$.

III) $\alpha > 1, \beta = 1$

Then q is a Fermat prime, $p = 2$, and $b = 1$. For example: if $x = 2^4$, $y = 17$, we need all values of t coprime to 34 satisfying $\varphi(t) = 2^{4-a}$ for some $1 \le a \le 4$.

These are $t_1 = 15$, $t_2 = 5$, $t_3 = 3$, and $t_4 = 1$. So all solutions for $x = 2^4$, $y = 17$ are $z_1 = 2 \cdot 17 \cdot 15$, $z_2 = 2^2 \cdot 17 \cdot 5$, $z_3 = 2^3 \cdot 17 \cdot 3$, and $z_4 = 2^4 \cdot 17$.

We always have at least one solution: for $b = 1$ and $a = \alpha$ we get $\varphi(t) = 1$, i.e. $t = 1$, so $z = xy$ similar to the case of Mersenne primes.

IV) $\alpha = \beta = 1$

Then $q = 3$, $p = 2$, and $a = b = 1$ yielding $\varphi(t) = 1$, i.e. $t = 1$. This gives the solution $x = 2$, $y = 3$, $z = 6$. ■

The hardest part of the proof was to show that the only consecutive prime powers are 8 and 9. This is a challenge even for the best students, though it is elementary but requires a sophisticated application of identities. Students generally dislike such identities but this application can convince them that these can be very efficient tools. And they will be even more proud of their achievement if they learn about the story of the natural generalization of the problem, namely that 8 and 9 are the only consecutive powers of integers. This famous conjecture of Catalan dating from 1844 was unsolved for more than a century. In 1976 Tijdeman proved that if n is large enough, then both n and $n+1$ cannot be powers. His bound was so enormous that computers were unable to check the possibility of consecutive powers up to this limit. Finally, in 2002, Mihailescu settled the question completely by proving that there are no other consecutive powers besides 8 and 9, indeed. [2]

The situation gets much more complicated when x and y are powers of two distinct primes, while z is relatively prime to at least one of them. We illustrate the increasing difficulties on powers of two concrete primes:

Theorem 1.13

Let $x = 3^\alpha$ and $y = 11^\beta$, where $\alpha, \beta \in \mathbb{Z}^+$. All solutions of equation $\varphi(xy) + \varphi(xz) = \varphi(yz)$ are obtained from $\alpha = 1, \beta = 2$, i.e. $x = 3$, $y = 121$, and the corresponding values of z are $z_1 = 93$ and $z_2 = 186$.

Proof

We distinguish three cases according to z being coprime to exactly one of x and y, or to both of them.

A) $z = 3^a \cdot t$, where $a \in \mathbb{Z}^+$ and $(t, 33) = 1$.

By substitution, simplifying and rearranging the equation we have $\varphi(t) = \dfrac{10 \cdot 3^{\alpha-a} \cdot 11^{\beta-1}}{10 \cdot 11^{\beta-1} - 3^\alpha}$. Since for any exponents $\alpha, \beta, a \in \mathbb{Z}^+$ the numerator and the denominator are coprime, their ratio is a positive integer if and only

[2]source: https://en.wikipedia.org/wiki/Catalan's_conjecture

if the denominator is 1. We show that $10 \cdot 11^{\beta-1} - 3^{\alpha} = 1$ can only hold for $\beta = 1$, $\alpha = 2$. If $\beta > 1$, then necessarily $3^{\alpha} \equiv -1\,(11)$. But there is no exponent α satisfying this congruence, because the remainders of 3^{α} mod 11 are 3, -2, 5, 4, and 1, respectively. So $\beta = 1$, $\alpha = 2$, and $\varphi(t) = 10 \cdot 3^{2-a}$ where $a = 1$ or $a = 2$, and $(t, 33) = 1$. We get a solution only for $a = 1$: $t_1 = 31$ and $t_2 = 2 \cdot 31$. So, $x = 3^2$, $y = 11$, $z_1 = 3 \cdot 31$, and $z_2 = 3 \cdot 2 \cdot 31$.

B) $z = 11^b \cdot t$, where $b \in \mathbb{Z}^+$ and $(t,\, 33) = 1$.

By substitution, simplifying and rearranging the equation we have $\varphi(t) = \dfrac{2 \cdot 3^{\alpha-1} \cdot 11^{\beta-b}}{11^{\beta} - 2 \cdot 3^{\alpha-1}}$. Using the same argument as in A), the denominator must equal 1. Equation $11^{\beta} - 2 \cdot 3^{\alpha-1} = 1$ cannot hold since for any $\alpha, \beta, b \in \mathbb{Z}^+$ the left-hand side and the right-hand side are incongruent mod 5: all powers of 11 are congruent to 1 mod 5, while the other term on the left-hand side is not divisible by 5.

