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Teaching Mathematics and Computer Science

Regula falsi in lower secondary school education II

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Abstract. The aim of this paper is to investigate the pupils' word problem solving strategies in lower secondary school education. Students prior experiences with solving word problems by arithmetic methods can create serious difficulties in the transition from arithmetic to algebra. The arithmetical methods are mainly based on manipulation with numbers. When pupils are faced with the methods of algebra they often have difficulty in formulating algebraic equations to represent the information given in word problems. Their troubles are manifested in the meaning they give to the unknown, their interpretation what an equation is, and the methods they choose to set up and solve equations. Therefore they mainly use arithmetical and numerical checking methods to solve word problems. In this situation it is necessary to introduce alternative methods which make the transition from arithmetic to algebra more smooth. In the following we will give a detailed presentation of the false position method. In our opinion this method is useful in the lower secondary school educational processes, especially to reduce the great number of random trial-and-error problem solving attempts among the lower secondary school pupils. We will also show the results of some problem solving activities among grade 6-8 pupils. We analysed their problem solving strategies and we compared our findings with the results of other research works.

Key words and phrases: False position method, regula falsi, guess-and-check, trial-and-error, arithmetical procedures, algebraic methods.

MSC Subject Classification: 97-03, 97-11, 97B10, 97B50, 97D40, 97F10, 97H10, 97H20, 97H30, 97N10, 97N20.

Introduction

Many pupils have difficulties in the transition from arithmetic to algebra. These difficulties firstly appear when pupils attempt to create algebraic equations to represent word problems. Teachers in order to help students the transition be more smooth have to overcome these difficulties. They must understand the pupils' cognitive processes when they solve these type of problems.

In order to underline the importance of the false position method in the lower secondary school education processes we have to examine a few aspects of the transition from arithmetic thinking to algebraic reasoning.

The transition from arithmetic to algebra involves a number of serious congitive difficulties (Tall & Thomas, 1991). The cognitive obstacles in learning algebra have been the subject of many studies which have shown that certain conceptual changes are necessary (L. Booth, 1984; L. R. Booth, 1984; Collis, 1974; Filloy & Rojani, 1984; Hersovics & Linchevski, 1994; Kieran, 1981).

The word-problem solving process is an interesting terrain for examining the two modes of thought and the conceptual changes that mark the transition from arithmetic to algebra, or the passage from "procedural thinking" to "structural thinking". The "procedural thinking" mainly means the arithmetic way of approaching problems. In this stage pupils can solve problems whose algebraic model is $A \cdot x + B = C$, and working backwards is the most efficient way of problem-solving. The "structural thinking" involves a whole overview of the problem structure, the pupils have to see the task entirely in order to adopt the most adequate problem-solving strategies. When pupils are able to think structurally, they posses an algebraic way of thinking and can solve word problems through equations where the unknown occurs on both sides. In the transition from arithmetic to algebra one of the important steps seems to be the solution of equations of the form $A \cdot x + B = C \cdot x + D$ (or the simpler form $A \cdot x + B = C \cdot x$). This particular form of equation has been a subject of dispute in international researches. Filloy and Rojano suggest that word problems whose algebraic model is this type of equation demands the teacher's intervention during the educational processes, namely it is a "didactic cut" (Filloy & Rojani, 1989). Herscovics and Linchevski locate their argument in the pupils' cognitive developement, from their point of view this transition involves the presence of a "cognitive gap" (Hersovics & Linchevski, 1994). It is also recognised that the teaching of algebraic thinking should not start when pupils begin to learn algebra in a formal way. In order to solve word problems by algebraic methods it is indispensable to adopt a structural way of reasoning.

One of the important themes that researchers have focused on is the problem solving procedure by the use of algebraic tools, especially the solution of linear equations and problems related to them. Previous research works has documented ways in which pupils' arithmetic experiences constitute obstacles for the learning of algebra. Most of these research has focussed on the differences between the two systems, for example, differing syntaxis (Londholz, 1993), closure (Collis, 1974; Kieran, 1992) use of letters as shorthand (L. Booth, 1984; L. R. Booth, 1984).

Kieran (1997) and Sfard & Linchevski (1994) have indicated difficulties related with the use and meaning of the symbol of equality. Arithmetic mainly is limited to numbers and numerical computations. In arithmetic computational procedures are separated from the objects obtained (Linchevski & Herscovics, 1999). A fundamental requirement of algebra is an understanding that the equal sign indicates equivalence and that information can be processed in either direction (Kieran, 1981; L. Linchevski, 1995). Many pupils' understanding reflects that the equal sign is action indication, such as "makes" or "gives" (Stacey & McGregor, 1997), or syntactic (showing the place where the answer can be written) (Filloy & Rojani, 1989). To the pupils the equal sign often means "carry out the operation". They are not aware that the equal sign expresses a relationship between the numbers or the algebraic expressions on each side of the equal sign. Misconceptions related to the equal symbol (algebraically this means an equivalence) make it very difficult for pupils to transform word-problems into algebraic equations and to solve these type of problems in an algebraic way (Kieran, 1992; L. Linchevski, 1995).