C) $(z, 33) = 1$

In this case we have $\varphi(t) = \dfrac{10 \cdot 3^{\alpha-1} \cdot 11^{\beta-1}}{5 \cdot 11^{\beta-1} - 3^{\alpha-1}}$. The greatest common divisor of the numerator and the denominator is 2, so their ratio is a positive integer if and only if the denominator equals 2. But then the right-hand side is an odd integer greater than 1, so it cannot be equal to $\varphi(t)$. ■

In each case of the proof, we had to deal with exponential Diophantine equations. It causes no difficulty if the students have not yet learned about exponential problems in general, as we are working only with non-negative integer exponents and use just divisibility and remainders. And it is a well-known "trick" for students from a very young age that they should search for patterns and repetition, e.g. that finding the last digit or the last two digits of 2^{2021} depends on realizing (and at a later stage also proving) that the sequence of the last or last two digits of powers of 2 is periodic. This refers to examinations just mod 10 or 100, so discussing the equations in the proof of Theorem 1.13 gives us a good opportunity to underline that we can use similar methods for other moduli, too. Trying to prove that there exists no solution to an equation of this type, it is enough to find a suitable modulus for which the two sides of the equation are incongruent, and we have some strategies to find such nominees, e.g. try the ones which are divisors of the base or are one less than the base of the power. Of course we do not have to use the concept of congruences, but it is certainly one of the concepts students understand and apply easily.

The sum of divisors

Several results show analogy to the ones obtained for $\varphi(n)$.

Theorem 1.14

Consider equation $\sigma(xy) + \sigma(xz) = \sigma(yz)$.

(i) There are no solutions where all variables are powers of the same prime.

(ii) A triplet $(p_1; p_2; p_3)$ is a solution in primes if and only if p_i are distinct and $(p_1 + 1)^2 = (p_2 - p_1)(p_3 - p_1)$.

(iii) Any prime triplet of the form $(4k - 1; 5k - 1; 20k - 1)$ is a solution.

Proof

(i) If $x = p^\alpha, y = p^\beta$, and $z = p^\gamma$, where $\alpha, \beta, \gamma \in \mathbb{Z}^+$, then $\dfrac{p^{\alpha+\beta+1} - 1}{p - 1} + {}$ $+ \dfrac{p^{\alpha+\gamma+1} - 1}{p - 1} = \dfrac{p^{\beta+\gamma+1} - 1}{p - 1}$. As $\dfrac{p^k - 1}{p - 1} = p^{k-1} + \cdots + p + 1 \equiv 1 \pmod{p}$ for every $k \in \mathbb{Z}^+$, the left-hand side is 2 (mod p) while the right-hand side is 1 (mod p). So equality cannot hold.

(ii) By Lemma 1.8, there are no solutions where $x = y$ or $x = z$. If $x = p$ and $y = z = q$, where $p \neq q$, then $2(p + 1)(q + 1) = q^2 + q + 1$, a contradiction, since the left-hand side is even, while the right-hand side is odd. So the primes p_i must be distinct, and we obtain the equality in the statement by (3) in Lemma 1.8.

Such prime triplets are e.g. $x = 5$, $y = 7$, $z = 23$, or more generally, if x and y are twin primes $x = p, y = p + 2$ and also $z = \dfrac{p^2 + 4p + 1}{2}$ is a prime.

(vi) This follows immediately from (ii).

We constructed these solutions using Lemma 1.9 with $r = 1, s = 4$. We can produce further similar types of triplets by choosing other values for r and s. E.g. if $r = 2, s = 3$, then we get $x = 6k - 1, y = 10k - 1$ and $z = 15k - 1$ yielding a solution of our equation when all three values are primes.

So, if there are infinitely many $k \in \mathbb{Z}^+$, for which $4k - 1, 5k - 1$ and $20k - 1$, or $6k - 1, 10k - 1$ and $15k - 1$ are primes – an unsolved problem so far –, then the equation $\sigma(xy) + \sigma(xz) = \sigma(yz)$ has infinitely many solutions in primes. ■

Earlier we showed that $d(xy) + d(xz) = d(yz)$ has no solutions where two variables are squares by comparing the parity of both sides. Since $\sigma(n)$ is odd if and only if $n = m^2$ or $n = 2m^2$ for some positive integer m, it follows that also $\sigma(xy) + \sigma(xz) = \sigma(yz)$ has no solutions where two variables are squares or doubles of a square.

Finally we illustrate that fixing the values of two variables, the number of solutions can vary from infinitely many to zero.

Theorem 1.15

Let $x = 2$. There exist values of y for which the number of solutions in z is
(i) infinitely many;
(ii) one;
(iii) zero.

Proof

(i) If $x = 2, y = 3$, then $z = 11 \cdot 3^{\alpha}$ is a solution for any $\alpha \in \mathbb{N}$.

By substitution we have $12 + 36 \cdot \dfrac{3^{\alpha+1} - 1}{2} = 12 \cdot \dfrac{3^{\alpha+2} - 1}{2}$. Simplifying and rearranging this equality we get $12 + 18 \cdot (3^{\alpha+1} - 1) - 6 \cdot (3 \cdot 3^{\alpha+1} - 1) = 0$, which holds for any $\alpha \in \mathbb{N}$, indeed.

(ii) If $x = 2$, $y = 11$, then we know from (i) that $z = 3$ is a solution (since the equation is symmetric in y and z). We prove that this is the only solution in this case.

Let the standard form of z be: $z = 2^{\alpha} \cdot 11^{\beta} \cdot t$, where $\alpha, \beta \in \mathbb{N}$ and $(t, 22) = 1$.

By substitution we have $(2^{\alpha+2} - 1) \cdot \dfrac{11^{\beta+1} - 1}{10} \cdot \sigma(t) + 36 = (2^{\alpha+1} - 1) \cdot \dfrac{11^{\beta+2} - 1}{10} \cdot \sigma(t)$. Simplifying and rearranging the equation we get $180 = [11^{\beta+1} \cdot (9 \cdot 2^{\alpha} - 5) + 2^{\alpha}] \cdot \sigma(t)$.

The right-hand side is not less than 485, so only $\beta = 0$ is possible. Then we have $180 = [100 \cdot 2^{\alpha} - 55] \cdot \sigma(t)$ yielding $\alpha = 0$. Finally $180 = 45 \cdot \sigma(t)$ implies $\sigma(t) = 4$, i.e. $t = 3$, so $z = 3$.

(iii) If $x = 2, y = 5$, then there are no solutions.

Let the standard form of z be: $z = 2^{\alpha} \cdot 5^{\beta} \cdot t$, where $\alpha, \beta \in \mathbb{N}$ and $(t, 10) = 1$.

By substitution, $(2^{\alpha+2} - 1) \cdot \dfrac{5^{\beta+1} - 1}{4} \cdot \sigma(t) + 18 = (2^{\alpha+1} - 1) \cdot \dfrac{5^{\beta+2} - 1}{4} \cdot \sigma(t)$. Simplifying and rearranging this equality we get $36 = [5^{\beta+1} \cdot (3 \cdot 2^{\alpha} - 2) + 2^{\alpha}] \cdot \sigma(t)$.

The right-hand side is not less than 126, so we only have to check two possible values for β.

If $\beta = 0$, then $36 = [16 \cdot 2^{\alpha} - 10] \cdot \sigma(t)$ implying $\alpha = 0$. So $\sigma(t) = 6$, i.e. $t = 5$, a contradiction, since it is not relatively prime to 10.

Checking $\beta = 1$, we have $36 = [76 \cdot 2^{\alpha} - 50] \cdot \sigma(t)$ yielding no solution in t for any non-negative integer α.

So there are no solutions for $x = 2$ and $y = 5$. ■

As students experience the similarities in handling the equation for φ and σ, it is a good occasion to point out some further related features. E.g. $|f(n) - n| = 1$ holds for both functions if and only if n is a prime; their average order of magnitude is symmetric to n: $\dfrac{6}{\pi^2}n$ and $\dfrac{\pi^2}{6}n$; both are connected to famous antique problems: constructing regular polygons and perfect numbers.

The equation $f(x^2) + f(y^2) = f(z^2)$ and its generalization

We investigate the equation first for the numbers of distinct and all prime divisors, respectively.

Proposition 2.1

Equations $\omega(x^k) + \omega(y^k) = \omega(z^k)$ and $\Omega(x^k) + \Omega(y^k) = \Omega(z^k)$ have no solutions in primes for $k > 1$. The first equation has no solutions even among the prime powers, whereas the second equation has infinitely many such solutions.