Kieran (Kieran, 1985) and Küchemann (Küchemann, 1981) found misunderstandings about the use and meaning of letters. Küchemann classified pupils' interpretations of algebraic letters into two major divisions: the letter is ignored, given an arbitrary value, or used as the name of an object; the letter is used as a specific unknown number or generalised number. MacGregor and Stacey (Stacey & McGregor, 1997) have shown, that difficulties in learning to use algebraic notation, beyond the cognitive difficulties, have several origins, including:

- intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system;
- analogies with symbol systems used in everyday life, in other parts of mathematics, or in other school subjects;
- interference from new learnings in mathematics;

- poorly-designed and misleading teaching materials.

Stacey and MacGregor (Stacey & McGregor, 2000) detailed examples which reveal some of the ways of thinking that pupils adopt when they have to deal with word problems. They analysed Australian pupils' solutions to word problems. The students involved were aged 13-16, mainly in their third or fourth year of algebra learning. Although the pupils were requested to solve the problems using algebra, they were generally more successful without algebra than with it. Even most of the clever pupils have adopted an arithmetic approach, working from the known numbers in stages towards the goal. The main conclusion was that pupils in solving mathematical word problems involving equations were found to apply the following different routes while they solve algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known number at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing an equation and solving it with the balance principle, and (e) algebraic route: solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning. According to this researchers we can distinguish two types of trial-and-error: random trial-and-error and sequential trial-and-error. There are a number of important factors that makes trial-and-error a good tool to use for solving problems. We have also to mention, that the purpose of trial-and-error is primarily to solve the problem, but in the same time the algebraic structure of the problem is ignored and we can not find out that the problem has other solutions.

Yerushalmy supports that the notion of function is the fundamental subject of algebra and that it ought to be present in the teaching and learning algebra from the very start (Yerushalmy, 2000). The concept of function, in a variety of representations, have to be one of the central ideas in the learning processes related to algebra. Many researchers suggest that teaching algebra based on functions is the most efficient way to develop the pupils' structural thinking (Kieran, 1992; Thorpe, 1989; Leitzel, 1989). Computers and graphing calculators make it easy to produce tables and graphs for functions, to describe real-word phenomena. The functional approach enables to construct formulas for functions and to perform algebraic operations on functions, so it is an important topic in algebra courses. Farmaki et al concluded that the functional approach gives a way to answer problems that are expressed by the equation with the unknown on both sides. Moreover pupils have at their disposal a suitable referential field in which the symbols acquire meaning, because of the familiarity of the situational context (Farmaki et al., 2004). At the same time the functional approach enables the developement of meta-cognitive processes, such as using different representations to find an answer.

False position method - Patterns and examples

The false position method or regula falsi method is a term for problem-solving methods in arithmetic, algebra, and calculus. In problems involving arithmetic or algebra, the false position method or regula falsi is confused with basic trial-and-error methods of solving problems by substituting test values for the unknown quantities. This is sometimes also referred to as guess-and-check. Versions of this method predate the advent of algebra and the use of equations. This is a specific arithmetical problem solving method used to solve word problems with two or three unknowns. In simple terms, this method begin by attempting to evaluate a problem using test values for the variables (which numbers, for the most part, happen to be false) and we try to compare the situation created in this way with the data and conditions of the problem. Taking into account this difference, we can conclude how to change the values of the variables to obtain the right answer in few steps. The importance of this method increases because research has revealed that pupils prefer to use arithmetic methods in solving algebraic word problems and show difficulties in setting up and using equations to solve such problems. There is also evidence that the most frequently used methods are guess-and-check or trial-and-error among 13-14 years old pupils.

In the Hungarian Mathematics education the Regula Falsi method has its own upholders. As I have mentioned in my own research works, György Maróthi in his book *Arithmetica* solved word problems by the method of *Regula Falsi Duarum Positionum*. I have also mentioned that for a present day mathematician or a mathematics teacher his problem solving methods are quite stereotyped and the problem solving process involves the use of some severe algorithms (Fülöp, 2016). Nonetheless the main idea of the false position method has its own right place in the arithmetical teaching activities, but it requires some modifications in order to follow the way of thinking of present day pupils.

In my own teaching activities I followed two main ways in the problem solving process by false position method, namely multiplication method and the increase/decrease method.

The multiplication method

We can introduce the multiplication method through the following exercise.

PROBLEM 2.1.: *A tickets for a soccer match for adults costs two times as much as a ticket for students. 253 tickets for adults and 358 tickets for students were sold, and the income from ticket sales was 561600 HUF in the aggregate. How much is a ticket for students and for adults, respectively?*

From the teachers point of view the algebraic model is the following system of equations with two unknowns:

$$358 \cdot x + 253 \cdot y = 561600y = 2 \cdot x.$$

The problem also could be solved by the use of the linear equation

$$358 \cdot x + 253 \cdot 2 \cdot x = 561600.$$

An important question is how we can solve this problem by false position method.

We take an arbitrary number for the price of a ticket for students, for example 200 HUF. In this way we make our *first position*, namely the tickets are 200 HUF for students, and 400 HUF for adults, respectively. So the income is $253 \cdot 400 + 358 \cdot 200 = 172800$ HUF in the aggregate. But the real income is $561600 : 172800 = 3.25$ times as much as the income from our first position, therefore the tickets are $3.25 \cdot 200 = 650$ HUF for students and $3.25 \cdot 400 = 1300$ HUF for adults.

PROBLEM 2.2.: *We bought red and white balls. A red ball was 3 euros and a white ball was 2 euros. We paid 634 euros in the aggregate. The number of red balls is 6 more than three times the number of white balls. How many red balls and white balls did we buy, respectively?*

The algebraic model is the following system of equations

$$2 \cdot x + 3 \cdot y = 634y = 3 \cdot x + 6$$

where x and y denote the number of white balls and the number of red balls, respectively.

In order to use the multiplication method we have to make the transformation $y_1 = y - 6$ so the system of equations becomes

$$2 \cdot x + 3 \cdot y_1 = 634 - 3 \cdot 6 y_1 = 3 \cdot x$$

where y_1 denotes the number of red balls diminished by 6.
But how can a teacher explain this train of thought to lower secondary school pupils? If we remove the extra 6 red balls, the number of the red balls is three times the number of the white balls, and value of balls becomes $634 - 6 \cdot 3 = 616$ euros im the aggregate. In this way, the problem becomes analogous to PROBLEM 2.1. and we can make our *first position* in the same way. Let us suppose, for example, that there are 20 white balls and 60 red balls. Their value is $2 \cdot 20 + 3 \cdot 60 = 220$ euros. But we paid $616 : 220 = 2.8$ times as much as the price from our first position, so we bought $2.8 \cdot 20 = 56$ white balls. The number of red balls is $3 \cdot 56 + 6 = 174$.

The above thread requires more ability than the increase/decrease method, as follows.

The increase/decrease method

This method is more available even for medium or low achievers. The main idea is to make two suppositions (the *first position* and the *second position* in the following) and we follow the variation of the error. Firstly we solve PROBLEM 2.2 in this way.

First position: Let us consider, for example, we bought 10 white balls. So from the first condition the number of red balls is $3 \cdot 10 + 6 = 36$ and in this way we had to pay $2 \cdot 10 + 3 \cdot 36 = 128$ euros. But this solution contradicts the second condition referring to the price of the balls. Therefore the error of the first position is $634 - 128 = 506$ euros.

Second position: Let us increase the number of white balls by one. So we bought 11 white balls and $3 \cdot 11 + 6 = 39$ red balls. In this case, the price of the balls is $2 \cdot 11 + 3 \cdot 39 = 139$ euros and the error of the second position becomes $634 - 139 = 495$.

We can see if the number of white balls increases by one, the error decreases by 11. Therefore to decrease the error by 506 (see the first position), we have to increase the number of white balls by $506 : 11 = 46$. So the number of white balls is 56 and the number of red balls is $3 \cdot 56 + 6 = 174$. This satisfy the second condition referring to the price of the balls.

We can summarise our calculations in the following table:

	white balls	red balls	price	error
First position	10	36	128	506
Second position	11	39	139	495
Increase/decrease	+1	+3	+11	-11
Right answer	56	174	634	0

In the following we will solve an exercise from the central national examination in Hungary, school-year 2012-2013, grade 8 (Oktatási Hivatal, Felvételi feladatsorok a 9. évfolyamra).