Proof

Since $\omega(x) = \omega(x^k)$, and $\Omega(x^k) = k \cdot \Omega(x)$ for any k, the equations are equivalent to $\omega(x) + \omega(y) = \omega(z)$ and $\Omega(x) + \Omega(y) = \Omega(z)$, respectively. The first equation cannot hold for prime powers since $1 + 1 \neq 1$. All prime power solutions of the second equation are $x = p_1^{\alpha}$, $y = p_2^{\beta}$, $z = p_3^{\alpha+\beta}$, where p_1, p_2, p_3 are primes, and α, $\beta \in \mathbb{Z}^+$ are arbitrary. Similar statements are true if we have more terms on the left-hand side. ■

Now we turn to multiplicative functions, starting with the number of divisors.

Proposition 2.2

Equation $d(x^k) + d(y^k) = d(z^k)$ has no solution in positive integers for any $k > 1$.

Proof

Since $d(n^k) \equiv 1 \pmod{k}$, the left-hand side is congruent to 2 mod k, while the right-hand side is 1 mod k. ■

When working with students, it is good to see some special cases before turning to the general problem. As an analogy to the Pythagorean triplets, let $k = 2$, so $d(x^2)+d(y^2) = d(z^2)$, and we can ask the students to search for solutions in primes first. They will immediately find that there are no such solutions as $3+3 \neq 3$. Then we can advance step by step to prime powers, to products of two prime powers, or whatever the students suggest to look at, and they will come up with longer and longer calculations to realize that there are no solutions in these sets either. Sooner or later they will formulate the conjecture that there are no solutions at all in positive integers. And it will be very instructive to see the one-line argument that as the number of divisors of a square is odd, the two sides of the equation are of opposite parity. This is a good example that a simple proof can be much more appreciated if it was found after hard struggles with complicated ideas. The students can extend then this argument for any even exponent k instead of 2. Turning to odd exponents k, we can start with $k = 3$. After characterizing the standard form of cubes, we can see that the number of their divisors has a remainder 1 mod 3, yielding immediately that there are no solutions for cubes either. It is important to point out that in contrast to the case of squares, $d(n) \equiv 1 \pmod 3$ holds not only for the cubes, but also e.g. for products of an even number of primes. And after the cubes it is straightforward to generalize the proof of insolvability to kth powers for any k.

Another direction of generalization is to increase the number of terms on the left-hand side of the equation. Taking three terms first, $d(x_1^2) + d(x_2^2) + d(x_3^2) = d(z^2)$ has infinitely many solutions, e.g. $x_1 = p_1, x_2 = p_2, x_3 = p_3$ and $z = p_4^4$, where p_i are primes. For four terms we have no solutions due to the opposite parity of the two sides. The same applies to any even number of terms, whereas there are infinitely many solutions if the number of terms is an odd integer n: $\sum_{i=1}^{n} d(x_i^2) = d(z^2)$ holds with $x_i = p_i^{\alpha_i}$, $z = q^\beta$, where p_i and q are primes, $\alpha_i \geq 1$ and $\sum_{i=1}^{n}(2\alpha_i + 1) = 2\beta + 1$ (and in prime powers these are the only solutions). Or more generally, when the exponents equal k, equation $\sum_{i=1}^{n} d(x_i^k) = d(z^k)$ is solvable if and only if the number of terms on the left-hand side is congruent to 1 mod k, and there are infinitely many solutions in this case: e.g. $x_i = p_i^{\alpha_i}$, $z = q^\beta$, where p_i and q are primes, $\alpha_i \geq 1$, and $\sum_{i=1}^{n}(k\alpha_i + 1) = k\beta + 1$ (and in prime powers these are the only solutions).

We can apply the parity argument also for the sum of divisors if the exponent k is even:

Proposition 2.3

Equation $\sigma(x^k) + \sigma(y^k) = \sigma(z^k)$ has no solution in positive integers for any *even* exponent k.

Proof

As the sum of divisors of a square is odd, the left-hand side is even while the right-hand side is odd. ■

It is surprising compared to and contrary to the number of divisors that there exist solutions of equation $\sigma(x^k) + \sigma(y^k) = \sigma(z^k)$ when k is odd, for example

$\sigma(3^3) + \sigma(10^3) = \sigma(13^3)$. This and other solutions can be found with the help of mathematical softwares. From this alone we can get infinitely many solutions with $x = 3n, y = 10n$ and $z = 13n$, $(n, 390) = 1$. Finding all solutions for a fixed odd exponent or – being more modest – finding all $(x, y, z) = 1$ solutions seems to be hopelessly difficult. The same applies to the case when the number of terms on the left-hand side is odd: it is probably beyond our means to give all solutions, but we can find some of them, for example $\sigma(11^2) + \sigma(17^2) + \sigma(23^2) = \sigma(31^2)$. Multiplying each variable by integers coprime to each of them generates again infinitely many solutions.