PROBLEM 2.3.: *A coffee roasting factory produces two types of coffee. The firsts type costs 2500 HUF/kg, and the second type costs 3300 HUF/kg. 80 kilos of coffee mixture was ordered from the factory on the unit price of 3000 HUF/kg. How much should be added from both types to obtain the desired mixture?*

First position: Let us suppose that we get equal quantities of both types. So the price of the mixture is $40 \cdot 2500 + 40 \cdot 3300 = 232000$ HUF. But 80 kilos of the ordered mixture worth $80 \cdot 3000 = 240000$ HUF. Therefore the error of the first position is $240000 - 232000 = 8000$.

Second position: The price of the mixture resulting from our first supposition is less than the actual price from the data of the exercise. Therefore we increase the amount of the second type by one kilo (so we have to decrease the amount of the first type by one kilo). In this way the price of the mixture becomes $39 \cdot 2500 + 41 \cdot 3300 = 232800$ and the error of the supposition becomes $240000 - 232800 = 7200$.

We can see that if we increase the amount of the second type by one kilo (the amount of the first type decreases by one kilo) the error decreases by 800 HUF. Therefore to decrease the error by 8000 (see the first position), we have to increase the amount of the second type by 10 kilos. So we have to take 50 kilos from the second type and 30 kilos from the first type to obtain the ordered mixture.

We can use the method of false position to solve more complicated word problems too, such as the following.

PROBLEM 2.4.: *A farmer has cows and pigs. The number of pigs is 17 more than three times the number of cows. A third of the cows and a half of the pigs were sent to another farm, and 82 cows and 78 pigs were sold. Now the number of pigs is 39 less than four times the number of cows. How many cows and how many pigs were on the farm at the beginning?*

The algebraic approach involves the following equation:

$$4 \cdot \left(x - \frac{x}{3} - 82\right) - 39 = 3 \cdot x + 17 - \frac{3 \cdot x + 17}{2} - 78$$

where x denotes the initial number of cows.

It may be a great challenge even for clever pupils to set up and solve this equation.

The false position method is a more reasonable way to approach this type of problem.

First position : At first we give the number of cows. It is a practical way to choose a number divisible by 3, so the number of cows is 147. Therefore the number of pigs is $3 \cdot 147 + 17 = 458$. After a third of the cows and a half of the pigs were sent to another farm, and 82 cows and 78 pigs were sold, there are 16 cows and 151 pigs on the farm. But according to the second condition the number of pigs must be by 39 less than four times the number of cows. In this way the number of pigs must be $4 \cdot 16 - 39 = 25$, and the error of the first position is $151 - 25 = 126$.

Second position : We increase the number of cows by 6, because in this way we can handle more easily the operations with fractions. So there are 153 cows and $3 \cdot 153 + 17 = 476$ pigs on the farm. As we perform the required operations, there are $2 \cdot 153 : 3 - 82 = 20$ cows and $476 : 2 - 78 = 160$ pigs on the farm. The error of the second position is $160 - (4 \cdot 20 - 39) = 119$.

If we increase the number of cows by 6 then the error will decrease by 7. The error of the first position is 126, therefore we have to increase the number of cows by $126 : 7 \cdot 6 = 108$. So there are $147 + 108 = 255$ cows and $3 \cdot 255 + 17 = 782$ pigs on the farm.

False position method and the introduction of algebraic thinking by a functional approach.

In the introduction we have mentioned that many research studies have been oriented to a functional approach in early algebra studies. Farmaki at al have investigated the teaching and learning of school algebra to 13 year-old pupils. They adopted a functional approach to algebra which widens the meaning of algebraic thinking. Through problems which are expressed by equations of the form $a \cdot x + b = c \cdot x + d$ or $a \cdot x + b = c \cdot x$, they examined the pupils' solution processes by the two approaches, functional and letter-symbolic. One of the problems they processed is the following.

PROBLEM 3.1.: *Mr. Georgiou goes by car every work day to the centre of Athens, where his office is located. Nearby there are two car parks. The first demands 4 euros to enter and 2 euros per hour. The second demands only 3 euros per hour. Mr. Georgiou does not have a regular timetable. So, his choice about where he parks his car depends on how many hours he will stay at his office. Questions: 1. Express the amount of money as a function of time for both car parks! 2. For how many hours can he park his car and pay the same amount of money at each car park?*

They used three ways to move from graphs to equations. First, giving a specific value of the dependent variable on a table and asking for the associate value of the independent variable x. Second, from the graph of $y = a \cdot x + b$, asking for the value of the independent variable x when y had a specific value. Third, equating the letter-symbolic representations of two lines, in order to find the coordinates of their point of intersection algebraically. In this way the pupils faced the equation through the function firstly using table, secondly graph and finally using the formal way to write the equation $2 \cdot x + 4 = 3 \cdot x$ by replacing $y = 3 \cdot x$ on the type of the function $y = 2 \cdot x + 4$. The authors's conclusion was that pupils can approach word problems more easily by the use of tables and graphs. One of the cognitive difficulties to the functional approach appeared when the pupils were faced with the presence of the two variables x and y, and they was trying to form the equation of the problem by some interesting conceptions, such as $y = 2 \cdot x + 4 = y = 3 \cdot x$. Authors consider this difficulty as natural for beginners, but attention must be paid in such cases to explain that y is a different "name" for the expression $2 \cdot x + 4$. The results of this research show that the algorithm of solving equations can hardly be considered as a way towards the development of algebraic thinking (Farmaki et al., 2004).

The above mentioned conclusions are in line with our ideas. To solve word problem by algebraic equations is a great challenge even to medium and high-achievers, especially in the phase of early algebra. Before we solve word problems using equations with unknowns on both sides the pupils have to pass from the "procedural thinking" to "structural thinking". The functional approach is beneficial way to accelerate this transition. The false position method shows some similarities with the functional approach. When pupils use the method of false position and make their own suppositions they have to deal with numbers instead letter symbolic representations and algebraic expressions. In fact they firstly give specific values of the independent variable and follow the variation of the error of their suppositions. Actually they have to search the value of the independent variable for which the two sides of a linear equation are equalised. In this way the transition to algebraic method is more smooth, because they have to replace the numbers with the symbol x (which denotes the independent variable) after they earlier gained a general overview about the situation context of the problem.

The main similarity between the functional approach and the false position method is the chain *specific number - variable - value of the unknown*, which follows the transition from procedural thinking (based on the operation with numbers), to the structural thinking (which requires a global overview and operations with the tools of algebra).

Moreover when students have got stronger algebraic knowledge both functional approach and false position method are useful tools in order to check the solution and to create new problem solving strategies. In this way students could use both of the methods even in the secondary school education until they have to solve word problems whose algebraic model is a linear equation.

Testing and results

In the Hungarian curriculum for the lower secondary school education equations and inequalities precede the fundamental notions of algebra, such as manipulation with letters, the basic knowledge regarding the notions of unknown or variable, etc. The solution of equations is presented in a typical way, based mainly on the balance principle. At the same time it is ignored the main purpose to set up and solve equations, namely to deal with real life problems. For this reason investigating the solution of word problems was another goal of our approach.

When I was a Mathematics teacher in the Reformed Primary School in Veresegyház (and I also was a PhD student at PhD School for Mathematics and Computational Sciences, University of Szeged), I set up the purpose to examine the possibilities to introduce the false position method in the secondary school educational processes. The mentioned school gather pupils from 15 localities, most of them have reformed religion. In all grades there are two classes, the average number of pupils is 25 per class. The pupils are well motivated, most of them high or medium achievers. In a set of extra-curricular activities I tested the grade 8 pupils' way of thinking and their affinity for the false position method. The results of the study were summarized in my recent paper (Fülöp, 2016).

Taking into account the encouraging results, I proposed to introduce the false position method in the grade 6 pupils' educational activities. 50 grade 6 pupils were involved in this study. After 18 hours of curricular activities regarding to solve word problems in arithmetic and algebraic ways, following the curriculum, I dedicated 2 hours to solve word problems by false position method (Fülöp, 2017). Their results we will mention as grade 6 experimental group in the following.

I am currently a high school teacher in the Reformed High School from Gödöllő This school gathers a part of the elite pupils in the region. The students are enrolled in the beginning of the 7th grade, and some of them has some previous elementary algebraic knowledge. All of these students have a serious attitude towards the study of Mathematics, generally their average is about 4,70. In this school I decided to continue the research work among the grade 7 pupils. In the Hungarian curriculum for the grade 7 pupils equations and inequations precede the word problem solving activities, namely the pupils have to use equations and inequations to solve word problems. For this reason investigating the pupils' word problem skills by algebraic and non-algebraic methods was the main goal of our approach. We also tracked out how the pupils use the false position method in solving word problems. In order to investigate the questions above, we completed the course on equations and word problems with the false position method. I coordinated this activity in three grade 7 classes, where I taught Mathematics. At the beginning of the solution of a problem, attention was paid to the false position method. So, problems which traditionally could be answered only by the solution of an equation were firstly treated in a non-algebraic way. Secondly, every word problem was solved by the formal use of linear equations. At the end of the course, a post-test was given to these students, considered from then on as the grade 7 experimental group, as well as to three grade 7 classes and two grade 8 classes, considered grade 7 control group and grade 8 control group, respectively. In the case of the control group the teaching of equations and word problems followed the textbook and my Mathematics teacher colleagues conducted these activities. We administered the test two weeks after we finished the algebra and word problem teaching activities. The test-paper contained 6 exercises, the same problems were solved by the grade 6 pupils mentioned above. The exercises were chosen by the author, and the pupils have 45 minutes to solve them. The pupils were asked to write in detail their attempts, to give reasons for their actions even though they could not solve the problem entirely. During the work on the solution, pupils were observed by their mathematics teachers. The results of the test is the following.