Finally we investigate Euler's totient function.

Theorem 2.4

Equation $\varphi(x^2) + \varphi(y^2) = \varphi(z^2)$ has no solution in primes.

Proof

If x, y, and z are primes, we have equation $x(x-1) + y(y-1) = z(z-1)$ which is equivalent to $x(x-1) = (z-y)(z+y-1)$. Since x is a prime, it must divide one of the factors of the right-hand side.

A) $x|z - y \Leftrightarrow kx = z - y$, for some $k \in \mathbb{Z}^+$.

This implies $x - 1 = k(kx + 2y - 1)$, a contradiction, since the left-hand side is less than the right-hand side.

B) $x|z + y - 1 \Leftrightarrow kx = z + y - 1$, where $k \in \mathbb{Z}^+$.

This implies $x - 1 = k(kx - 2y + 1)$, so

$$y = \frac{(k+1)((k-1)x+1)}{2k} \quad \text{and} \quad z = \frac{k^2x + x + k - 1}{2k}$$

follows.

As $(k, k+1) = 1$, y is a prime exactly in the following three cases:

(a) $\dfrac{k+1}{2} = 1$ and $\dfrac{(k-1)x+1}{k}$ is a prime, a contradiction, since we have $k = 1$ from the first equation implying $y = 1$, which is not a prime.

(b) $\dfrac{k+1}{2}$ is a prime and $\dfrac{(k-1)x+1}{k} = 1$. Then $(k-1)(x-1) = 0$ giving $k = 1$ or $x = 1$ contradicting to $\dfrac{k+1}{2}$ and x being a prime.

(c) $k + 1$ is a prime and $\dfrac{(k-1)x+1}{2k} = 1$

From the second equality $k \neq 1$, so $x = 2 + \dfrac{1}{k-1}$. Then x is an integer only for $k = 2$. So $x = 3$, $y = 3$, and $z = 4$, but 4 is not a prime, a contradiction. ■

When working with students first we should show that the equation has many solutions of different types. The students will probably find easily e.g.

$\varphi(2^2) + \varphi(3^2) = \varphi(4^2)$ or $x = 4, y = 6$ and $z = 5$. Multiplying each variable by integers coprime to each of them generates again infinitely many solutions. Using mathematical softwares or programming they can also find pairwise coprime solutions, for instance $x = 3, y = 25$ and $z = 23$ or with $(x, y, z) = d > 1$ that is not obtained by multiplying all variables of another solution by the same integer, e.g. $\varphi(3^2) + \varphi(12^2) = \varphi(9^2)$. They might come up with solutions containing two primes (other than the first one, e.g. $\varphi(2^2) + \varphi(10^2) = \varphi(7^2)$). After finding solutions with two primes, the question arises naturally whether there exist solutions in primes. We should try to find the answer for a fixed value of x, for instance $x = 2$. Supposing that both y and z are primes, we get $2 + y^2 - y = z^2 - z$. It can be solved as a quadratic equation of parameter z. So we write it as $y^2 - y - (z^2 - z - 2) = 0$, and the quadratic formula gives $y_{1;\,2} = \dfrac{1 \pm \sqrt{1 + 4(z^2 - z - 2)}}{2}$. The condition of y being an integer is for $1 + 4(z^2 - z - 2)$ to be an odd square. Since $1 + 4(z^2 - z - 2) = (2z - 1)^2 - 8$ and 9 and 1 are the only squares with a difference of 8, so $(2z - 1)^2 = 9$. Thus $z = 2$ and $y = 1$, a contradiction. Another way of solving the problem is rearranging the equation to $2 = z^2 - y^2 - (z - y)$, equivalently $2 = (z - y)(z + y - 1)$. The only possible factorization gives $z = 2$, $y = 1$, a contradiction. The second method can be generalized and we used it to prove that there is no solution in primes.

Turning to higher powers we can look for solutions of equation $\varphi(x^3) + \varphi(y^3) = \varphi(z^3)$. Quite surprisingly, with both sides not being greater than $\max(\varphi(n^3);\ 0 < n < 10^5 + 1)$, the first solution is $x = 107, y = 354$ and $z = 251$. We can also have more terms on the left-hand side with a solution in primes, e.g. $\varphi(3^2) + \varphi(11^2) + \varphi(13^2) = \varphi(17^2)$, but answering these questions in general is certainly out of our reach.

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