PROBLEM 1: *Ann has collected spiders and cockchafers, 38 altogether. These insects have 250 legs. A spider has 8 legs and a cockchafer has 6 legs. How many spiders and how many cockchafers has Ann collected?*

Tabel 1 shows the repartition of right and wrong answers.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	38	62	19	30
Wrong answer	11	7	34	20
No response	1	1	10	3
Total	50	70	63	53

Table 1

We also analysed the correct answers taking into account the methods used to solve the exercise.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	0	2	1	9
Arithmetic	0	0	4	0
Groping	11	1	14	18
False position	27	59	0	3
Total	38	62	19	30

Table 2

As *Table 2* shows most of the pupils used non-algebraic tools to solve the exercise, the most commonly used methods were groping and false position. This confirmed our previous experiences. When pupils have to deal with word problems which can be described algebraically by the following system of equations

$$a \cdot x + b \cdot y = cx + y = d$$

they mainly use groping (guess-and-check or trial-and-error) to solve the exercise.

Most of the pupils from the experimental group made the first position as "38 spiders and 0 cockchafers" or "0 spiders and 38 cockchafers". Thereafter they noticed that if we change a spider with a cockchafer (or a cockchafer with a spider) then the error will decrease by 2 and gave the right answer.

Although the pupils from the control group have not teached anything about the false position method previously, three of them used the same algorithm: they made their suppositions and then they searched consciously the variation of the error.

We have to underline the frequent appearance of the equation "$8 \cdot x + 6 \cdot x = 250$", where the number of spiders and cockchafers were denoted by the same letter. This shows that the symbol manipulation is difficult in this type of problems.

PROBLEM 2: *Mary bought a toy for 2410 HUF. She paid for this toy using 20 HUF and 50 HUF coins. She used 5 more 20 HUF coins than 50 HUF coins. How many coins did she use from each, respectively?*

The pupils' answers show the following.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	40	54	20	35
Wrong answer	10	15	32	16
No response	0	1	11	2
Total	50	70	63	53

Table 3

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	0	12	2	18
Arithmetic	0	0	2	2
Groping	24	4	16	14
False position	16	38	0	1
Total	40	54	20	35

Table 4

We have to mention the 6 graders' efficiency. 16 of them gave the right answer by the false position method, they summarised their attempts in tables, as follows (one pupil's work).

	50 HUF	20 HUF	value	error
First position	20	25	1500	910
Second position	21	26	1570	840
Solution	33	38	2410	0

Table 5

24 used groping, but most of them begun with the false position method and did not find the relation between their supposition and the error variation. The 7 graders from the experimental group did not use any groping, most of them gave the right answer by false position method.

The 8 graders from the control group gave the right answer by groping and by algebraic methods. They solved equations, such as $20 \cdot (x+5) + 50 \cdot x = 2410$ or $20 \cdot x + 100 + 50 \cdot x = 2401$. The most frequent erroneous equations were "$x + x + 5 = 2410$" ; "$20 \cdot x + 5 + 50 \cdot x = 2410$" or "$50 \cdot x + 5 = 20 \cdot x$". We have to underline the lack of the arithmetical methods.

PROBLEM 3: *A bus on the first day covered four-times-longer route than on the second day. The number of kilometers covered the first day is 135 more than the second day. How many kilometers did the bus cover each day?*

The pupils' answers show the following.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	31	68	26	46
Wrong answer	14	2	19	6
No response	5	0	18	1
Total	50	70	63	53

Table 6

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	0	37	13	38
Arithmetic	9	25	13	6
Groping	6	0	0	2
False position	16	6	0	0
Total	31	68	26	46

Table 7

Grade 7 and grade 8 pupils solved this problem very straightforward by algebraic methods. They solved the equation $4 \cdot x = x + 135$ or $4 \cdot x - x = 135$, some of them simply wrote the equation $3 \cdot x = 135$. The arithmetical method was also useful, the pupils gave the right answer by drawing line segments. It is interesting to follow the grade 6 pupils' works by false position method. 10 pupils gave the right answer in the following way (one pupil's work).

	First day	Second day	difference	error
First position	80	20	60	$135 - 60 = 75$
Second position	84	21	63	$135 - 63 = 72$
Solution	180	45	135	$135 - 135 = 0$

Table 8

These pupils made their suppositions taking into account that the first day route is four times the second day route, and they calculated the error from the difference of 135 km.

The others solved the problem thinking as follows (one pupil's work).

	First day	Second day	Four times the second day	error
First position	136	1	4	$136 - 4 = 132$
Second position	137	2	8	$137 - 8 = 129$
Solution	180	45	180	$180 - 180 = 0$

Table 9

In this case "the first day route is by 135 km longer then the second day route" was the starting point to make the suppositions, and the error was calculated from "the first day route is four times the second day route" condition.

PROBLEM 4: *Peter and Paul collect stamps. Paul has got 12 stamps more than twice the number of Peter's stamps. They have got 168 stamps altogether. How many stamps does each boy have?*

The pupils' answers show the following.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	26	64	34	37
Wrong answer	20	5	19	12
No response	4	1	10	4
Total	50	70	63	53

Table 10

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	1	51	25	35
Arithmetic	4	6	8	1
Groping	9	1	1	1
False position	12	6	0	0
Total	26	64	34	37

Table 11

This problem can be easily solved using algebraic methods. This is why most of grade 7 and grade 8 pupils wrote and solved the equation $x + 2x + 12 = 168$.

The grade 6 pupils solved the exercise mainly by false position method. In the following we will show one pupil's work.

	Peter's stamps	Paul's stamps	In the aggregate	error
First position	15	42	57	$168 - 57 = 111$
Second position	16	44	60	$168 - 60 = 108$
Solution	52	116	168	$168 - 168 = 0$

Table 12

This pupil used the condition "Paul has got 12 stamps more than twice the number of Peter's stamps" to make suppositions, and the condition "they have got 168 stamps in the aggregate" to calculate the error. The other pupils proceeded in the same way, choosing other numbers to make the suppositions. We have to underline the great number of wright answers by false position method or by simple groping among grade 6 students.

Although grade 7 and grade 8 students gave most of the correct answers by algebraic methods, we have to mention several erroneously written equations, such as "$x+12 = 2\cdot x$ " ; "$2\cdot x+2\cdot x+12 = 168$ " or "$\frac{2 \cdot x - 12}{2} + 12 + 2\cdot x = 168$ ". We have to mention the appearance of some incorrect arithmetic reasoning, for example "$168 - 12 = 156$; $156 : 2 = 78$; Peter has 78 stamps and Paul has 90 stamps" or "$168 : 3 = 56$, so Peter has 56 stamps and Paul has $2 \cdot 56 = 112$ stamps".

PROBLEM 5: *In a bookcase there are two shelves. On the second shelf there are three times as many books as on the first shelf. If we put 10 more books on the first shelf and we remove 13 books from the second shelf, then the number of books on the second shelf will be twice the number of books on the first shelf. Find the number of books on each shelf!*

The pupils' answers show the following.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	27	48	14	31
Wrong answer	13	20	33	15
No response	10	2	16	7
Total	50	70	63	53

Table 13

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	0	26	13	29
Arithmetic	0	0	0	0
Groping	13	1	1	2
False position	14	21	0	0
Total	27	48	14	31

Table 14

Examining the rate of right answers, we can see that grade 7 pupils from the experimental group were the most successful ones (68 %). Also we have to underline the efficiency of the grade 6 pupils (54 %), in comparison with the grade 8 pupils' wright answers rate (58 %). Examining the grade 7 pupils' answers from the control group, we have to mention the relatively low average of right answers (22%).

The correct use of the false position method, and the simply groping (we mean trial-and-error and guess-and check) lead the grade 6 pupils to the wright answer. The grade 8 pupils from the control group were successful with algebraic methods, but misleadings in writing equations occurred in several cases. The main mistake was the frequent occurrence of the reversal error, many pupils placed the constant of proportionality on the wrong side of the equation sign(they wrote the equation "$x + 10 = 2 \cdot (3x - 13)$"). Other wrong written equations were "$3x-13+x+10 = 2x+x$ ", "$3x-13+x+10 = 2\cdot(3x-13)$ ", "$x+10 = 2\cdot(x+10)$ " and "$x + 10 = 3x - 13$ ". This shows a lack of ability to generate and perceive a global overview about the situation structure of the problem. This pupils have difficulties to translate the word problem into the formal language of algebra and to deal with different parts of an equation including the equal sign due to poor understanding of the structural aspects of algebra. In some cases the pupils omitted the parenthesis, and wrote the equation $x + 10 \cdot 2 = 3 \cdot x - 13$.

PROBLEM 6: *Andrew has read a quarter of a book and 12 more pages. Two-thirds of the book is still to be read. How many pages does the book have?*

The pupils' answers show the following.

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Right answer	14	55	20	35
Wrong answer	16	13	24	9
No response	20	2	19	9
Total	50	70	63	53

Table 15

	grade 6 (E.G.)	grade 7 (E.G.)	grade 7 (C.G.)	grade 8 (C.G.)
Algebra	0	11	5	15
Arithmetic	5	41	15	20
Groping	3	0	0	0
False position	6	3	0	0
Total	14	55	20	35

Table 16

40 % of the grade 6 pupils did not have any time to deal with PROBLEM 6. This is due to the fact that grade 6 students solved all of the problems mainly by the false position method, which requires a relatively great amount of calculus and they did not have any time to deal with PROBLEM 6.

This problem can be solved more easily by arithmetic methods, and we can observed this in our survey too. Most of the pupils gave the wright answer in an arithmetical way of thinking (59 % of the grade 7 E.G., 24 % of the grade 7 C.G., 38 % of the grade 8 C.G.).

We also have to mention the increasing rate of the usage of the algebraic methods among grade 8 pupils. The grade 6 pupils completely omitted the algebraic tools. The grade 7 pupils' percentage regarding the right answers by algebraic method shows a relatively low average, namely 16 % (grade 7 E.G.) and 8 % (grade 7 C.G.). In the same time 28 % of the grade 8 pupils worked properly with algebraic tools. Several grade 8 pupils wrote the equation $\frac{x}{4} + 12 = \frac{x}{3}$, so in their case the algebraic skills are fully formed.

Nevertheless we need to highlight a few mistakes and erroneous written "equations", such as "$\frac{1}{4} + 12 + \frac{2}{3} = x$" and "$\frac{1}{4} + 12 + \frac{2}{3} = 1$" (thereafter these pupils gave the right answer by an arithmetical way of reasoning), which denotes that these pupils were not able to break away from the arithmetical conventions and to adopt an algebraic way of reasoning and the proper operations with letters. This is a real evidence, that lower secondary school pupils' skills to deal with algebraic symbolism are not completely formed. Some pupils wrote the equation "$\frac{x}{4} + 12 = \frac{2x}{3}$" which shows a misinterpretation of the situation context of the problem.

Summary and conclusions

Based on the analysis of pupils' works, we can conclude that the false position method helps students to develop their algebraic thinking. The false position method that we applied has given the chance for students to solve problems with different approaches and strategies. This teaching approach encourages the pupils to manipulate the situation context of the problem giving specific values to unknown quantities, and to check the variation of the error. The pupils could seek and recognize the relation between co-variable quantities and thereafter to write this relation in a letter-symbolic representation. The symbols gained meaning and importance for the pupils, after they treat the situation context of the problem with concrete numbers. In this way it is more smooth the transition from procedural thinking to structural thinking, that predate the transition from arithmetic to algebra. We observed that pupils gained a greater understanding when they set up an equation to solve the word problem after they firstly analysed the situation context in a procedural way than in the formal way, based on the letter-representation and on the properties of the equality and the operations. It is well known that for students of this age a great obstacle is the non consolidation of the operations and the meaning of the equality symbol. In this way a lot of difficulties and misunderstandings appear, such as reversal error, closure, misinterpretation of the relation among two or more variables, the errors in performing algebraic operations, the improper understanding of the notion of unknown and variable, etc. The above mentioned also emerges from our survey, where the results of the experimental group were better than the results of the control group.

In our opinion, the false position method has its own right place in the lower secondary school educational processes. Pupils mainly appeal to guess-and-check and trial-and-error when they are faced with unusual problems or some exercises which they did not met before. When the pupils are initiated in false position method they will adopt a more systematic way of thinking which can eliminate the traditional numerical checking methods that requires a huge amount of calculus. This method is strongly recommended in the early algebra teaching activities. In spite of the fact that we teach algebric methods to grade 6 pupils, they are not able to set up equations in order to solve word problems individually. This fact also emerges from our research work, so we have to underline the necessity to introduce the false possition method in the pre-algebra problem solving activities. At the same time there is the risk that the excessive use of the false position method can overwhelm the traditional algebraic methods in word problem solving activities.

From our teaching experience, after the consolidation of algebraic skills, pupils set up equations and the false position method becomes embarrassing for them in opposition to the straightforward methods of algebra. So the false position method is useful in the phase of early algebra but later the students may ignore it.

We recommend the false position method to lower secondary school teachers, in the word problems solving educational processes before and during the introduction of elementary algebraic procedures among grade 6 and 7 pupils.

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